NOTE ON A THEOREM OF GUPTA

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In the present note a theorem on the Cesàro summability of the ultraspherical series defined on a sphere has been established for $0 < \lambda < 1$, which is a generalization of a theorem of Gupta (1958) proved for the range $0 < \lambda \leqslant \frac{1}{2}$. This theorem includes as a particular case that of Du Plessis (1952) also on the Cesàro summability of Laplace series for the value $\frac{1}{2}$ of the parameter λ.

1. The ultraspherical series corresponding to a function $F(\theta, \phi)$, which is defined for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ and is assumed to be integrable (L) over the whole surface of the unit sphere S, is given by

$$(1.1) \qquad F(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \iint_S \frac{P_n^{(\lambda)}(\cos \gamma) F(\theta', \phi') d\sigma'}{[\sin^2 \theta' \sin^2 (\phi-\phi')]^{\frac{1}{2}-\lambda}}, \ \lambda > 0,$$

where γ is the spherical distance between the points (θ, ϕ) and (θ', ϕ'),

$$d\sigma' = \sin \theta' d\theta' d\phi'$$

and the ultraspherical polynomials $P_n^{(\lambda)}(x)$ is defined by the following expansion:

$$(1.2) \qquad (1-2xz+z^2)^{-\lambda} = \sum_{n=0}^{\infty} z^n P_n^{(\lambda)}(x).$$

The Laplace series is a particular case of the series (1.1) for the value $\frac{1}{2}$ of the parameter λ and in view of the relation

$$\lim_{\lambda \to 0} \frac{1}{\lambda} P_n^{(\lambda)} (\cos \theta) = \frac{2}{n} \cos n\theta, \ n \geqslant 1,$$

the series (1.1) reduces to the trigonometric series of a function as $\lambda \to 0$ in the limit. Throughout this note we assume that the function

$$F(\theta', \phi') \ [\sin^2 \theta' \sin^2 (\phi-\phi')]^{\lambda-\frac{1}{2}}$$

is integrable (L) over the whole surface of the unit sphere, and define the generalized mean value (2) of $F(\theta, \phi)$ as follows:

$$f(w) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)}{2\pi\Gamma(\lambda)\ (\sin w)^{2\lambda}} \int_{C_W} \frac{F(\theta', \phi') ds'}{[\sin^2 \theta' \sin^2 (\phi-\phi')]^{\frac{1}{2}-\lambda}},$$

where the integral is taken along the small circle whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is w. We also write

$$F(w) = f(w)\,(\sin w)^{2\lambda-1}.$$

2. Positive order Cesàro summability of the series (1.1) has been discussed in detail by Kogbetliantz (1924), Obrechkoff (1936) and Gupta (1958). It has been observed by almost all the researchers in this line of work that in passing from Fourier series to Laplace series a jump of $\frac{1}{2}$ in the Cesàro scale of summation always persists and the origin of this kind of behaviour is due to the particular values of the parameter λ for the Laplace and Fourier series.

Du Plessis (1952) has shown that for a function satisfying a Lipschitz condition also this jump of $\frac{1}{2}$ exists. He proved the following theorem:

THEOREM A. *For* $-\frac{1}{2} < k < \frac{1}{2}$, the Laplace series of $f(\theta, \phi)$ on a sphere S is summable (c, k) at the point (θ, ϕ) of the sphere to the sum $F_\rho(\theta')$, provided that

$$F_\rho(\theta') \equiv \int_0^{2\pi} f(\theta', \phi')d\theta' \in \text{lip}^*\,(\tfrac{1}{2}-k), \tag{2.1}$$

(θ, ϕ) being the pole.

This theorem is an analogue of a well-known result of Hardy and Littlewood (1928) for Fourier series.

