STABILITY OF A STRATIFIED, COMPRESSIBLE, INVISCID FLUID CONFINED BETWEEN TWO FREE BOUNDARIES

by M. M. GUPTA,* P. V. BAKORE *and* J. M. GANDHI, *Department of Physics, University of Rajasthan, Jaipur*

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The equilibrium of a compressible, inviscid fluid of density ρ varying in the direction of gravity and confined between two free boundaries, subjected to an initial infinitesimal perturbation is discussed. The solution is expressed in the form of integrals and a variational principle characterizing the solution of the problem is shown to exist. An approximate solution is obtained for the fluid in which the stratification of density ρ is according to the law

$$\rho = \rho_0 \exp(\beta z),$$

where ρ_0 and β are constants. The stratification is assumed to be so small so as to consider the velocity of sound in the fluid to be constant. It is shown that for all values of wavelengths, the waves are generated and that for stable stratification we have undamped waves.

1. INTRODUCTION

Vandervoort (1961) studied the equilibrium of a heavy, non-viscous and compressible fluid. He had shown that a variational principle characterizing the solution of the problem exists. However, as the integrals occurring in the solution of the problem have a complicated dependence on growth rate n and wave number k, he concluded that the variational principle may be of little importance. As such, he had discussed only the case of two superposed fluids of constant densities. In this paper we assume that the speed of sound in the fluid is a constant quantity. Following Hide (1956) and others we assume that

$$\beta d \ll 1 \qquad \ldots \quad (1)$$

where d is the distance between the two boundaries within which the fluid is confined. This will include many problems of physical interest.

Recently Mitchner and Landshoff (1964) have discussed the problem of the stability of the plane interface separating two compressible fluids. They have mentioned that their results depart from those of Vandervoort based on a physically unrealistic assumption that the density of the fluid and the speed of sound in it are both constant in each fluid. However, in the present

* *Present address :* Ramjas College, Delhi University, Delhi.

problem, we have assumed the density to be variable, but for the tractibility of the problem (If the assumption of the constant speed of sound is not made, we cannot apply the variational principle. Moreover, without the variational principle, the problem appears as untractable), we assume the speed of sound to be constant, which will have some justification, because of equation (1), *i.e.* the stratification of the fluid is very small. In this paper we consider the case of a stratified fluid confined between two free boundaries. For all wavelengths of the disturbance, waves are generated and for stable stratification we have undamped waves, which is justified as the fluid is assumed to be non-viscous. The results have also been graphically illustrated.

2. The Basic Equations and the Variational Principle

The characteristic equation for the problem in hand as derived by Vandervoort (1961) is

$$n^2 D\left[\rho\left(k^2+\frac{n^2}{V^2}\right)^{-1} Dw\right] - n^2\rho w + gk^2\left[\left(D+\frac{g}{V^2}\right)\rho\left(k^2+\frac{n^2}{V^2}\right)^{-1}\right] w = 0 \quad \ldots \quad (2)$$

where

$$\left.\begin{aligned} n &= \partial/\partial t \\ D &= d/dz \\ \\ k^2 &= k_x^2+k_y^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

and

k being the total wave number of the disturbance in the horizontal plane and is always taken to be positive, g is the acceleration due to gravity acting vertically along z axis, w is the z-component of the fluid velocity generated because of the infinitesimal perturbation and V is the speed of sound in the fluid.

The solution of equation (2) is to be sought subjected to some boundary conditions. As the fluid is assumed to be confined between two free boundaries, the kinematical condition to be imposed is that a particle which is at the surface at one instant of time remains there indefinitely. However, in problems in which the potential energy of a disturbed free surface is not a major consideration, it is convenient to follow Rayleigh (1916) and many others and forbid the vertical motion at the free surface. Thus

$$w = 0 \text{ at } z = 0 \text{ and } z = d. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Since the fluid is inviscid, as such there are no dissipative forces, no further boundary condition can be derived, as there are no tangential stresses. However, there will be no loss of generality if we assume

$$D^2 w = 0 \text{ at } z = 0 \text{ and } z = d \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

the latter boundary condition being strictly true in the absence of a magnetic field. The requirement that a solution of equation (2) satisfies the necessary boundary conditions will lead to a determinate sequence of possible values of n. Let n_i and n_j be two such characteristic values and let the solutions belonging to these characteristic values be distinguished by the subscripts i and j respectively.

