UNION CONGRUENCES OF A SUBSPACE V_n EMBEDDED IN A RIEMANNIAN V_m

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Springer (1945, 1947, 1950) has defined a union curve on an ordinary surface as a curve such that the osculating plane at each point of the curve contains the line of a specified rectilinear congruence. He further developed the subject for a hypersurface in a Riemannian space by introducing a totally geodesic surface as an analogue for the osculating plane in ordinary space, and studied some properties of these curves. Mishra (1951, 1952) extended the notion of union curves to a subspace V_n embedded in a Riemannian V_m. He also discussed hyperasymptotic curves on an ordinary surface and in a V_n embedded in V_m. In this paper we introduce a union congruence of curves and some other congruences on similar lines.

1. Let V_n be a subspace with the coordinates x^i, and the fundamental metric $g_{ij}dx^idx^j$ $(i, j = 1, 2, \ldots, n)$, embedded in a Riemannian V_m of m dimensions with y^α as the coordinates of any point, and $a_{\alpha\beta}dy^\alpha dy^\beta (\alpha, \beta = 1, 2, \ldots, m)$ as the fundamental metric. We have the relation

$$a_{\alpha\beta}y^\alpha_{;i}y^\beta_{;j} = g_{ij}. \qquad (1)$$

Let $N^\alpha_{\nu|}$ $(\nu = 1, 2, \ldots, m-n)$ be the contravariant components of a system of $m-n$ linearly independent orthogonal unit normal vectors to V_n, so that

$$a_{\alpha\beta}N^\alpha_{\nu|}N^\beta_{\mu|} = \delta^\nu_\mu \qquad (2)$$

$$a_{\alpha\beta}y^\alpha_{;i}N^\beta_{\nu|} = 0. \qquad (3)$$

Let us consider a congruence of curves λ tangential to V_n, such that one curve of the congruence passes through each point of the subspace V_n. Let t^i and λ^α be the contravariant components of the unit tangent vector to a curve of the congruence λ in terms of the coordinates of V_n and V_m respectively.

We have

$$\lambda^\alpha = t^iy^\alpha_{;i}. \qquad (4)$$

The tensor derivative of (4) gives

$$\lambda^\alpha_{;i} = t^ly^\alpha_{;il}+t^l_{;i}y^\alpha_{;l}.$$

Using the known formula (Weatherburn 1957)

$$y^{\alpha}_{;ij} = \sum_{\nu} \Omega_{\nu|ij} N^{\alpha}_{\nu|},$$

where $\Omega_{\nu|ij}$ are symmetric covariant tensors of the second order, the above simplifies to

$$\lambda^{\alpha}_{;i} = \sum_{\nu} \Omega_{\nu|il} t^{l} N^{\alpha}_{\nu|} + t^{l}_{;i} y^{\alpha}_{;l}.$$

Let $x^i = x^i(s)$ be any given curve C in V_n and P any point on it. The derived vector of λ along the curve C considered as a curve in V_m is given by

$$\frac{\delta\lambda^{\alpha}}{\delta s} = \sum_{\nu} \Omega_{\nu|il} t^{l} \frac{dx^i}{ds} N^{\alpha}_{\nu|} + \frac{\delta(t^l)}{\delta s} y^{\alpha}_{;l}. \qquad \ldots \quad \ldots \quad (5)$$

This equation can be put in the following form (Nirmala 1963):

$$\frac{\delta\lambda^{\alpha}}{\delta s} = \kappa_{\lambda|} \omega^{\alpha}_{\lambda|1} = \kappa_{\lambda|n} N^{\alpha} + \kappa_{\lambda|g} a^{\alpha}, \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

where (i) $\kappa_{\lambda|}$ is the absolute curvature of the congruence λ w.r.t. C, (ii) $\kappa_{\lambda|n}$ and $\kappa_{\lambda|g}$ are the normal curvature and geodesic curvature of the congruence λ w.r.t. C respectively. Their magnitudes are given by

$$\left.\begin{aligned} \kappa^2_{\lambda|n} &= \sum_{\nu} \Omega_{\nu|li} \Omega_{\nu|mj} t^l t^m \frac{dx^i}{ds} \frac{dx^j}{ds} \\ \text{and} \qquad \kappa^2_{\lambda|g} &= g_{lm} \frac{\delta(t^l)}{\delta s} \frac{\delta(t^m)}{\delta s}. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

