VISCOUS INCOMPRESSIBLE FLOW BETWEEN TWO COAXIAL ROTATING POROUS CYLINDERS

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An attempt has been made to find the solution of the Navier-Stokes equations for the steady laminar flow of a viscous incompressible fluid between two coaxial porous cylinders rotating with constant angular velocities. A solution has been obtained under the assumption of uniform conditions along the axis of the cylinders. The cylinders being porous, a hyperbolic radial velocity distribution has been superimposed over the circumferential velocity produced due to rotation. There is a Bernoulli type pressure variation in the radial direction. When the inner cylinder is at rest, the shearing stress at it and the torque transmitted to it decrease as $\sigma\left(=\frac{v_0 y_1}{\nu}\right)$ increases. When $\sigma = 0$, the results transform to the known results for Couette flow between uniformly rotating coaxial cylinders.

NOTATIONS

x	Coordinate along the axis of the cylinders.
y	Coordinate along the radius.
ϕ	Azimuthal coordinate.
y_1	Radius of the inner cylinder.
y_2	Radius of the outer cylinder ($y_2 > y_1$).
u	Axial velocity.
v	Radial velocity.
w	Azimuthal velocity.
ω_1	Angular velocity of the inner cylinder.
ω_2	Angular velocity of the outer cylinder.
v_0	Uniform radial velocity at the inner cylinder.
p	Pressure.
ρ	Density of the fluid.
μ	Coefficient of viscosity.
$\nu = \frac{\mu}{\rho}$	Kinematic viscosity.
$\eta = \frac{y}{y_1}$	Dimensionless y coordinate.
$\sigma = \frac{v_0 y_1}{\nu}$	Suction parameter.
$k = \frac{y_2}{y_1}$	

$\tau_1 = \mu\left(\frac{\partial w}{\partial y}\right)_{y=y_1}$ Shearing stress at the inner cylinder.

M Torque per unit length of the inner cylinder.

INTRODUCTION

Couette first obtained the exact solution of the Navier-Stokes equations for steady laminar flow of a viscous incompressible fluid between two coaxial rotating cylinders. The solution proved of importance and it was used to determine the coefficient of viscosity of a fluid.

In this paper a solution of the Navier-Stokes equations has been obtained for laminar flow of a viscous incompressible fluid between two coaxial rotating cylinders with uniform radial velocity imposed at the surfaces. It has been assumed that the pressure is uniform along the axis of the cylinders. The surfaces being porous, a radial velocity is superimposed over the circumferential velocity due to rotation.

The case when the inner cylinder is at rest and the outer is rotating has some practical significance. The circumferential velocity distribution and the torque transmitted to the inner cylinder decrease as $\sigma\left(=\frac{v_0 y_1}{\nu}\right)$ increases. The shearing stress at the inner cylinder decreases as the annular gap between the cylinders increases.

EQUATIONS OF MOTION AND THEIR SOLUTION

With x, y, ϕ as the axial, radial and azimuthal coordinates, and u, v, w the axial, radial and azimuthal velocities, the Navier-Stokes equations of a viscous incompressible fluid in cylindrical polar coordinates are

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+\frac{w}{y}\frac{\partial v}{\partial \phi}-\frac{w^2}{y}+u\frac{\partial v}{\partial x}\right)$$
$$=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial y^2}+\frac{1}{y}\frac{\partial v}{\partial y}-\frac{v}{y^2}+\frac{1}{y^2}\frac{\partial^2 v}{\partial \phi^2}-\frac{2}{y^2}\frac{\partial w}{\partial \phi}-\frac{\partial^2 v}{\partial x^2}\right) \quad .. \quad (1)$$

$$\rho\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+\frac{w}{y}\frac{\partial w}{\partial \phi}+\frac{vw}{y}+u\frac{\partial w}{\partial x}\right)$$
$$=-\frac{1}{y}\frac{\partial p}{\partial \phi}+\mu\left(\frac{\partial^2 w}{\partial y^2}+\frac{1}{y}\frac{\partial w}{\partial y}-\frac{w}{y^2}+\frac{1}{y^2}\frac{\partial^2 w}{\partial \phi^2}+\frac{2}{y^2}\frac{\partial v}{\partial \phi}+\frac{\partial^2 w}{\partial x^2}\right) \quad .. \quad (2)$$

$$\rho\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}+\frac{w}{y}\frac{\partial u}{\partial \phi}+u\frac{\partial u}{\partial x}\right)$$
$$=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{y^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial x^2}\right) \quad .. \quad .. \quad .. \quad (3)$$

and the equation of continuity is

$$\frac{\partial v}{\partial y}+\frac{v}{y}+\frac{1}{y}\frac{\partial w}{\partial \phi}+\frac{\partial u}{\partial x}=0 \quad .. \quad .. \quad .. \quad .. \quad (4)$$

For steady motion between two rotating porous cylinders

$\frac{\partial}{\partial t} = 0$, for steady motion.

