TWO-DIMENSIONAL RECTANGULAR INCLUSION

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The problem of a rectangular inclusion is considered. The inclusion tends to undergo spontaneous dimensional changes of the order of elastic displacements in an infinite isotropic elastic medium. Locked-up accommodation stresses develop in the surrounding material and the inclusion. The problem has been solved by the point force method first suggested by J. D. Eshelby and further developed by one of the authors for plane strain case using complex variable methods. The equilibrium boundary shows again some surprising features. This has been schematically drawn for a few cases.

J. Frankel (1946) and Mott and Nabarro (1940) studied the 'inclusion problem' in elasticity theory in connection with their theory of alloys. No progress could be made thereafter, since the problem solved by them possessed a very high degree of symmetry. Eshelby (1957, 1959), using ingenious point force method gave a systematic account of solving such problems and in particular he obtained the solution for ellipsoidal inclusions. Bhargava (1959), using the point force concept and complex variable techniques developed by Stevenson (1945) in England and Muskhelishvili (1953) in the U.S.S.R., dealt with plane inclusions. Jaswon and Bhargava (1961) further extended their results to include some problems connected with elliptic inclusions. In this paper we consider the case of a rectangular inclusion and use essentially the method developed by Bhargava (1959) with some modifications.

For the sake of clarity we have to briefly go over a few arguments of Bhargava (1959).

An infinite isotropic elastic medium contains a specified region (the inclusion) which spontaneously tends to undergo a homogeneous non-elastic deformation. But owing to the constraints of the surrounding material (the matrix), this homogeneous deformation is restrained and stresses develop within the inclusion and the matrix. The problem in the main is to find the equilibrium shape and the allied stress distribution. The problem is tackled by the following hypothetical operations: cut out the inclusion from the matrix and allow it to undergo the non-elastic deformation. Next apply the surface tractions that restore it to its original shape. Then replace the 'stressed inclusion' into the cavity left behind and rejoin the material across the cut. Finally introduce a layer of point forces equal and opposite to the impressed surface tractions. Since the point forces are acting within an elastic infinite medium, they generate a deformation which brings the inclusion to the equilibrium configuration.

A point force $P = P_1+iP_2$ acting at the point $z = x+iy$ produces a known displacement at any point $\zeta = \xi+i\eta$ in the infinite medium. The functional equations which define $\phi(\zeta)$ and $\psi(\xi)$ are

$$\phi'(\zeta) = \frac{-1}{2\pi(\alpha+1)}\frac{P_1+iP_2}{\zeta-z},$$

$$\psi''(\zeta) = \frac{\alpha}{2\pi(\alpha+1)}\frac{P_1-iP_2}{\zeta-z}+\frac{1}{2\pi(\alpha+1)}\frac{d}{d\zeta}\left[\frac{\bar{z}(P_1+iP_2)}{\zeta-z}\right],$$

where the complex functions $\phi(\zeta)$ and $\psi(\zeta)$ are such that

$$R[\bar{\zeta}\phi(\zeta)+\psi(\zeta)]$$

is a solution of biharmonic equation. Dashes denote differentiation and $\alpha = 3-4\nu$ for plane strain and $\alpha = (3-\nu)/1+\nu$ for plane stress, ν is Poisson's ratio.

The components of stress are given by

$$p_{\xi\xi}+p_{\eta\eta} = 4Rl[\phi'(\zeta)], \tag{1}$$

$$p_{\eta\eta}-p_{\xi\xi}+2ip_{\xi\eta} = 2[\bar{\zeta}\phi''(\zeta)+\psi''(\zeta)].$$

The displacement field is given by

$$2\mu(u+iv) = \alpha\phi(\zeta)-\zeta\overline{\phi'(\zeta)}-\overline{\psi'(\zeta)}, \tag{2}$$

bar signifying the complex conjugate.

For a continuous distribution of point forces round the contour τ of the 'inclusion', we introduce the integrals

$$\phi'(\zeta) = -\frac{1}{2\pi(1+\alpha)}\int_\tau \frac{P_1+iP_2}{\zeta-z}\,ds, \tag{3}$$

$$\psi''(\zeta) = \frac{\alpha}{2\pi(\alpha+1)}\int_\tau \frac{P_1-iP_2}{\zeta-z}\,ds+\frac{1}{2\pi(\alpha+1)}\frac{d}{d\zeta}\int_\tau \frac{\bar{z}(P_1+iP_2)}{\zeta-z}\,ds$$

to get the cumulative effect of the point forces. $P_1 \pm iP_2$, z, $\bar{z}$ vary along τ, and ds is the differential arc length along τ.