Considering equation (2) for the characteristic value n_i, multiplying by w_j (belonging to n_j) and integrating over the whole vertical extent of the fluid $(0 \leqslant z \leqslant d)$ we get

$$n_i^2 \int_0^d w_j D(\rho w_i)\, dz - n_i^2 \int_0^d \rho w_j \left(k^2 + \frac{n2}{V^2}\right) w_i \, dz + gk^2 \int_0^d \left(D + \frac{g}{V^2}\right) \rho w_i w_j \, dz = 0. \quad (6)$$

Since

$$\int_0^d w_j D(\rho D w_i)\, dz = -\int_0^d \rho D w_i D w_j \, dz$$

equation (6) can be written as

$$\left(\frac{ni^4}{V^2} + n_i^2 k^2\right) I_1(i, j) + n_i^2 I_2(i, j) - gk^2 I_3(i, j) = 0 \quad \ldots \quad \ldots \quad (7)$$

where

$$I_1(i, j) = \int_0^d \rho w_i w_j \, dz \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$I_2(i, j) = \int_0^d \rho D w_i D w_j \, dz \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

and

$$I_3(i, j) = \int_0^d \left(D + \frac{g}{V^2}\right) \rho w_i w_j \, dz. \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Setting $i = j$ and considering the effect on n of an arbitrary variation δw in w compatible with the boundary condition on w, we have from equation (7) to the first order

$$\left(\frac{N^2}{V^2} + Nk^2\right)\delta I_1 + \left(\frac{2N}{V^2} + k^2\right) I_1 \delta N + N \delta I_2 + I_2 \delta N - gk^2 \delta I_3 = 0 \quad \ldots \quad (11)$$

where $N \equiv n^2$; δI_1, δI_2 and δI_3 are the corresponding variations in I_1, I_2 and I_3 and are given by

$$\delta I_1 = 2\int_0^d \rho w \delta w \, dz \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\delta I_2 = -2\int_0^d \rho D^2 w \delta w \, dz \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$\delta I_3 = 2\int_0^d \left(D + \frac{g}{V^2}\right) \rho w \delta w \, dz. \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Using equations (12) to (14), equation (11) can be written as

$$\tfrac{1}{2}\delta N\left[\left(\frac{2N}{V^2}+k^2\right)I_1+I_2\right] = -\left(\frac{N^2}{V^2}+Nk^2\right)\int_0^d \rho w\delta w\, dz$$

$$+N\int_0^d \rho D^2 w\delta w\, dz + gk^2\int_0^d \left(D+\frac{g}{V^2}\right)\rho w\delta w\, dz. \quad \text{.. (15)}$$

We observe that the right-hand side of equation (15) vanishes because equation (7) is satisfied, and since

$$\left(\frac{2N}{V^2}+k^2\right)I_1+I_2 \neq 0$$

$$\delta N \equiv \delta(n^2) = 0. \quad \text{.. .. .. .. (16)}$$

Thus the solution of equation (7) is characterized by a variational principle by virtue of which a simple integration procedure based on equation (7) is evolved for determining n as a function of total wave number k and other parameters of the problem. This is the suitable form of the variational principle to be applied to problems of practical interest.

