IMPERFECT BOSE GAS IN AN ARBITRARY NUMBER OF DIMENSIONS

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(*Received March 19, 1965*)

The thermodynamic properties of an imperfect gas in n dimensions (where n is any positive integer) are derived by the method of pseudopotentials. The special case of two dimensions is discussed. A formula for the scattering cross-section in an arbitrary number of dimensions is derived.

Kothari and Singh (1941) have given an analysis of the perfect Bose gas and derived the thermodynamic parameters in the general case of n dimensions. Recently, May (1964) has shown that the case $n = 2$ is a special one in the continuum of dimensions. In this case no condensation occurs, and the specific heat at constant volume in perfect Bose and Fermi gases in two dimensions is identical. Using Huang's (1957, 1959) method of pseudopotentials, we calculate the various thermodynamic parameters for an imperfect Bose gas in an arbitrary number of dimensions. The motivation for such a calculation is purely theoretical. The practical cases are given by $n \leqslant 3$. However, the general case will, we hope, give a deeper insight into the problem. The first section gives the derivation of the approximate pseudopotential. In the second section the exact pseudopotential is derived. The thermodynamic parameters such as free energy, specific heat, pressure, etc., are derived in the third section. The fourth section deals with the special case of two dimensions. The method to draw the curves for the case of four dimensions is indicated to illustrate the general procedure. Finally, the scattering cross-section in n dimensions is derived.

I. Consider a system of N particles obeying B.E. statistics enclosed in a large $(p+2)$ dimensional box ($p = 1$, for three dimensions; $p = 2$, for four dimensions), interacting through a two-body potential of core diameter a, and having no bound states. We take the system of units in which, $\hbar = c = 1$. We shall first discuss the two-body problem. The Schrödinger equation in relative coordinates can be written as

$$(\nabla^2+k^2)\psi(\vec{r}) = 0,\ r > a \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

with the boundary condition $\psi(\vec{r}) = 0,\ r < a$ where r is the inter-particle separation and $\frac{\hbar^2 k^2}{m}$ is the energy of the relative motion. Let us first consider the s-wave solutions. The solution to (1) may be represented by an extended wave function $\psi_{ex}(\vec{r})$ which for $r > a$, $= \psi(\vec{r})$ and vanishes at $r = a$. We postulate that $\psi_{ex}(\vec{r})$ satisfies

$$(\nabla^2 + k^2)\psi_{ex}(\vec{r}) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

(everywhere except at $\vec{r} = 0$).

We wish to find the necessary modification at $r = 0$ so that the boundary condition is satisfied ($\psi_{ex}(a) = 0$). For $k \to 0$ (i.e. in the limit of very low energies), (2) becomes

$$\frac{1}{r^{p+1}} \frac{d}{dr}\left(r^{p+1} \frac{d\psi_{ex}}{dr}\right) = 0,$$

so that we obtain

$$\psi_{ex}(r) \underset{r \to 0}{\to} \left[1 - \left(\frac{a}{r}\right)^p\right] \chi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where χ can be written as

$$\chi = \frac{1}{p!}\left[\left(\frac{\partial}{\partial r}\right)^p (r^p \psi_{ex})\right]_{r=0} \quad \ldots \quad \ldots \quad \ldots \quad (3a)$$

which follows from (3).

The boundary condition (3) implies that for very small values of r

$$r^{p+1} \frac{\partial \psi_{ex}}{\partial r} \to \frac{pa^p}{p!}\left[\left(\frac{\partial}{\partial r}\right)^p (r^p \psi_{ex})\right]. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Following Blatt and Weisskopf (1962) we integrate both sides of (4) over the full solid angle. The left-hand side becomes

$$\int r^{p+1} \frac{\partial \psi_{ex}}{\partial r} d\Omega = \int (\nabla \psi_{ex}) \cdot \vec{n}\, ds$$

$$= \int (\nabla^2 \psi_{ex}) \cdot dv \quad \ldots \quad \ldots \quad \ldots \quad (4a)$$

where the last integral is over the volume of a $(p+2)$ dimensional sphere of radius r. The right-hand side of (4) becomes

$$\frac{pa^p}{p!} \int \left(\frac{\partial}{\partial r}\right)^p (r^p \psi_{ex})\, d\Omega = s(p+2) pa^p \chi \int \delta^{p+2}(\vec{r})\, dv \quad \ldots \quad \ldots \quad (4b)$$

where we have used the fact that for very small r,

$$\frac{1}{p!}\left(\frac{\partial}{\partial r}\right)^p (r^p \psi_{ex}) = \chi$$

which is independent of r.

$s(p+2)$ is the surface area of the $(p+2)$ dimensional unit sphere, given by

$$s(p) = \frac{2\Pi^{p/2}}{\Gamma(p/2)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Hence identifying the integrands of (4a) and (4b) as $r \to 0$, we get

$$\nabla^2 \psi_{ex}(r) = \frac{s(p+2)pa^p}{p!} \delta^{p+2}(\vec{r}) \left\{ \left(\frac{\partial}{\partial r}\right)^p (r^p \psi_{ex}) \right\}_{r=0} . \quad \text{(6)}$$

Therefore as $k \to 0$ the function $\psi_{ex}(r)$ everywhere satisfies the equation,

$$(\nabla^2 + k^2)\psi_{ex}(r) = \frac{s(p+2)pa^p}{p!} \delta^{p+2}(\vec{r}) \left(\frac{\partial}{\partial r}\right)^p (r^p \psi_{ex}). \quad \text{(7)}$$

In analogy with the three-dimensional case, the operator $\delta^{p+2}(\vec{r}) \left(\frac{\partial}{\partial r}\right)^p r^p$ is the pseudopotential.