In a previous paper, we have worked out the Serret-Frenet formulae (Nirmala 1964) for a congruence. We have thereby

$$\frac{\delta\omega^{\alpha}_{\lambda|r}}{\delta s} = \omega^{\alpha}_{\lambda|r+1} \kappa_{\lambda|r+1} - \omega^{\alpha}_{\lambda|r-1} \kappa_{\lambda|r}$$

$$(r = 0, 1, 2, \ldots, m-1),$$

where

$$\omega^{\alpha}_{\lambda|0} = \lambda^{\alpha},\ \kappa_{\lambda|1} = \kappa_{\lambda|},\ \kappa_{\lambda|0} = 0,\ \kappa_{\lambda|m} = 0,$$

$\omega_{\lambda|0-1}$, $\omega_{\lambda|m}$ are the zero or null vectors,
and the vectors $\lambda^{\alpha}, \omega^{\alpha}_{\lambda|1}, \omega^{\alpha}_{\lambda|2}, \ldots, \omega^{\alpha}_{\lambda|m-1}$ are mutually orthogonal.

The vectors with contravariant components $\omega^{\alpha}_{\lambda|1}, \omega^{\alpha}_{\lambda|2}, \ldots, \omega^{\alpha}_{\lambda|m-1}$ have been called the unit principal normal vector, the first unit binormal vector, the second unit binormal vector, ..., to the congruence λ in V_m relative to the curve C at the point P. We can express these vectors in terms of the tangential and normal components on V_n. Hence we have

$$\omega^{\alpha}_{\lambda|r} = t^{i}_{r|} y^{\alpha}_{;i} + \sum_{\nu} a_{\nu|r} N^{\alpha}_{\nu|} \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$(r = 0, 1, 2, \ldots, m-1),$$

where $t^i_{r|}$ and $a_{\nu|r}$ are the tangential and normal components of $\omega_{\lambda|r}$ on V_n and

$$\omega^{\alpha}_{\lambda|0} = \lambda^{\alpha},\ t^i_{0|} = t^i,\ a_{\nu|0} = 0.$$

Also we have from (8),

$$1-\sum_{\nu} a^2_{\nu|r} = g_{ij}t^i_{r|}t^j_{r|} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

$$(r = 1, 2, \ldots, m-1).$$

Now let us consider a congruence μ of curves in V_m such that one curve of the congruence passes through each point of V_n. Let μ^{α} be the contravariant components of the unit vector tangential to a curve of the congruence μ at the point P on C. Resolving μ^{α} tangentially and normally to V_n, we have

$$\mu^{\alpha} = p^i y^{\alpha}_{;i} + \sum_{\nu} c_{\nu|} N^{\alpha}_{\nu|}. \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where p^i are the tangential components of μ in V_n and $c_{\nu|} = \cos\ \theta_{\nu|}$, $\theta_{\nu|}$ being the angle between the vectors μ and $N_{\nu|}$.
From (10), we have

$$1-p^i p_i = \sum_{\nu} c^2_{\nu|} = \sum_{\nu} \cos^2\ \theta_{\nu|}. \quad \ldots \quad \ldots \quad (11)$$

We shall call the surface generated by the geodesics tangential at P to the pencil of directions determined by the unit tangent vector λ^{α} and the unit principal normal vector $\omega^{\alpha}_{\lambda|1}$ as the osculating geodesic surface of the congruence λ w.r.t. C. If the congruence μ is such that the osculating geodesic surface of the congruence λ at each point of the curve C contains the unit tangent vector μ^{α} to a curve of the congruence μ, we shall then *call the congruence λ as a union congruence of the subspace V_n relative to the congruence μ.*

Hence we can express μ^{α} in the following form :

$$\mu^{\alpha} = r\lambda^{\alpha} + v\omega^{\alpha}_{\lambda|1}. \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

From (4), (8) and (10), this can be written as

$$p^i y^{\alpha}_{;i} + \sum_{\nu} c_{\nu|} N^{\alpha}_{\nu|} = rt^l y^{\alpha}_{;l} + v\Big(t^l_{1|} y^{\alpha}_{;l} + \sum_{\nu} a_{\nu|1} N^{\alpha}_{\nu|}\Big). \quad \ldots \quad (13)$$