3. Application of Variational Principle

Since the fluid is confined between two free boundaries, choose w as

$$w = A \sin\frac{\pi s}{d}\,.\, z \quad \text{.. .. .. .. (17)}$$

so that the boundary conditions (4) and (5) are satisfied, s being a positive integer. The stratification of the fluid is assumed to obey the law

$$\rho = \rho_0 \exp(\beta z) \quad \text{.. .. .. .. (18)}$$

with the assumption $\beta d \ll 1$. Equation (7) then reduces to

$$N^2+N(k^2V^2+V^2l^2)-k^2(g\beta V^2+g^2) = 0 \quad \text{.. .. (19)}$$

where

$$l = \frac{\pi s}{d}. \quad \text{.. .. .. .. (20)}$$

Measuring n and k in the dimensionless form given by

$$y = \frac{nd}{V\sqrt{R}} \quad \text{.. .. .. .. (21)}$$

$$x = \frac{kd}{R} \quad \text{.. .. .. (22)}$$

and introducing the dimensionless parameters

$$R = \pi s \quad \text{.. .. .. (23)}$$

and

$$T = \frac{d^2}{V^2}\left(g\beta + \frac{g^2}{V^2}\right) \quad \ldots \quad (24)$$

equation (19) becomes

$$y^4 + R(x^2+1)y^2 - x^2T = 0 \quad \ldots \quad (25)$$

the solution of which can immediately be written as

$$y = \frac{1}{\sqrt{2}}\left[\pm\sqrt{R^2(x^2+1)^2+4x^2T} - R(x^2+1)\right]^{\frac{1}{2}}. \quad \ldots \quad (26)$$

Case I ($T = 0$):

Taking negative sign before the radical in equation (26) we have undamped oscillations of the fluid with angular frequency $\vartheta(y)$, wave velocity V_w and group velocity V_g as

$$\vartheta(y) = \sqrt{R(x^2+1)} \quad \ldots \quad (27)$$

$$V_w^2 = \left[\frac{\vartheta(y)}{x}\right]^2 = \frac{R(x^2+1)}{x^2} \quad \ldots \quad (28)$$

and

$$V_g^2 = \left[\frac{d}{dx}\vartheta(y)\right]^2 = \frac{Rx^2}{(x^2+1)}. \quad \ldots \quad (29)$$

Fig. 1 shows the variation of $\vartheta(y)$, V_w and V_g as a function of x for $R = \pi$ (since s is assumed to be equal to 1) and $T = 0$.

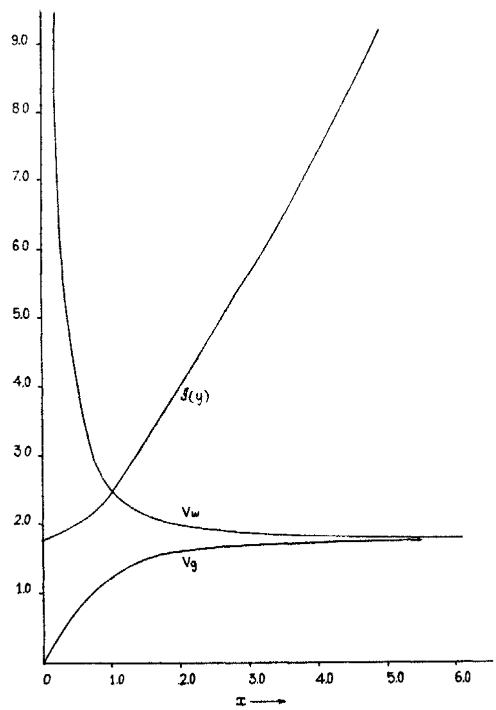

FIG. 1. The group velocity V_g, wave velocity V_w and angular frequency $\vartheta(y)$ as a function of the wave number x for $T = 0$ and $R = \pi$ ($s = 1$) according to equations (27) to (29).