In a recent paper Gupta (1958) has proved the following more general result:

THEOREM B. For $\lambda - 1 < k < \lambda$ and $0 < \lambda \leqslant \frac{1}{2}$, the ultraspherical series (1.1) is (c, k) summable at the point (θ, ϕ) to the sum $f(w)$, provided that

$$F(w) \in \text{lip}^*\,(\lambda-k). \tag{2.2}$$

For $\lambda = \frac{1}{2}$ Theorem B includes as a particular case Theorem A for Laplace series. But very recently in a review of the above result Kogbetliantz (1961) has remarked that Gupta's theorem should hold for $0 < \lambda < 1$. The object of this note is to change slightly the hypothesis of Theorem B and establish the following more general result, which is in agreement with the suggestion given by Kogbetliantz:

THEOREM. *If* $\lambda-1 < k < 1-\lambda$, $0 < \lambda < 1$ *and*

$$F(w) \in \text{lip}^*\,(\lambda-k), \tag{2.3}$$

then the ultraspherical series (1.1) *is summable* (c, k) *at the point* (θ, ϕ) *to the sum* $f(w)$.

It is known (Gupta 1958, p. 420) that the mth partial sum of the series (1.1) is given by

$$S_m = \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \int_0^\pi F(w) \left[\frac{d}{dx}\left\{P_{m+1}^{(\lambda)}(x) + P_m^{(\lambda)}(x)\right\}\right]_{x=\cos w} .\sin w dw.$$

Thus the (c, k) mean $\sigma_n^k(P)$ of the series (1.1) at the point P can be written as

$$\sigma_n^k(P) = \int_0^\pi F(w) L_n^k(w) dw,$$

where

$$L_n^k(w) = \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} (A_n^k)^{-1} \sum_{m=0}^{n} A_{n-m}^{k-1} \left[\frac{d}{dx}\left\{P_{m+1}^{(\lambda)}(x) + P_m^{(\lambda)}(x)\right\}\right]_{x=\cos w} . \sin w.$$

3. The proof of the theorem depends upon the following lemmas proved by Gupta (1958):

Lemma 1. For $0 < \lambda < 1$,

$$L_n^k(w) = O(n^{2\lambda+1} w). \tag{3.1}$$

Lemma 2. For $0 < \lambda < 1$, $0 < k < 1-\lambda$ and $\pi - \frac{1}{n} \leq w \leq \pi$,

$$L_n^k(w) = O(n^{2\lambda} \sin w). \tag{3.2}$$

Lemma 3. If $\alpha_n \leq w \leq \pi - \frac{1}{n}$, $\left(\alpha_n \geqslant \frac{1}{n}\right)$, $0 < k < 1-\lambda$,

$$0 < \lambda < 1 \text{ and } \mu_n = \frac{\pi}{n+\lambda+\frac{1}{2}}, \text{ then}$$

$$\begin{aligned} L_n^k(w) = {} & \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \left\{A_n^k\right\}^{-1} R\{w^{-\lambda}\phi(w) e^{i(n+\lambda+\frac{1}{2})w}\} \\ & + O\{n^{\lambda-1} . w^{-1}(\sin w)^{-\lambda}\} + O\{n^{-1}(\sin w)^{-\lambda-1}\} \\ & + O\{n^{\lambda-1} w^{-\lambda}(\sin w)^{-1}\}, \end{aligned} \tag{3.3}$$

where $\phi(w)$ is such that

$$\phi(w) = O(n^\lambda w^{-k});\ \phi(w+\mu_n) - \phi(w) = O(n^{k+\lambda-1} . w^{-1} \log n).$$

Lemma 4. If

$$0 < \lambda < 1, \lambda-1 < k \leqslant 0 \text{ and } \pi - \frac{1}{n} \leqslant w \leqslant \pi, \text{ then}$$

$$L_n^k(w) = O(n^{2\lambda-k} . \sin w). \tag{3.4}$$

Lemma 5. If

$$\alpha_n \leqslant w \leqslant \pi - \frac{1}{n}, 0 < \lambda < 1 \text{ and } \lambda - 1 < k \leqslant 0, \text{ then}$$

$$\begin{aligned} L_n^k(w) = {} & \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \left\{A_n^k\right\}^{-1} . R\{w^{-\lambda}\phi(w) e^{i(n+\lambda+\frac{1}{2})w}\} \\ & + O\{n^{\lambda-k-1} . w^{-1}(\sin w)^{-\lambda}\} + O\{n^{-1}(\sin w)^{-\lambda-1}\} \\ & + O\{n^{\lambda-k-1} . w^{-\lambda}(\sin w)^{-1}\}. \end{aligned} \tag{3.5}$$

where

$$\phi(w) = O(n^{\lambda}w^{-k})\,;\ \phi(w+\mu_n)-\phi(w) = O(n^{\lambda-1}\,.\,w^{-k-1}).$$

Now we divide the proof of the theorem into parts, *viz.*

$$0 < k < 1-\lambda \text{ and } \lambda-1 < k \leqslant 0.$$

4. *Proof of the theorem for* $0 < k < 1-\lambda$.