The expressions for point force distribution as shown in (Bhargava 1959) are

$$(P_1+iP_2)\,ds = -\tfrac{1}{2}i\left\{\left(\mathring{p}_{\xi\xi}+\mathring{p}_{\eta\eta}\right)dz-\left(\mathring{p}_{\xi\xi}-\mathring{p}_{\eta\eta}\right)d\bar{z}\right\}+\mathring{p}_{\xi\eta}\,d\bar{z} \tag{4}$$

$$(P_1-iP_2)\,ds = -\tfrac{1}{2}i\left\{\left(\mathring{p}_{\xi\xi}-\mathring{p}_{\eta\eta}\right)dz-\left(\mathring{p}_{\xi\xi}+\mathring{p}_{\eta\eta}\right)d\bar{z}\right\}+\mathring{p}_{\xi\eta}\,dz$$

where superscript '0' denotes the initially impressed stress field.

Consider a rectangular inclusion region of sides $2a$ and $2b$ tending to go into a similarly situated rectangular region of sides $2a(1+\delta_1)$, $2b(1+\delta_2)$; δ_1 and δ_2 being of the order of elastic strains. The expanded inclusion is now reduced to the size of the hole by the application of surface tractions. The reversed surface tractions at the boundary are accounted by the stress field:

$$\overset{\circ}{p}_{\xi\xi} = \lambda(\delta_1+\delta_2)+2\mu\delta_1, \tag{5}$$

$$\overset{\circ}{p}_{\eta\eta} = \lambda(\delta_1+\delta_2)+2\mu\delta_2,$$

$$\overset{\circ}{p}_{\xi\eta} = 0.$$

These relations are now substituted in (4) whence $(P_1 \pm iP_2)ds$ is evaluated. These in turn are substituted in (3) and the integrals evaluated. The integrals are not of Cauchy type, but can be evaluated in this case, since the equations of the four sides are

$$z+\bar{z} = \pm 2a,$$

and

$$z-\bar{z} = \pm 2ib$$

whence $\bar{z}$ can be put in terms of z. Care should be taken to evaluate the integrals, as they involve multivalued logarithmic functions. It is necessary to keep to the same branch. For this it seems to be necessary to consider the anticlockwise direction positive. If the radius vector joining a fixed point (either in the matrix or in the inclusion) to a variable point on the boundary traces an angle in anticlockwise direction, the angle will be positive, otherwise negative.

We take $\zeta = \xi + i\eta$ any point either in the matrix or in the inclusion and evaluate the above integrals. We name the angles subtended by AB, BC, CD and DA (*see* Fig. 1) as θ_1, θ_2, θ_3 and θ_4 respectively. θ's are algebraic quantities, positive or negative sign to be attached to them with the help of the figure and in the manner already stated. Evaluation of integrals leads to

$$\phi'(\zeta) = \frac{\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}}{4\pi(\alpha+1)}(\theta_1+\theta_2+\theta_3+\theta_4) \tag{6}$$

$$-\frac{\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta}}{4\pi(\alpha-1)}\Bigg[(\theta_1-\theta_2+\theta_3-\theta_4)$$

$$+i\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}\Bigg].$$

$$(7)\quad \Psi''(\zeta) = -\frac{\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta}}{4\pi}\frac{\alpha-1}{\alpha+1}(\theta_1+\theta_2+\theta_3+\theta_4)$$
$$+\frac{\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}}{4\pi}\frac{\alpha-1}{\alpha+1}(\theta_1-\theta_2+\theta_3-\theta_4)$$
$$+\frac{i(\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta})}{4\pi}\frac{\alpha-1}{\alpha+1}\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}$$
$$-\frac{i(\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta})}{\pi}\frac{ab}{\alpha+1}\left[\frac{2\xi(\eta-b)-i(a^2-\xi^2+(\eta-b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta-b)^2)}\right.$$
$$-\frac{2\eta(\xi+a)+i(b^2-\eta^2+(\xi+a)^2)}{((\xi+a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}$$
$$+\frac{2\xi(\eta+b)-i(a^2-\xi^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}$$
$$\left.-\frac{2\eta(\xi-a)+i(b^2-\eta^2+(\xi-a)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi-a)^2+(\eta-b)^2)}\right].$$

It is obvious that

$$\theta_1+\theta_2+\theta_3+\theta_4 = 0 \text{ for } \zeta \text{ in the matrix,}$$
$$= 2\pi \text{ for } \zeta \text{ in the inclusion.}$$