Case II $(T > 0)$:

(*a*) Taking positive sign before the radical, y becomes real and there is no generation of waves but the fluid is aperiodically damped with the damping coefficient

$$y = \frac{1}{\sqrt{2}}\left[\sqrt{R^2(x^2+1)^2+4x^2T} - R(x^2+1)\right]^{\frac{1}{2}}. \qquad \ldots \qquad \ldots \quad (30)$$

(*b*) Taking negative sign before the radical in equation (26), y is imaginary leading to the generation of hydrodynamic waves with wave and group velocities V_w and V_g respectively given by

$$V_w^2 = \frac{1}{2x^2}\left[\sqrt{R^2(x^2+1)^2+4x^2T} + R(x^2+1)\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

$$V_g^2 = \frac{x^2}{2}\left[R + \frac{R^2(x^2+1)+2T}{\sqrt{R^2(x^2+1)^2+4x^2T}}\right]^2\left[\sqrt{R^2(x^2+1)^2+4x^2T} + R(x^2+1)\right]^{-1} \quad \ldots \quad (32)$$

and angular frequency

$$\vartheta(y) = \frac{1}{\sqrt{2}}\left[\sqrt{R^2(x^2+1)^2+4x^2T} + R(x^2+1)\right]^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

Fig. 2 gives the variation of $\vartheta(y)$, V_w and V_g as a function of x for $T = 1$ and $R = \pi$ $(s = 1)$.

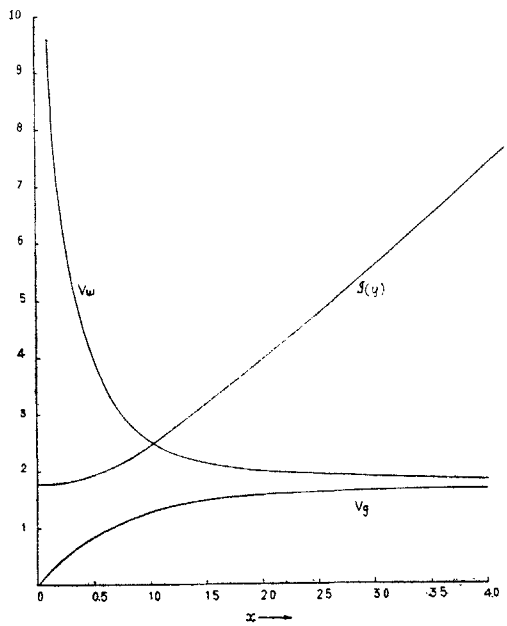

FIG. 2. The group velocity V_g, wave velocity V_w and angular frequency $\vartheta(y)$ for $T = 1$ and $R = \pi$ $(s = 1)$ as a function of the wave number x according to equations (31) to (33).

From case II(b) we conclude that we have undamped waves while from II(a) we have aperiodic damping for the same wave numbers of the disturbance, which is physically unjustified. Since viscosity is absent, there can be no damping and as such case II(a) is to be rejected.

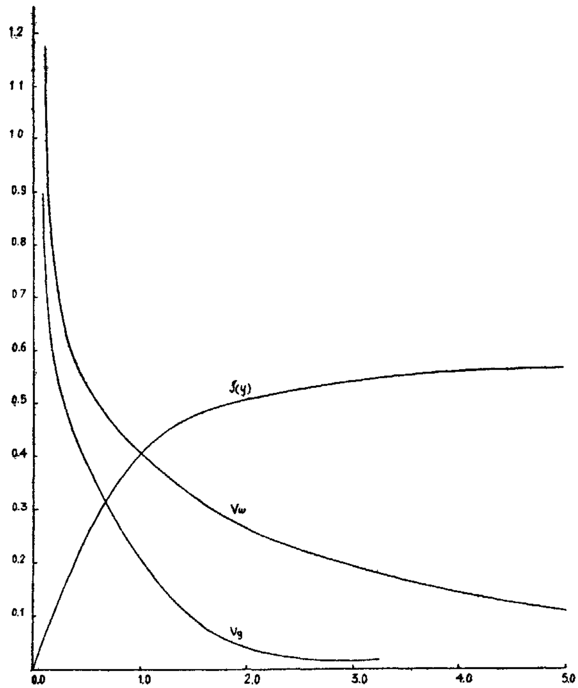

FIG. 3. The group velocity V_g, wave velocity V_w and angular frequency $\vartheta(y)$ for $T = -1$ and $R = \pi$ ($s = 1$) as a function of the wave number x according to equations (34) to (36).