$\frac{\partial}{\partial \phi} = 0$, for axial symmetry.

$u = 0$, for motion due to rotation only.

Under these conditions, equations (1), (2), (3) and (4) reduce to

$$\rho\left(v\frac{\partial v}{\partial y} - \frac{w^2}{y} + u\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial y^2} + \frac{1}{y}\frac{\partial v}{\partial y} - \frac{v}{y^2} - \frac{\partial^2 v}{\partial x^2}\right) \quad \ldots \quad (5)$$

$$\rho\left(v\frac{\partial w}{\partial y} + \frac{vw}{y}\right) = \mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{y}\frac{\partial w}{\partial y} - \frac{w}{y^2} + \frac{\partial^2 w}{\partial x^2}\right) \quad \ldots \quad (6)$$

$$\frac{\partial p}{\partial x} = 0 \quad \ldots \quad (7)$$

and

$$\frac{\partial v}{\partial y} + \frac{v}{y} = 0 \quad \ldots \quad (8)$$

Equation (7) states the condition of uniform pressure distribution along the axis of the cylinder.

Assuming

$\frac{\partial v}{\partial x} = 0$, for uniform radial velocity

and

$\frac{\partial w}{\partial x} = 0$, for circumferential velocity produced due to rotation only,

the equations (5), (6) and (8) reduce to

$$\rho\left(v\frac{\partial v}{\partial y} - \frac{w^2}{y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial y^2} + \frac{1}{y}\frac{\partial v}{\partial y} - \frac{v}{y^2}\right) \quad \ldots \quad (9)$$

$$\rho\left(v\frac{\partial w}{\partial y} + \frac{vw}{y}\right) = \mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{y}\frac{\partial w}{\partial y} - \frac{w}{y^2}\right) \quad \ldots \quad (10)$$

and

$$\frac{\partial v}{\partial y} + \frac{v}{y} = 0 \quad \ldots \quad (11)$$

Substituting from (11) into (9), we get

$$\rho\left(\frac{v^2 + w^2}{y}\right) = \frac{\partial p}{\partial y} \quad \ldots \quad (12)$$

Equation (12) states Bernoulli type pressure variation in the radial direction. Equation (11) gives the distribution of radial velocity as

$$v = \frac{A}{y}, \text{ where } A \text{ is a constant.}$$

Assuming $v = v_0$ when $y = y_1$ (inner radius),

we get
$$v = v_0 \frac{y_1}{y} \qquad \ldots \qquad (13)$$

The radial volume flux per unit length of the inner cylinder is $2\pi y_1 v_0$. An equal amount of volume flux must be imposed at the outer cylinder to maintain continuity.

Introducing $\eta = \dfrac{y}{y_1}$, we have
$$\frac{v}{v_0} = \frac{1}{\eta} \qquad \ldots \qquad (14)$$

The variation of the radial velocity against η is shown in Fig. 1.