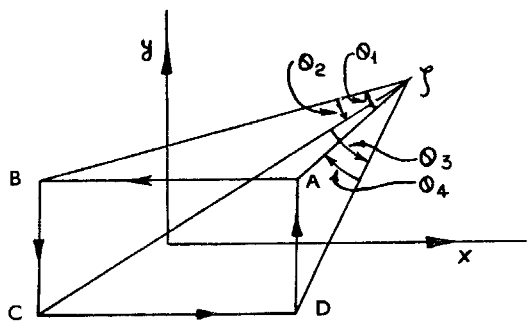


FIG. 1.

By differentiating (6) with respect to ζ we obtain

$$(8)\qquad \phi''(\zeta) = \frac{\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta}}{2\pi(\alpha+1)}\left[\frac{-2a\xi(\eta-b)+ia(a^2-\xi^2+(\eta-b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta-b)^2)}\right.$$
$$+\frac{b(b^2-\eta^2+(\xi+a)^2)-2ib\eta(\xi+a)}{((\xi+a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}$$
$$+\frac{2a\xi(\eta+b)-ia(a^2-\xi^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta+b)^2)}$$
$$\left.+\frac{-b(b^2-\eta^2+(\xi-a)^2)+2ib\eta(\xi-a)}{((\xi-a)^2+(\eta+b)^2)((\xi-a)^2+(\eta-b)^2)}\right].$$

The stresses in the matrix are given by directly substituting the values of $\phi'(\zeta)$, $\phi''(\zeta)$ and $\psi''(\zeta)$ in (1). However, for inclusion we should add (1) to the stress field which it had, when surface tractions were applied to reduce it to the size of the hole. Hence the stress field in the inclusion is given by

$$P_{\xi\xi}+P_{\eta\eta} = \frac{1-\alpha}{1+\alpha}(\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}) - \frac{\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta}}{\pi(1+\alpha)}(\theta_1-\theta_2+\theta_3-\theta_4),$$

$$\begin{aligned} P_{\eta\eta}-P_{\xi\xi}+2iP_{\xi\eta} &= \frac{2(\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta})}{\alpha+1} + \frac{\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}}{2\pi}\,\frac{\alpha-1}{\alpha+1}(\theta_1-\theta_2+\theta_3-\theta_4) \\ &+ \frac{i(\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta})}{2\pi}\,\frac{\alpha-1}{\alpha+1}\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)} \\ &+ \frac{\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta}}{\pi(\alpha+1)}\Bigg[(\zeta+2ib)\left\{\frac{-2a\xi(\eta-b)+ia(a^2-\xi^2+(\eta-b)^2)}{((\xi+a)^2+(\eta-b)^2)((\xi-a)^2+(\eta-b)^2)}\right\} \\ &+(\zeta-2a)\left\{\frac{b(b^2-\eta^2+(\xi+a)^2)-2ib\eta(\xi+a)}{((\xi+a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}\right\} \\ &+(\zeta-2ib)\left\{\frac{2a\xi(\eta+b)-ia(a^2-\xi^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta+b)^2)}\right\} \\ &+(\zeta+2a)\left\{\frac{-b(b^2-\eta^2+(\xi-a)^2)+2ib\eta(\xi-a)}{((\xi-a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)}\right\}\Bigg], \end{aligned}$$

where to distinguish the stress components from those in the matrix we have used capital letters. It can be readily checked that the normal and tangential stress components are continuous. This provides an excellent check on the analysis. On the other hand, hoop stress is discontinuous which is obvious on physical grounds. The jump in the hoop stress as we cross the boundary is

$$\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}(\delta_1+\delta_2) \text{ for plane strain}$$

and

$$(\lambda+2\mu)(\delta_1+\delta_2) \quad \text{for plane stress.}$$

An important corollary can be derived, when $\delta_1 = \delta_2$ which means that the initial stress field in the inclusion is uniform. It is at once obvious that the stress field in the matrix is free from dilatation, and is only of distortion. In the matrix the stress field is

(9) $$p_{\xi\xi}+p_{\eta\eta} = 0,$$

$$\begin{aligned} p_{\eta\eta}-p_{\xi\xi}+2ip_{\xi\eta} = \frac{\overset{\circ}{p}_{\xi\xi}}{\pi}\,\frac{\alpha-1}{\alpha+1}\Bigg[&(\theta_1-\theta_2+\theta_3-\theta_4) \\ &+i\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}\Bigg]. \end{aligned}$$

It is of interest to observe the variation of the stress $p_{\xi\zeta}$ and $p_{\xi\eta}$ on the boundary. In Figs. 2 and 3 we have drawn the graphs of $p_{\xi\xi}|\overset{\circ}{p}_{\xi\xi}$ and $p_{\xi\eta}|\overset{\circ}{p}_{\xi\xi}$ for different values of a/b, taking Poisson's ratio $v = 1/3$ in plane stress case.