Multiplying (13) by $a_{\alpha\beta} y^{\beta}_{;j}$,

$$p_j = rt_j + vt_{1|j}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Multiplying (13) by $a_{\alpha\beta} N^{\beta}_{\mu|}$,

$$c_{\mu|} = va_{\mu|1}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Taking the scalar products of (12) w.r.t. λ,

$$
\begin{aligned}
r &= a_{\alpha\beta}\mu^{\alpha}\lambda^{\beta} \\
&= a_{\alpha\beta}\Big(p^{i}y^{\alpha}_{;i}+\sum_{\nu} c_{\nu|}N^{\alpha}_{\nu|}\Big)t^{j}y^{\beta}_{;j} \\
&= p_{j}t^{j}. \qquad \ldots \qquad (16)
\end{aligned}
$$

From (15) and (16), (14) can be written as

$$a_{\nu|1}[p_{j}-p_{h}t^{h}t_{j}] = c_{\nu|}t_{1|j}. \qquad \ldots \qquad (17)$$

In this equation the suffix j can take any of the values $1, 2, \ldots, n$ and ν can take any of the values $1, 2, \ldots, m-n$. Hence we get altogether $n(m-n)$ equations in the quantities t and their solutions will determine the directions at P of the union congruences of a subspace relative to a given congruence μ of V_m. But on account of (15), the $(m-n)$ equations given by (17), for different values of ν, are identical for any one value of j. Hence we get only n differential equations in the quantities t. The equations (17) are thus always consistent and hence *union congruences associated with any given congruence μ exist.* We shall call these equations as *the differential equations of the union congruence of a subspace relative to the congruence μ.*

From (5), (6) and (8) we have

$$
\begin{aligned}
\frac{\delta\lambda^{\alpha}}{\delta s} &= \kappa_{\lambda|1}\Big(t^{l}_{1|}\, y^{\alpha}_{;l}+\sum_{\nu} a_{\nu|1}N^{\alpha}_{\nu|}\Big) \\
&= \sum_{\nu} \Omega_{\nu|li}t^{l}\frac{dx^{i}}{ds}N^{\alpha}_{\nu|}+\frac{\delta}{\delta s}(t^{l})\, y^{\alpha}_{;l}. \qquad \ldots \qquad (18)
\end{aligned}
$$

Multiplying (18) by $a_{\alpha\beta}y^{\beta}_{;j}$,

$$\kappa_{\lambda|1}t_{1|j} = g_{lj}\frac{\delta}{\delta s}(t^{l}). \qquad \ldots \qquad (19)$$

Multiplying (18) by $a_{\alpha\beta}N^{\beta}_{\nu|}$, we have

$$\kappa_{\lambda|1}a_{\nu|1} = \Omega_{\nu|li}t^{l}\frac{dx^{i}}{ds}. \qquad \ldots \qquad (20)$$

Using (19) and (20), *equation (17) can be written in the alternative form*

$$\Omega_{\nu|li}t^{l}\frac{dx^{i}}{ds}[p_{j}-p_{h}t^{h}t_{j}] = c_{\nu|}g_{lj}\frac{\delta}{\delta s}(t^{l}) \qquad \ldots \qquad (21)$$

which is obviously a differential equation of the first order in the t's and the derivatives of t.

Kaul (1957) has dealt with the converse of this problem for a hypersurface. Given a vector field in V_n, he discusses the determination of a curve C in V_n such that the totally indicatrix surface of the vector field contains the unit tangent vector to a curve of the congruence μ. The problem studied in this paper is the converse, *viz.* the determination of a congruence λ in V_n such that the osculating geodesic surface of the congruence λ w.r.t. a given curve C contains the unit tangent vector to a curve of the given congruence μ. The differential equations (21) for a hypersurface are identical with the differential equations (3.7) of Kaul's paper. But they are used in Kaul's paper to determine $\frac{dx^i}{ds}$, given t^i, whereas we consider them as determining t^i, given $\frac{dx^i}{ds}$.