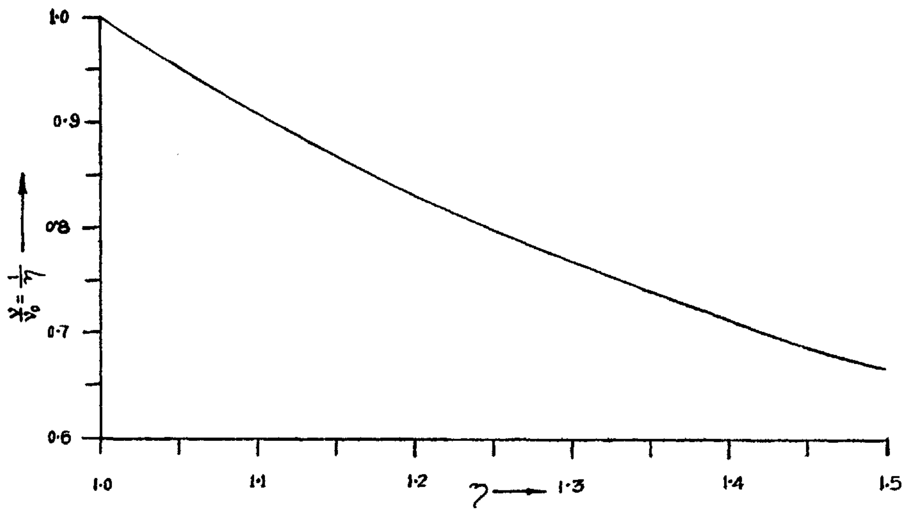

FIG. 1
Flow between two coaxial rotating porous cylinders.
Radial velocity $\frac{v}{v_0} = \frac{1}{\eta}$ plotted against $\eta = \frac{y}{y_1}$.

Substituting from (13) into (10) we get
$$y^2 \frac{\partial^2 w}{\partial y^2} + y \frac{\partial w}{\partial y} - \frac{v_0 y_1}{\nu} y \frac{\partial w}{\partial y} - \frac{v_0 y_1}{\nu} w - w = 0$$

i.e.
$$y^2 \frac{\partial^2 w}{\partial y^2} + (1-\sigma) y \frac{\partial w}{\partial y} - (1+\sigma) w = 0 \qquad \ldots \qquad (15)$$

The solution of equation (15) with the boundary conditions
$$w = y_1 \omega_1 \text{ when } y = y_1$$
and
$$w = y_2 \omega_2 \text{ when } y = y_2$$
is
$$w = \frac{1}{y_2^{2+\sigma} - y_1^{2+\sigma}} \left\{ y^{1+\sigma} (y_2^2 \omega_2 - y_1^2 \omega_1) - \frac{y_1^2 y_2^2}{y} (y_1^\sigma \omega_2 - y_2^\sigma \omega_1) \right\} \qquad \ldots \qquad (16)$$

The case, when the inner cylinder is at rest and the outer rotates, has some practical importance.

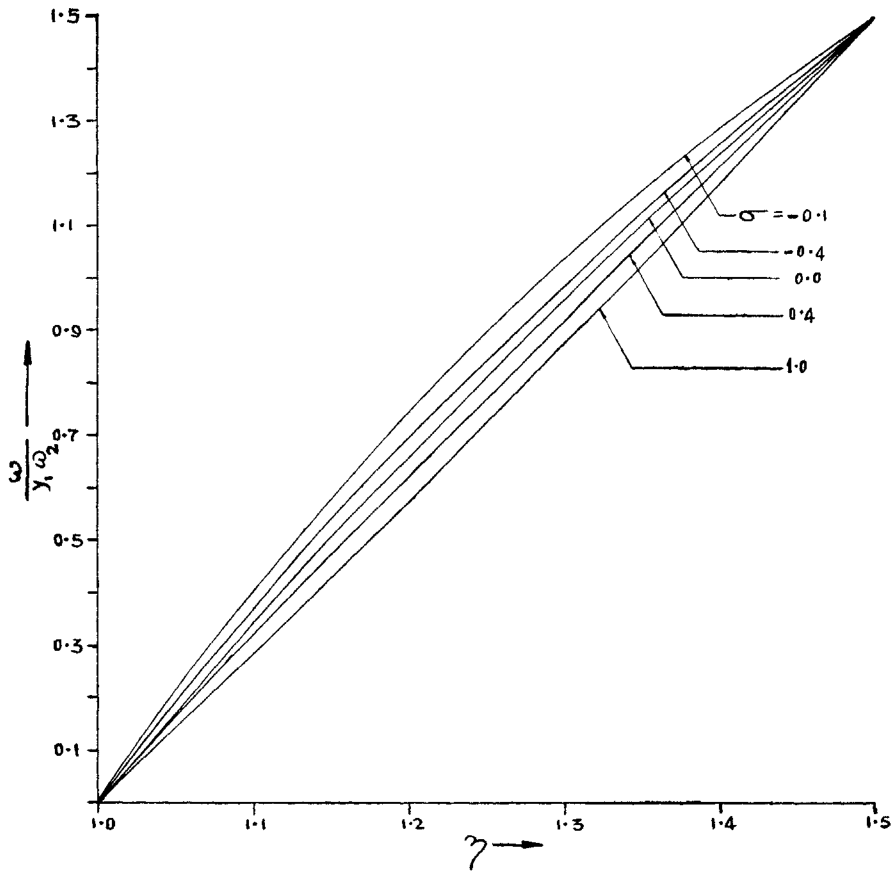

FIG. 2

Flow between two coaxial rotating porous cylinders.