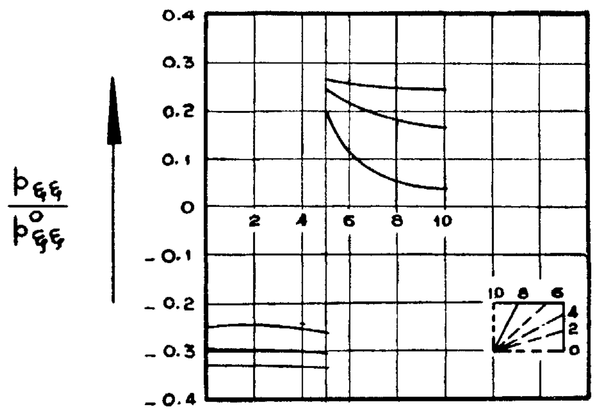


FIG. 2.

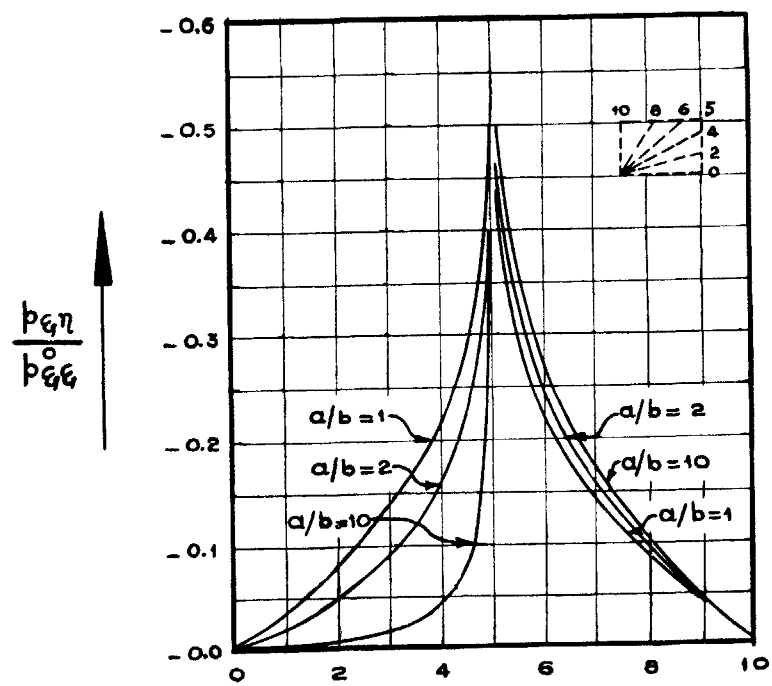


FIG. 3.

As regards inclusion (for $\delta_1 = \delta_2$), the stress field is given by

$$P_{\xi\xi}+P_{\eta\eta} = -2\overset{\circ}{p}_{\xi\xi}\frac{\alpha-1}{\alpha+1},$$

$$P_{\eta\eta}-P_{\xi\xi}+2iP_{\xi\eta} = \frac{\overset{\circ}{p}_{\xi\xi}}{\pi}\,\frac{\alpha-1}{\alpha+1}\Big[(\theta_1-\theta_2+\theta_3-\theta_4) +i\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}\Big].$$