When λ defines a curve tangential to C, $t^i = \frac{dx^i}{ds}$, the equation (21) reduces to

$$\Omega_{\nu|li} \frac{dx^l}{ds} \frac{dx^i}{ds} \left[p_j - g_{jk} p_h \frac{dx^h}{ds} \frac{dx^k}{ds}\right] = c_{\nu|} g_{lj} \frac{\delta}{\delta s} \left(\frac{dx^l}{ds}\right).$$

The differential equations of the union curves of a subspace relative to a congruence, as given by Mishra, thus follow as a special case of the above. From (15) and (20), we have

$$\frac{1}{v} = \frac{a_{\nu|1}}{c_{\nu|1}} = \frac{\Omega_{\nu|li} t^l \frac{dx^i}{ds}}{c_{\nu|} \kappa_{\lambda|1}} = \frac{\Omega_{\mu|li} t^l \frac{dx^i}{ds}}{c_{\mu|} \kappa_{\lambda|1}}$$

$$\nu, \mu = 1, 2, \ldots, m-n \,.\, \nu \neq \mu.$$

Using (7), each of the above ratios is equal to

$$\frac{\kappa_{\lambda|n}}{\left[\sum_\nu c^2_{\nu|}\right]^{\frac{1}{2}} \kappa_{\lambda|1}}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Let α be the angle between the vectors p^i and t^i.

$$\cos \alpha = \frac{g_{ij} p^i t^j}{\left[1 - \sum_\nu c^2_{\nu|}\right]^{\frac{1}{2}}}. \qquad \ldots \quad \ldots \quad \ldots \quad (23)$$

Let β be the angle between the vectors p^i and $t^i_{1|}$.

$$\cos \beta = \frac{g_{ij} p^i t^j_{1|}}{\left(1 - \sum_\nu c^2_{\nu|}\right)^{\frac{1}{2}} \left(1 - \sum_\nu a^2_{\nu|1}\right)^{\frac{1}{2}}}. \qquad \ldots \quad \ldots \quad (24)$$

Multiplying (14) by g^{jk}, we have

$$p^k = rt^k + vt^k_{1|}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Multiplying (25) by $g_{lk} t^l_{1|}$ and using (9) we have

$$g_{lk} t^l_{1|} p^k = v g_{lk} t^l_{1|} t^k_{1|} = v\left(1 - \sum_\nu a^2_{\nu|1}\right). \qquad \ldots \quad \ldots \quad (26)$$

From (24) and (26),

$$v\left(1-\sum_{\nu} a^2_{\nu|1}\right)^{\frac{1}{2}} = \cos\beta\left(1-\sum_{\nu} c^2_{\nu|}\right)^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad (27)$$

Multiplying (25) by $g_{lk}t^l$,

$$g_{lk}t^l p^k = r = \cos\alpha\left(1-\sum_{\nu} c^2_{\nu|}\right)^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad (28)$$

Squaring (25), we have

$$\left(1-\sum_{\nu} c^2_{\nu|}\right) = r^2+v^2\left(1-\sum_{\nu} a^2_{\nu|1}\right).$$

Using (27) and (28), this can be written as

$$\left(1-\sum_{\nu} c^2_{\nu|}\right) = \cos^2\alpha\left(1-\sum_{\nu} c^2_{\nu|}\right)+\cos^2\beta\left(1-\sum_{\nu} c^2_{\nu|}\right).$$

$$\cos^2\alpha+\cos^2\beta = 1$$

$$i.e.\quad \cos^2\beta = \sin^2\alpha. \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

Let us denote

$$\eta_j = \kappa_{\lambda|1}\left[t_{1|j}-\frac{a_{\nu|1}}{c_{\nu|}}\left(p_j-p_h t^h t_j\right)\right]. \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

We shall call the vector defined by the components η_j as *the union curvature vector of the congruence* λ *w.r.t.* C *relative to the congruence* μ. From (17), it is obvious that for a union congruence, the union curvature vector is the null or zero vector.