Circumferential velocity distribution $\frac{\omega}{y_1\omega_2}$ plotted against $\eta = \frac{y}{y_1}$ for various values of $\sigma = \frac{v_0 y_1}{\nu}$ when the inner cylinder is at rest.

Then $\omega_1 = 0$ and

$$w = \frac{1}{\left(\frac{y_2}{y_1}\right)^{2+\sigma} - 1}\left\{\left(\frac{y_2}{y_1}\right)^2 y_1 \left(\frac{y}{y_1}\right)^{1+\sigma}\omega_2 - \frac{\left(\frac{y_2}{y_1}\right)^2 y_1^2\omega_2}{\frac{y}{y_1}\cdot y_1}\right\}$$

i.e.,
$$\frac{w}{y_1\omega_2} = \frac{K^2}{K^{2+\sigma}-1}\left\{\eta^{1+\sigma} - \frac{1}{\eta}\right\} \qquad \ldots \quad (17)$$

where

$$K = \frac{y_2}{y_1}.$$

The distribution of circumferential velocity for various values of σ and $K = 1{\cdot}5$ is shown in Fig. 2.

The shearing stress at the inner cylinder is

$$\tau_1 = \mu\left(\frac{\partial w}{\partial y}\right)_{y=y_1} = \mu \frac{y_1^{\sigma} y_2^{2}}{y_2^{2+\sigma} - y^{2+\sigma}} \cdot (2+\sigma)\omega_2$$

i.e.,

$$\frac{\tau_1}{\mu\omega_2} = \frac{K^2}{K^{2+\sigma}-1}(2+\sigma) \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

The shearing stress has been calculated for various values of σ and K and the results of calculation have been plotted in Fig. 3.

The torque transmitted by the fluid to unit length of the inner cylinder is

$$M = 2\pi\mu \frac{y_1^{2+\sigma} y_2^{2}}{y_2^{2+\sigma} - y_1^{2+\sigma}} \cdot (2+\sigma)\omega_2$$

i.e.,

$$\frac{M}{2\pi\mu\omega_2 y_2^2} = \frac{2+\sigma}{K^{2+\sigma}-1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

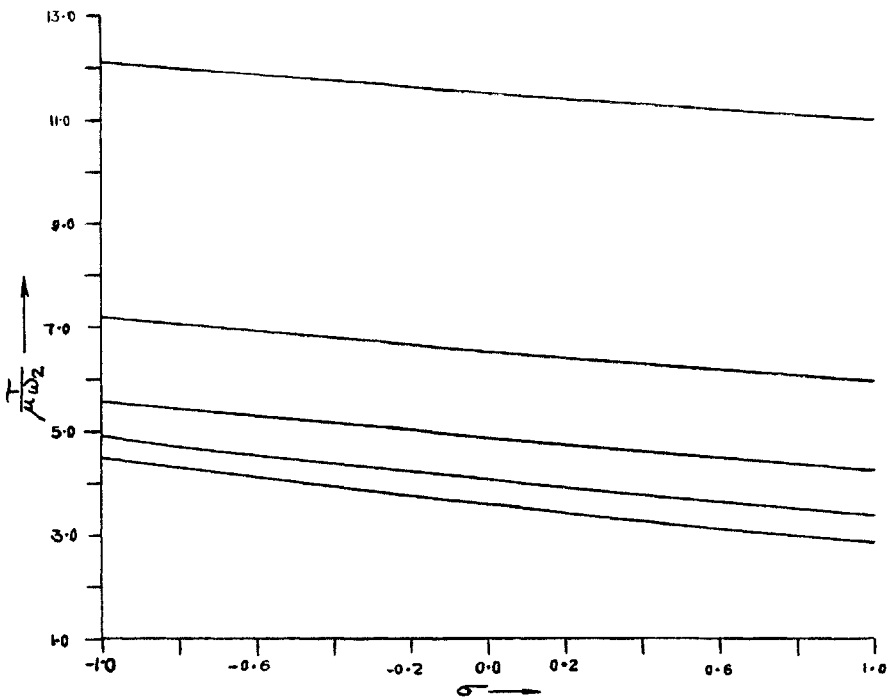

FIG. 3

Flow between two coaxial rotating porous cylinders.

Shearing stress at the inner cylinder $\frac{\tau}{\mu\omega_2}$ plotted against $\sigma = \frac{v_0 y_1}{\nu}$ when it is at rest.