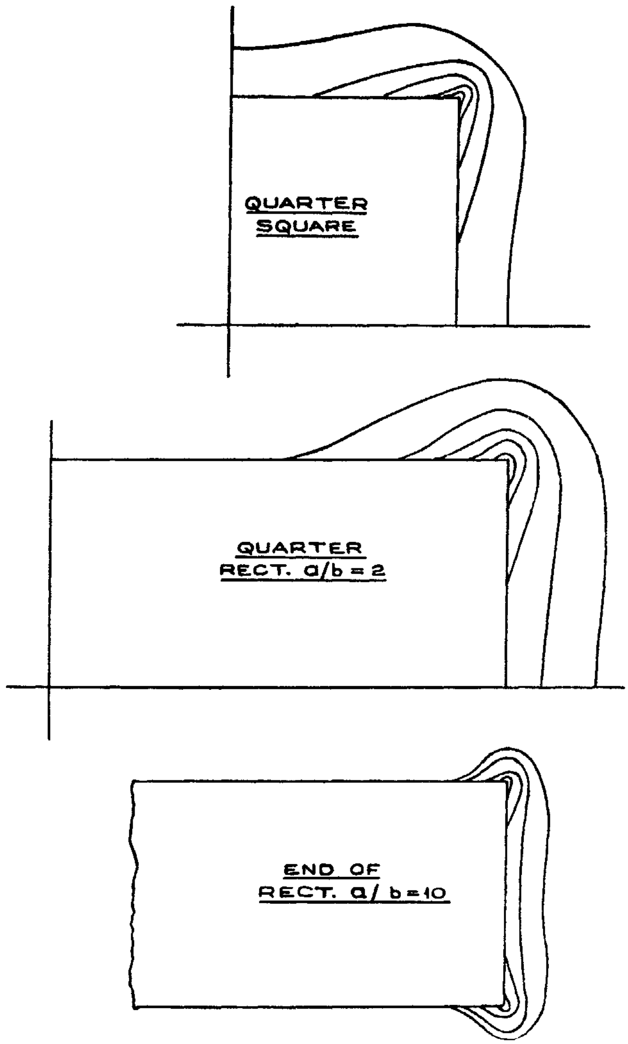
The maximum shearing stress in the matrix is given by

$$\left[\left(\frac{p_{\xi\xi}-p_{\eta\eta}}{2}\right)^2+p_{\xi\eta}^2\right]^{\frac{1}{2}}.$$

Numerical evaluation of the expression can easily be made from (9). Lines of constant maximum shearing stress have been drawn for plane stress case taking $v = 1/3$ for different values of a/b (Fig. 4). Clearly, the shearing stress is highly concentrated near the corners and in fact it tends to infinity at the corners. This may be explained by saying that at points very near the corner the shearing stress is sufficiently large to produce plastic deformations and that linear elasticity theory cannot be applied to the region very near the corners. It can also be seen that the shearing stresses are dominant on the narrow side of the rectangle. Similar results are seen in the case of an elliptic inclusion. When $a/b = 10$ the lines of maximum shearing stress are substantially the same as in the case of an ellipse with axial ratio equal to 10. (The latter lines were drawn for the case of an elliptic inclusion from Bhargava 1959.) This is explained by the fact that rectangular and elliptic inclusions are essentially the same for large values of a/b.

As regards the displacement field, we integrate (6) and (7) w.r.t. ζ and obtain the complex functions:

$$\begin{aligned}\phi(\zeta) = {} & \frac{\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}}{4\pi(\alpha+1)}(\theta_1+\theta_2+\theta_3+\theta_4)\zeta \\ & +\frac{\overset{\circ}{p}_{\xi\xi}-\overset{\circ}{p}_{\eta\eta}}{4\pi(\alpha+1)}\Big[\Big\{\eta\log\frac{((\xi+a)^2+(\eta+b)^2)((\xi-a)^2+(\eta-b)^2)}{((\xi+a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)} \\ & +b\log\frac{((\xi+a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)} \\ & -\xi(\theta_1-\theta_2+\theta_3-\theta_4)+2a(\theta_2-\theta_4)\Big\} \\ & +i\Big\{\xi\log\frac{((\xi+a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)} \\ & +a\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta-b)^2)}{((\xi+a)^2+(\eta+b)^2)((\xi-a)^2+(\eta+b)^2)} \\ & -\eta(\theta_1-\theta_2+\theta_3-\theta_4)+2b(\theta_1-\theta_3)\Big\}\Big],\end{aligned}$$



