ON UNIQUE EXTENSION OF LINEAR FUNCTIONALS

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The Hahn-Banach theorem which extends a linear functional on a linear subspace A of a linear space B to the whole of B without change of norm is well known. However, this extension is not unique. In this paper a necessary and sufficient condition for a unique extension has been obtained as Theorem 2 which proves that a normed linear functional defined on a normed linear subspace A extends to the whole space uniquely if for every point $x_0 \epsilon A$, corresponding to $\epsilon > 0$, there exist points x' and x'' in A such that

$$[\{f(x')-f(x'')\}+\{\| x'+x_0 \| + \| x''+x_0 \|\}] < \epsilon.$$

As an application of this theorem a simple proof of the following Theorem 3 has been given:

(Theorem 3) Let A be a dense subspace of a normed linear space B. Then any normed linear functional on A can be uniquely extended to B.

INTRODUCTION

The Hahn-Banach theorem on the extension of linear functionals is well known. But the usual proofs do not discuss any necessary and sufficient conditions under which the said extension could be unique. Here we study a necessary and sufficient condition for the extension to be unique. Since our condition comes out of the proof of the Hahn-Banach theorem, for the sake of completeness we give an outline of the proof of the Hahn-Banach theorem and then take up the problem of uniqueness.

THEOREM 1. (Hahn-Banach theorem). If f is a normed linear functional on a linear subspace A of a normed linear space B, then f can be extended to a normed linear functional on the whole space without changing its norm.

Proof: Let $x_0 \notin A$ and let $\| f \| = 1$ without any loss of generality. We also let the scalars be real; the case when scalars are complex being an easy generalization.

Now by the condition of linearity we can obtain $f(x+ax_0)$ by setting it equal to $f(x)+af(x_0)$, where $x \epsilon A$. Hence it suffices to show that there exists $f(x_0)$ such that

$| f(x)+af(x_0) | \leqslant \| x+ax_0 \|$ for every $x \epsilon A$ and every scalar a.

Now A being a linear subspace, we can replace x by ax and, by letting x vary, the condition becomes

$$\text{(I)} \qquad \sup_x \{-\| x+x_0 \| - f(x)\} \leqslant f(x_0) \leqslant \inf_x \{\| x+x_0 \| - f(x)\}.$$

Hence acceptable values of $f(x_0)$ exist if the above supremum is no greater than the above infimum.
i.e. If whatever be $x', x'' \epsilon A$,

$$-\| x'+x_0 \| - f(x') \leqslant \| x''+x_0 \| - f(x'')$$

or

$$f(x'')-f(x') \leqslant \| x''+x_0 \| + \| x'+x_0 \|.$$

But by the linearity of f and the triangular inequality for norm, we have

$$f(x'')-f(x') = f(x''-x') \leqslant \| x''-x' \| \leqslant \| x''+x_0 \| + \| x'+x_0 \|.$$

Therefore, acceptable values exist.

Thus we can extend the domain point by point. And by applying Zorn's lemma the functional extends to the whole space.*

The extension of linear functionals outlined above need not be unique. We now proceed to prove the following :

THEOREM 2. (Uniqueness of extension). The extension of the linear functional by the Hahn-Banach theorem is unique if, and only if, for every point $x_0 \notin A$ corresponding to $\epsilon > 0$ there exist points x' and x'' in A such that

$$[\{f(x')-f(x'')\} + \{\| x'+x_0 \| + \| x''+x_0 \|\}] < \epsilon.$$

Proof: From (I) it is clear that uniqueness of extension is attained if there is a single acceptable value, *i.e.*

$$\text{(II)} \qquad \operatorname*{Sup}_{x} \{-\| x+x_0 \| - f(x)\} = \inf_{x} \{\| x+x_0 \| - f(x)\}.$$

Now (II) is equivalent to

$$\text{(III)} \qquad [\{\| x''+x_0 \| - f(x'')\} - \{-\| x'+x_0 \| - f(x')\}] < \epsilon$$

for some $x', x'' \epsilon A$. And (III) is equivalent to

$$\text{(IV)} \qquad [\{f(x')-f(x'')\} + \{\| x'+x_0 \| + \| x''+x_0 \|\}] \text{ for some } x', x'' \epsilon A.$$

This completes the proof. As an application of Theorem 2 we prove the following well-known result on uniqueness of extensions :

THEOREM 3. Let A be a dense subspace of a normed linear space B. Then any normed linear functional on A can be uniquely extended to B.
Proof: Let us consider the set A which is everywhere dense in B and take any $x_0 \notin A$. Then take $\{x_n\}$ in A such that $x_n \to -x_0$.

Now for any $\epsilon > 0$ $\exists N$ such that for $m, n > N$, we have

$$\| x_n+x_0 \| < \frac{\epsilon}{3}, \ \| x_m+x_0 \| < \frac{\epsilon}{3} \text{ and } \| x_n-x_m \| < \frac{\epsilon}{3};$$

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* For a detailed proof, *see*, for example, Loéve (1963).

so that

$$\{f(x_n)-f(x_m)\}+\{\| x_n+x_0 \| + \| x_m+x_0 \|\}$$
$$\leqslant \| x_n-x_m \| + \| x_n+x_0 \| + \| x_m+x_0 \| < \epsilon.$$

Hence by Theorem 2, the extension, which exists by the Hahn-Banach theorem, is unique.

Reference

Loéve, Michael (1963). Probability Theory (3rd edition), D. Van Nostrand Company Inc., N.Y., pp. 80-81.