The torque transmitted to the inner cylinder has been calculated for various values of σ and K, and the results are shown in Fig. 4.

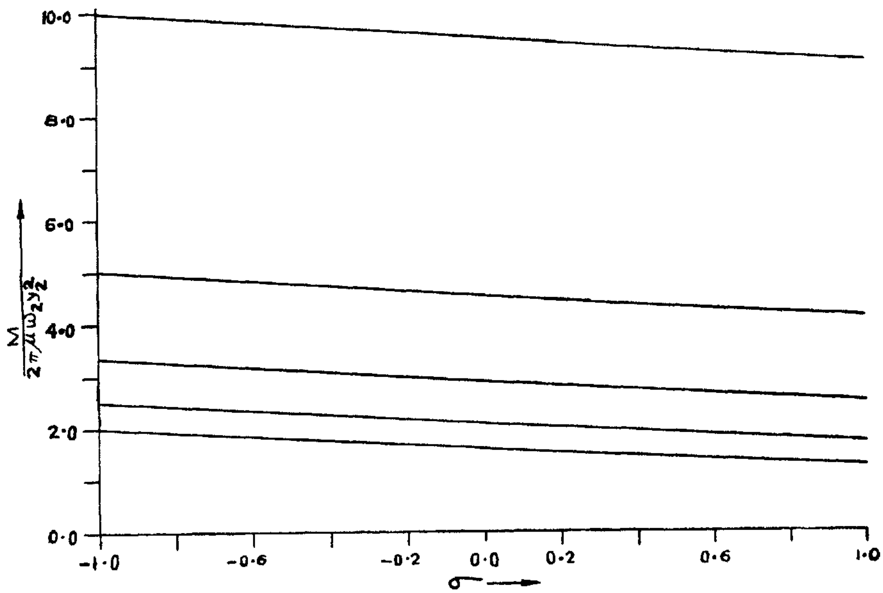

FIG. 4

Flow between two coaxial rotating porous cylinders.

Torque on the inner cylinder $\frac{M}{2\pi\mu\omega_2 y_2^2}$ plotted against $\sigma = \frac{v_0 y_1}{\nu}$ when it is at rest.

CONCLUSIONS

The following is the distribution of circumferential velocity for flow between two coaxial porous cylinders rotating uniformly.

$$w = \frac{1}{y_2^{2+\sigma} - y_1^{2+\sigma}} \left\{ y^{1+\sigma} (y_2^2 \omega_2 - y_1^2 \omega_1) - \frac{y_1^2 y_2^2}{y} (y_1^\sigma \omega_2 - y_2^\sigma \omega_1) \right\}$$

The radial velocity distribution is $\frac{v}{v_0} = \frac{1}{\eta}$.

When the inner cylinder is at rest and the outer is rotating,

$$\frac{w}{y_1 \omega_2} = \frac{K^2}{K^{2+\sigma} - 1} \left(\eta^{1+\sigma} - \frac{1}{\eta} \right)$$

$$\frac{\tau_1}{\mu \omega_2} = \frac{K^2}{K^{2+\sigma} - 1} (2+\sigma)$$

and

$$\frac{M}{2\pi\mu\omega_2 y_2^2} = \frac{2+\sigma}{K^{2+\sigma} - 1}.$$

Proceeding to the limit when $\sigma = 0$, we get

$$\frac{w}{y_1\omega_2} = \frac{K^2}{K^2-1}\left(\eta - \frac{1}{\eta}\right)$$

$$\frac{\tau_1}{\mu\omega_2} = \frac{2K^2}{K^2-1}$$

and

$$\frac{M}{2\pi\mu\omega_2 y_2^2} = \frac{2}{K^2-1}$$

as in the Couette flow between two uniformly rotating cylinders with solid surfaces ($\sigma = 0$).

Due to the superimposition of a small radial velocity over the circumferential velocity, the shearing stress at the inner cylinder and the torque transmitted to it decrease as $\sigma\left(=\frac{v_0 y_1}{\nu}\right)$ increases.

It is seen that $\sigma = -2$ is a critical value at which the results take indeterminate forms. Calculations have been made for small values of $\sigma(-1 \leqslant \sigma \leqslant 1)$.

Considering a small annular gap between the cylinders, $k\left(=\frac{y_2}{y_1}\right)$ has been assumed to be small ($1 < k \leqslant 1{\cdot}5$).

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