LINES OF MAXIMUM SHEARING STRESS

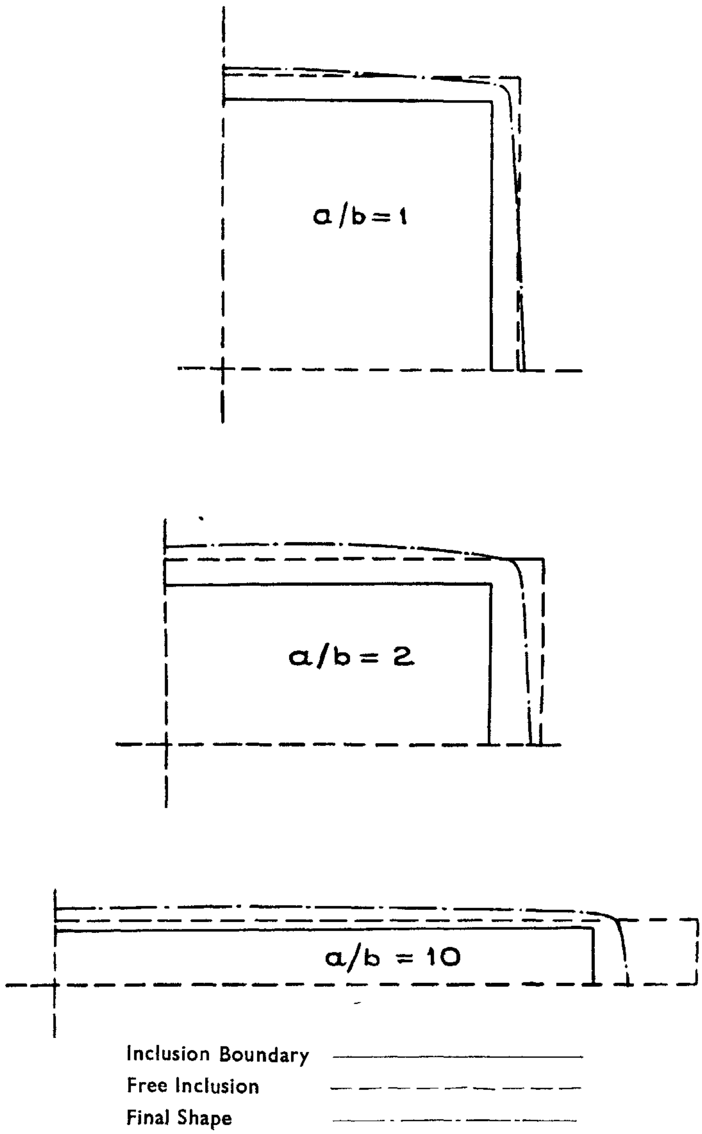
FIG. 4.

$$\psi'(\zeta) = \frac{-\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}}{4\pi}\,\frac{\alpha-1}{\alpha+1}(\theta_1+\theta_2+\theta_3+\theta_4)\zeta$$

$$-\frac{\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}}{4\pi}\,\frac{\alpha-1}{\alpha+1}\left[\left\{\eta\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi+a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)}\right.\right.$$

$$+b\log\frac{((\xi+a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)}-\xi(\theta_1-\theta_2+\theta_3-\theta_4)$$

$$\left.+2a(\theta_2-\theta_4)\right\}+i\left\{\xi\log\frac{((\xi+a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta+b)^2)}\right.$$



SCHEMATIC DRAWING OF THE EQUIL. SHAPE

Fig. 5.

$$+a\log\frac{((\xi-a)^2+(\eta-b)^2)((\xi+a)^2+(\eta-b)^2)}{((\xi+a)^2+(\eta+b)^2)((\xi-a)^2+(\eta+b)^2)}$$

$$\left.\left.-\eta(\theta_1-\theta_2+\theta_3-\theta_4)+2b(\theta_1-\theta_3)\right\}\right]$$

$$-\frac{\overset{\circ}{p}_{\xi\xi}+\overset{\circ}{p}_{\eta\eta}}{4\pi(\alpha+1)}\left[\left\{b\log\frac{((\xi+a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)}\right.\right.$$

$$\left.-2a(\theta_2-\theta_4)\right\}+i\left\{a\log\frac{((\xi+a)^2+(\eta+b)^2)((\xi+a)^2+(\eta-b)^2)}{((\xi-a)^2+(\eta-b)^2)((\xi-a)^2+(\eta+b)^2)}\right.$$

$$\left.\left.+2b(\theta_1-\theta_3)\right\}\right].$$

The displacements are obtained by substituting the above functions along with (6) in (2). In particular, the displacement of the internal boundary of the matrix can be evaluated by taking the point ζ on the boundary. As regards inclusion the elastic displacement field is given by the same functions. But care has to be taken for the proper interpretation of θ_1, θ_2, θ_3, θ_4. If that be done, it can be shown that displacement is continuous across the boundary, which provides another check on the analysis. It may be added that the displacement field exhibits no singularity at the corners.

In the case when $\delta_1 = \delta_2$ we have drawn the equilibrium shapes of inclusions for which $a/b = 1$ (square), 2 and 10, for the case of plane stress taking $v = \frac{1}{3}$ (Fig. 5).

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