Let κ_μ be the magnitude of the vector η_j.

$$\begin{aligned}\kappa^2_\mu &= g^{jk}\eta_j\eta_k \\ &= \kappa^2_{\lambda|1} g^{jk}\left[t_{1|j}-\frac{a_{\nu|1}}{c_{\nu|}}\left(p_j-p_h t^h t_j\right)\right] \\ &\quad \times\left[t_{1|k}-\frac{a_{\nu|1}}{c_{\nu|}}\left(p_k-p_h t^h t_k\right)\right] \\ &= \kappa^2_{\lambda|1}\left[g^{jk}t_{1|j}t_{1|k}-2\frac{a_{\nu|1}}{c_{\nu|}}g^{jk}t_{1|j}p_k\right. \\ &\quad \left.+\left(\frac{a_{\nu|1}}{c_{\nu|}}\right)^2 g^{jk}\left(p_j-p_h t^h t_j\right)\left(p_k-p_l t^l t_k\right)\right].\end{aligned}$$

Using (9), (23) and (24), this simplifies to

$$\begin{aligned}\kappa^2_u = \kappa^2_{\lambda|1}&\left[\left(1-\sum_{\nu} a^2_{\nu|1}\right)-2\frac{a_{\nu|1}}{c_{\nu|}}\cos\beta\left(1-\sum_{\nu} c^2_{\nu|}\right)^{\frac{1}{2}}\right. \\ &\left.\times\left(1-\sum_{\nu} a^2_{\nu|1}\right)^{\frac{1}{2}}+\left(\frac{a_{\nu|1}}{c_{\nu|}}\right)^2\left(1-\sum_{\nu} c^2_{\nu|}\right)\sin^2\alpha\right]. \quad \ldots \quad (31)\end{aligned}$$

But from (7) and (19) we have

$$\left(1-\sum_{\nu} a^2_{\nu|1}\right) = g_{ij}t^i_{1|}t^j_{1|} = \frac{\kappa^2_{\lambda|g}}{\kappa^2_{\lambda|1}}, \qquad \ldots \quad \ldots \quad (32)$$

Substituting in (31), we have

$$\kappa^2_u = \kappa^2_{\lambda|g} - 2\kappa_{\lambda|g}\kappa_{\lambda|1}\cos\beta\left(1-\sum_{\nu} c^2_{\nu|}\right)^{\frac{1}{2}}\frac{a_{\nu|1}}{c_{\nu|}}$$

$$+\kappa^2_{\lambda|1}\left(\frac{a_{\nu|1}}{c_{\nu|}}\right)^2\left(1-\sum_{\nu} c^2_{\nu|}\right)\sin^2\alpha. \qquad \ldots \quad \ldots \quad (33)$$

We shall call κ_u as *the union curvature of a congruence* λ *w.r.t.* C *relative to the congruence* μ. Hence the expression (33) *gives a relation connecting the union curvature, geodesic curvature and absolute curvature of the congruence* λ *w.r.t.* C.

Using (22) and (29), the general expression (33) for the union curvature reduces in the case of a union congruence to the form

$$\kappa^2_u = \left[\kappa_{\lambda|g} - \kappa_{\lambda|n}\left(\frac{1}{\sum c^2_{\nu|}} - 1\right)^{\frac{1}{2}}\sin\alpha\right]^2 = 0.$$

Hence we have *for a union congruence*

$$\kappa_{\lambda|g} = \kappa_{\lambda|n}\left(\frac{1}{\sum_{\nu} c^2_{\nu|}} - 1\right)^{\frac{1}{2}}\sin\alpha \qquad \ldots \quad \ldots \quad \ldots \quad (34)$$

a relation connecting the geodesic curvature and the normal curvature of the congruence.

Special cases.

(i) Choose the congruence μ to be normal to V_n.

$$\therefore \quad \sum_{\nu} c^2_{\nu|} = 1.$$

From (33), we have

$$\kappa^2_u = \kappa^2_{\lambda|g}.$$

i.e. The union curvature of the congruence λ *relative to a normal congruence* μ *is equal to the geodesic curvature of the congruence* λ *w.r.t.* C. Hence the curve C will be a λ-geodesic (Nirmala 1963) w.r.t. the union congruences.

(ii) Choose the curve C to be an asymptotic line of the first type (Nirmala 1963) of the congruence λ. Then $\kappa_{\lambda|n} = 0$.

If λ is a union congruence, it follows from (34), that

$$\kappa_{\lambda|g} = 0.$$

The curve C is therefore an absolute geodesic.