Case III ($T < 0$, $T = -T_1$, T_1 being positive):

(a) Taking positive sign before the radical in equation (26), y becomes imaginary and hydrodynamic waves of angular frequency $\vartheta(y)$ and with wave and group velocities V_w and V_g respectively as

$$\vartheta(y) = \frac{1}{\sqrt{2}}\left[R(x^2+1) - \sqrt{R^2(x^2+1)^2 - 4x^2 T_1}\right]^{\frac{1}{2}} \quad \ldots \quad (34)$$

$$V_w^2 = \frac{1}{2x^2}\left[R(x^2+1) - \sqrt{R^2(x^2+1)^2 - 4x^2 T_1}\right] \quad \ldots \quad (35)$$

and

$$V_g^2 = \frac{x^2}{2}\left[R - \frac{R^2(x^2+1) - 2T_1}{\sqrt{R^2(x^2+1)^2 - 4x^2 T_1}}\right]^2 \left[R(x^2+1) - \sqrt{R^2(x^2+1)^2 - 4x^2 T_1}\right]^{-1} \quad (36)$$

are generated, there being no damping of the fluid. Fig. 3 depicts the variations of $\vartheta(y)$, V_w and V_g with x for $T_1 = 1$ and $R = \pi$ $(s = 1)$.

(*b*) Taking negative sign before the radical in equation (26), again there is no damping of the fluid but the waves are generated. The result is illustrated in Fig. 4 for $T_1 = 1$ and $R = \pi$ $(s = 1)$. The waves have angular frequency, wave and group velocities respectively as

$$\vartheta(y) = \frac{1}{\sqrt{2}}\left[R(x^2+1)+\sqrt{R^2(x^2+1)^2-4x^2T_1}\right]^{\frac{1}{2}} \quad \text{.. .. .. ..} \quad (37)$$

$$V_w^2 = \frac{1}{2x^2}\left[R(x^2+1)+\sqrt{R^2(x^2+1)^2-4x^2T_1}\right] \quad \text{.. .. .. ..} \quad (38)$$

and

$$V_g^2 = \frac{x^2}{2}\left[R+\frac{R^2(x^2+1)-2T_1}{\sqrt{R^2(x^2+1)^2-4x^2T_1}}\right]^2\left[R(x^2+1)+\sqrt{R^2(x^2+1)^2-4x^2T_1}\right]^{-1}. \quad (39)$$

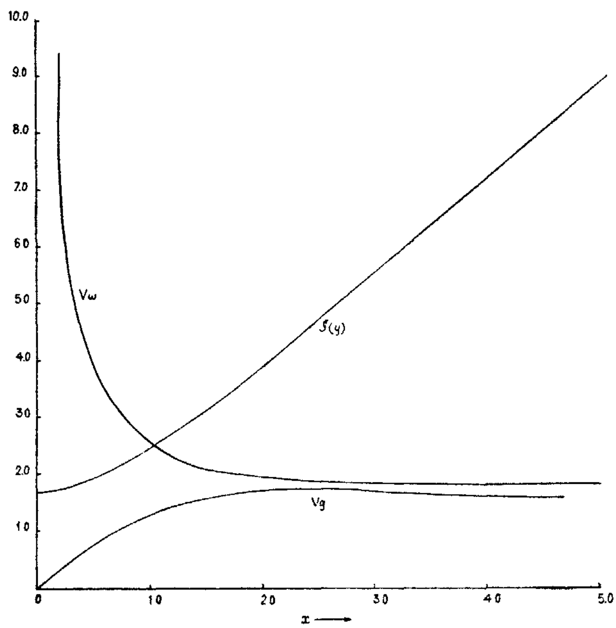

FIG. 4. The group velocity V_g, wave velocity V_w and angular frequency $\vartheta(y)$ as a function of the wave number x for $T = -1$ and $R = \pi$ $(s = 1)$ according to equations (37) to (39).

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