We have

$$\sigma_n^k = \int_0^{\pi} F(w)L_n^k(w)dw$$

$$= \left[\int_0^{\alpha_n} + \int_{\alpha_n}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi}\right] F(w)L_n^k(w)dw$$

(4.1) $$= I_1+I_2+I_3,$$

say. By the hypothesis of the theorem we have

$$I_1 = o\left\{n^{2\lambda+1}\int_0^{\alpha_n} w\,.\,w^{\lambda-k}dw\right\}$$

by lemma 1.

$$= o\left(n^{2\lambda+1}\,.\,\alpha_n^{2+\lambda-k}\right)$$

(4.2) $$= o(1),$$

where

$$\alpha_n = n^{-(2\lambda+1)(2+\lambda-k)^{-1}};$$

which is greater than $\frac{1}{n}$ for $0 < k < 1-\lambda$.

Also

$$I_3 = O\left\{n^{2\lambda}\int_{\pi-\frac{1}{n}}^{\pi} \sin w\, dw\right\},$$

by lemma 2.

$$= O\left[n^{2\lambda}\left\{\frac{1}{n}\,.\,\sin\frac{1}{n}\right\}\right]$$

$$= O\left\{n^{2\lambda}\,.\,\frac{1}{n^2}\right\}$$

(4.3) $$= o(1), \text{ for } \sin\frac{1}{n} \leqslant \frac{1}{n}.$$

Finally, using lemma 3, we have

$$I_2 = R\left\{\int_{\alpha n}^{\pi-\frac{1}{n}} F(w)\phi(w)w^{-\lambda}e^{i(n+\lambda+\frac{1}{2})w}\,dw\right\}$$

$$+O\left\{\int_{\alpha n}^{\pi-\frac{1}{n}} |F(w)|\,n^{\lambda-1}\,.\,w^{-1}(\sin w)^{-\lambda}dw\right\}$$

$$+O\left\{\int_{\alpha n}^{\pi-\frac{1}{n}} |F(w)|\,.\,n^{-1}(\sin w)^{-\lambda-1}\,dw\right\}$$

$$+O\left\{\int_{\alpha n}^{\pi-\frac{1}{n}} |F(w)|\,n^{\lambda-1}\,.\,w^{-\lambda}(\sin w)^{-1}\right\}$$

$$= I_{2,\,1}+I_{2,\,2}+I_{2,\,3}+I_{2,\,4},$$

say. It is easy to see that, for $0 < k < 1-\lambda$; $I_{2,\,2}$, $I_{2,\,3}$ and $I_{2,\,4}$ each is $o(1)$. Also, on the lines of Gupta, we have

$$I_{2,\,1} = o(1), \text{ for } 0 < k < 1-\lambda.$$

This completes the proof of the theorem for $0 < k < 1-\lambda$.

5. *Proof of the theorem for* $\lambda-1 < k \leqslant 0$.

We write $k = -p$ and hence we have to prove the theorem for $0 \leqslant p < 1-\lambda$.

$$\sigma_n^{-p} = \left\{\int_0^{\alpha_n} + \int_{\alpha_n}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi}\right\} F(w)L_n^{-p}(w)dw$$

(5.1) $$= J_1+J_2+J_3 \text{ say.}$$

$$|J_1| = o\left(n^{2\lambda+1}\int_0^{\alpha_n} w\,.\,w^{\lambda+p}\,dw\right)$$

(5.2) $$= o(1), \text{ for } \alpha_n = n^{-(2\lambda+1)(\lambda+p+2)-1}.$$

Also

$$J_3 = O\left\{n^{2\lambda+p}\int_{\pi-\frac{1}{n}}^{\pi} \sin w\,dw\right\}$$

(5.3) $$= o(1).$$

Finally, in view of the inequality $p+\lambda < 1$ and following Gupta (1958) again, it is evident that

(5.4) $$J_2 = o(1).$$

Thus the theorem is completely established.

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