Hence a curve C *which is an asymptotic line of the first type w.r.t. a union congruence* λ *is an absolute geodesic of the congruence* λ.

2. In a similar manner, we can introduce hyperasymptotic congruences of a subspace.

We shall call the surface generated by the geodesics tangential to the pencil of directions determined by the directions of the tangent and the first binormal vector to a curve of the congruence λ as the rectifying geodesic surface of the congruence λ w.r.t. C. Suppose the rectifying geodesic surface at each point of a curve of the congruence λ contains the tangent vector to a curve of the congruence μ. Then we shall call the congruence λ as a *hyperasymptotic congruence of the subspace relative to the congruence* μ. Proceeding on lines analogous to the above discussion for a union congruence, we obtain the differential equations of the hyperasymptotic congruences of a subspace relative to the congruence μ, in the form

$$a_{\nu|2}[p_j - p_h t^h t_j] = c_{\nu|} t_{2|};$$

$$j = 1, 2, \ldots, n \,.\, \nu = 1, 2, \ldots, m-n.$$

The intrinsic derivative of $\omega_{\lambda|1}$ w.r.t. the curve C can be put in the following form:

$$\frac{\delta\omega_{\lambda|1}}{\delta s} = (\kappa_{\omega 1|n} N^{\alpha}_{1|} + \kappa_{\omega 1|g} a^{\alpha}_{1|}) t_{1|},$$

where (i) $t_{1|}$ is the magnitude of the vector $t^{i}_{1|}$, (ii) $\kappa_{\omega 1|n}$ and $\kappa_{\omega 1|g}$ are the normal curvature and geodesic curvature of the congruence $\omega_{\lambda|1}$ w.r.t. C. Their magnitudes are given by

$$\kappa^2_{\omega 1|n} = \sum_{\nu} \left(t^l_{1|} \Omega_{\nu|li} + a_{\nu|1;i} + \sum_{\mu} a_{\nu|1} \theta_{\mu\nu|i} \right)$$

$$\left(t^m_{1|} \Omega_{\nu|mj} + a_{\nu|1;j} + \sum_{\mu} a_{\nu|1} \theta_{\mu\nu|j} \right) \frac{dx^i}{ds} \frac{dx^j}{ds} \bigg/ g_{lm} t^l_{1|} t^m_{1|}$$

and

$$\kappa^2_{\omega 1|g} = g_{lm} \left(t^l_{1|;i} - \sum_{\nu} a_{\nu|1} \Omega_{\nu|ik} g^{kl} \right)$$

$$\times \left(t^m_{1|;j} - \sum_{\nu} a_{\nu|1} \Omega_{\nu|jk} g^{km} \right) \frac{dx^i}{ds} \frac{dx^j}{ds} \bigg/ g_{lm} t^l_{1|} t^m_{1|}.$$

We further obtain

$$\frac{c_{\nu|}}{a_{\nu|2}} = \frac{c_{\mu|}}{a_{\mu|2}} = \frac{\left[\sum_{\nu} c^2_{\nu|} \right]^{\frac{1}{2}} \kappa_{\lambda|2}}{\kappa_{\omega 1|n} t_{1|}} \qquad \ldots \quad \ldots \quad \ldots \quad (35)$$

and

$$\sin \alpha = \cos \gamma, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

where α and γ are the angles made by the vector p^i with the vectors t^i and $t^i_{2|}$ respectively.

Let us denote

$$\xi_j = t_{2|j} - \frac{a_{\nu|2}}{c_{\nu|}}(p_j - p_h t^h t_j) \qquad \ldots \quad \ldots \quad \ldots \quad (37)$$

and call ξ_j as the components of the hyperasymptotic curvature vector of the congruence λ w.r.t. C.

From (37), it follows that for a h.a.* congruence, the h.a. curvature vector is the zero or null vector.

Let κ_h be the magnitude of the vector ξ_j.

$$\begin{aligned}\kappa_h^2 &= g^{jk}\xi_j\xi_k \\ &= g^{jk}\left[t_{2|j} - \frac{a_{\nu|2}}{c_{\nu|}}(p_j - p_h t^h t_j)\right] \\ &\quad \times \left[t_{2|k} - \frac{a_{\nu|2}}{c_{\nu|}}(p_k - p_l t^l t_k)\right].\end{aligned}$$

This simplifies to

$$\kappa_h^2 = \left(1 - \sum_\nu a_{\nu|2}^2\right) - 2\frac{a_{\nu|2}}{c_{\nu|}}\cos\gamma\left(1 - \sum_\nu c_{\nu|}^2\right)^{\frac{1}{2}}$$
$$\left(1 - \sum_\nu a_{\nu|2}^2\right)^{\frac{1}{2}} + \left(\frac{a_{\nu|2}}{c_{\nu|}}\right)^2\left(1 - \sum_\nu c_{\nu|}^2\right)\sin^2\alpha. \qquad \ldots \quad \ldots \quad (38)$$

We shall call κ_h as the h.a. curvature of the congruence λ w.r.t. C. For an h.a. congruence, the h.a. curvature vanishes. Hence using (35) and (36), we have for a h.a. congruence,

$$\kappa_h^2 = \left[\kappa_{\lambda|2}\left(1 - \sum_\nu a_{\nu|2}^2\right)^{\frac{1}{2}} - \kappa_{\omega 1|n} t_{1|}\left[\frac{1}{\sum_\nu c_{\nu|}^2} - 1\right]^{\frac{1}{2}}\sin\alpha\right]^2 = 0,$$

i.e.

$$\kappa_{\lambda|2}\left(1 - \sum_\nu a_{\nu|2}^2\right)^{\frac{1}{2}} = \kappa_{\omega 1|n} t_{1|}\left[\frac{1}{\sum_\nu c_{\nu|}^2} - 1\right]^{\frac{1}{2}}\sin\alpha$$

a relation connecting the absolute torsion of the congruence λ w.r.t. C and the normal curvature of the congruence $\omega_{\lambda|1}$ w.r.t. C.

Special case.

Choose the congruence μ to be normal to V_n.

$$\sum_\nu c_{\nu|}^2 = 1.$$

* We use the abbreviation h.a. to denote 'hyperasymptotic'.

The general expression (38) for the h.a. curvature reduces to

$$\kappa_h = 1-\sum_{\nu} a^2_{\nu|2} \quad \text{[using (35)]}$$

$$= 1-\frac{\kappa^2_{\omega 1|n} t^2_{1|}}{\kappa^2_{\lambda|2}}.$$

Hence we have *for an h.a. congruence relative to a normal congruence* μ,

$$\kappa^2_{\lambda|2} = \kappa^2_{\omega 1|n} t^2_{1|}.$$

From (32), this can be written as

$$\underline{\kappa_{\lambda|1}\kappa_{\lambda|2} = \kappa_{\omega 1|n}\kappa_{\lambda|g}.}$$

3. From (30) and (37), we have

$$(\kappa_{\lambda|1} t_{1|j} - \eta_j)\frac{1}{a_{\nu|1}} = (t_{2|j} - \xi_j)\frac{1}{a_{\nu|2}}.$$

On squaring,

$$g^{jk}[(\kappa_{\lambda|1} t_{1|j} - \eta_j)(\kappa_{\lambda|1} t_{1|k} - \eta_k)]a^2_{\nu|2}$$
$$= g^{jk}[(t_{2|j} - \xi_j)(t_{2|k} - \xi_k)]a^2_{\nu|1}.$$

This simplifies to

$$[\kappa^2_{\lambda|1} g^{jk} t_{1|j} t_{1|k} - 2g^{jk} t_{1|j}\eta_k \kappa_{\lambda|1} + \kappa^2_u]a^2_{\nu|2}$$
$$= [g^{jk} t_{2|j} t_{2|k} - 2g^{jk} t_{2|j}\xi_k + \kappa^2_h]a^2_{\nu|1}. \qquad \ldots \quad \ldots \quad (39)$$

Let ϕ be the angle between the vectors $t_{1|}$ and η_j.

$$\cos\phi = \frac{g^{jk} t_{1|k}\eta_j}{\kappa_u\left(1-\sum_{\nu} a^2_{\nu|1}\right)^{\frac{1}{2}}}$$

$$= \frac{\kappa_{\lambda|1} g^{jk} t_{1|k}\eta_j}{\kappa_u \kappa_{\lambda|g}}. \qquad \ldots \quad \ldots \quad \ldots \quad (40)$$

Let ψ be the angle between the vectors $t_{2|}$ and ξ_j.

$$\cos\psi = \frac{g^{jk} t_{2|j}\xi_k}{\kappa_h\left(1-\sum_{\nu} a^2_{\nu|2}\right)^{\frac{1}{2}}}. \qquad \ldots \quad \ldots \quad \ldots \quad (41)$$

Using (32), (40) and (41), (39) can be written as

$$(\kappa^2_{\lambda|g} - 2\kappa_u\kappa_{\lambda|g}\cos\phi + \kappa^2_u)a^2_{\nu|2}$$
$$= \left[\left(1-\sum_{\nu} a^2_{\nu|2}\right) - 2\kappa_h\left(1-\sum_{\nu} a^2_{\nu|2}\right)^{\frac{1}{2}}\cos\psi + \kappa^2_h\right]a^2_{\nu|1}$$

which gives a *relation between the union curvature, hyperasymptotic curvature and geodesic curvature of the congruence* λ *w.r.t.* C.

4. Pravanovitch (1955, 1956) has defined hyper-Darboux lines on an ordinary surface by the property that the plane determined by the tangent to the curve and the vector $R_1 n^i + R_2 \frac{dR_1}{ds} b^i$ at all points of the curve contains the tangent to the congruence μ through that point, n^i and b^i being the components of the unit principal normal vector and the binormal vector, R_1 and R_2, the radii of the first and second curvatures of the curve at that point. In this section, we consider a congruence λ of curves on V_n such that the surface generated by the geodesics tangential to the pencil of directions determined by the tangent to a curve of the congruence λ and the vector $\kappa_{\lambda|1}\omega^{\alpha}_{\lambda|1}+\kappa_{\lambda|2}\omega^{\alpha}_{\lambda|2}$ contains the unit tangent to a curve of the congruence μ, $\omega_{\lambda|1}$ and $\omega_{\lambda|2}$ being the unit principal normal vector and the first binormal vector to the congruence λ, $\kappa_{\lambda|1}$ and $\kappa_{\lambda|2}$ the first and second curvatures of the congruence λ w.r.t. C. We shall call the congruence λ with the above property as a *hyper-Darboux congruence of the subspace relative to the congruence* μ.

Proceeding on lines similar to the above, the differential equations of a hyper-Darboux congruence of the subspace relative to the congruence μ are obtained in the form

$$\begin{aligned}(\kappa_{\lambda|1}a_{\nu|1}+\kappa_{\lambda|2}a_{\nu|2})(p_j-p_h t^h t_j) &\\ = c_{\nu|}(\kappa_{\lambda|1}t_{1|j}+\kappa_{\lambda|2}t_{2|j}). &\\ j=1, 2, \ldots, n.\ \nu = 1, 2, \ldots, m-n. &\end{aligned}$$

The vector

$$\xi_j = (\kappa_{\lambda|1}t^j_{1|}+\kappa_{\lambda|2}t_{2|j}) - \frac{\kappa_{\lambda|1}a_{\nu|1}+\kappa_{\lambda|2}a_{\nu|2}}{c_{\nu|}}(p_j - p_h t^h t_j)$$

will be said to form the hyper-Darboux curvature vector of the congruence λ w.r.t. C. We establish on lines similar to the above, the relation

$$\left(\kappa^2_{\lambda|n}+\kappa^2_{\omega 1|n}t^2_{1|}-2\kappa_{\lambda|1}\kappa_{\lambda|2}g_{ij}t^i_{1|}t^j_{2|}\right)\left(\frac{1}{\sum_{\nu} c^2_{\nu|}}-1\right)\sin^2\alpha$$

$$= \left[\kappa^2_{\lambda|g}+\kappa^2_{\lambda|2}\left(1-\sum_{\nu} a^2_{\nu|2}\right)+2\kappa_{\lambda|1}\kappa_{\lambda|2}g^{jk}t_{1|j}t_{2|k}\right]$$

connecting the normal curvature, geodesic curvature, absolute curvature and absolute torsion of the congruence λ and normal curvature of the congruence $\omega_{\lambda|1}$.

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