II. Let us now proceed to calculate the exact pseudopotentials. We shall make the assumption that ψ depends on r and θ only. Since ultimately, the detailed calculations are carried out by considering the 's-wave' contribution only, the above assumption does not entail any error. We shall denote the Laplacian in our space by Δ_{p+2} so that the usual three-dimensional Laplacian would be represented by Δ_3. Following Sommerfeld (1949), the Laplacian in our space is given by

$$\Delta_{p+2} = \frac{1}{\bar{g}} \frac{d}{dr} \left(\frac{\bar{g}}{g_1^2} \frac{d}{dr}\right) + \frac{1}{\bar{g}} \frac{d}{d\theta} \left(\frac{\bar{g}}{g_2^2} \frac{d}{d\theta}\right) + \ldots + \frac{1}{\bar{g}} \frac{d}{d\phi_p} \left(\frac{\bar{g}}{g_{p+2}^2} \frac{d}{d\phi_p}\right) \quad \text{(8)}$$

where

$$g_1 = 1,\ g_2 = r,\ g_3 = r \sin\theta, \ldots g_{p+2} = r \sin\theta \sin\phi_1 \ldots \sin\phi_{p-1} \quad \text{(9)}$$

and

$$\bar{g} = \prod_{i=1}^{p+2} g_i = r^{p+1} \sin^p\theta \sin^{p-1}\phi_1 \ldots \sin\phi_{p-1}. \quad \text{(10)}$$

We come back to (1) and write its solution as ($r \geqslant a$)

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} \psi_l(kr) P_l(\cos\theta | p) \quad \text{(11)}$$

where $P_l(\cos\theta | p)$ are the 'p-dimensional' zonal harmonics which are also called the 'Gegenbauer polynomials'. They satisfy the equation

$$\left[\frac{d}{d\theta} \sin^p\theta \frac{d}{d\theta} + l(l+p) \sin^p\theta\right] P_l(\cos\theta | p) = 0. \quad \text{(12)}$$

The Legendre polynomials may then be denoted by $P_l(\cos\theta | 1)$. We shall only state the form of $\psi_l(kr)$. The derivation follows easily from Sommerfeld (1949), and any standard book on scattering, *e.g.* Ohmura and Wu (1962). If $p/2$ is non-integral ($p = 1, 3, 5, \ldots$)

$$\psi_l(kr) = A_l r^{-p/2} [\cos\eta_l J_{l+p/2}(kr) - \sin\eta_l J_{-l-p/2}(kr)] \quad \text{(13)}$$

where the J's are Bessel functions and η_l is the phase-shift for the lth partial wave, given by

$$\tan \eta_l = \frac{J_{l+p/2}(ka)}{J_{-l-p/2}(ka)} \quad \ldots \quad (13a)$$

which follows from the vanishing of the wave function at $r = a$. For $p/2$ integral ($p = 2, 4, \ldots$)

$$\psi_l(kr) = r^{-p/2} A_l [\cos \eta_l J_{l+p/2}(kr) - N_{l+p/2}(kr) \sin \eta_l] \quad \ldots \quad (14)$$

where N denotes the Neumann Bessel function. The boundary condition at a, gives

$$\tan \eta_l = \frac{J_{l+p/2}(ka)}{N_{l+p/2}(ka)}. \quad \ldots \quad (14a)$$

Using the well-known asymptotic formulae, Jackson (1962)

$$J_\nu(x) \underset{x \to 0}{=} \frac{x^\nu}{2^\nu \Gamma(\nu+1)} N_\nu(x) \to -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu \quad \ldots \quad (14b)$$

we find

$$\psi_l(kr) \underset{r \to 0}{\to} r^l B_l \left[1 + \frac{(-1)^l}{\pi} \frac{\tan \eta_l}{(kr)^{2l+p}} 2^{2l+p} \sin p\pi/2 \Gamma(l+p/2) \Gamma(l+p/2+1)\right]$$

$$(p/2 \text{ non-integral}) \quad \ldots \quad (15)$$

and

$$\psi_l(kr) \underset{r \to 0}{\to} r^l B_l \left[1 + \frac{\tan \eta_l}{\pi} \Gamma(l+p/2) \Gamma(l+p/2+1) \frac{2^{2l+p}}{(kr)^{2l+p}}\right]; \ (p/2 \text{ integral}) \ (15a)$$

where

$$B_l = \frac{A_l k^{l+p/2} \cos \eta_l}{2^{l+p/2} \Gamma(l+p/2+1)}. \quad \ldots \quad (15b)$$

Using (15) or (15a) we may express B_l as

$$B_l = \frac{1}{(2l+p)!} \left[\left(\frac{d}{dr}\right)^{2l+p} (r^{l+p} \psi_l)\right]_{r=0}. \quad \ldots \quad (16)$$

Thus we have obviated the necessity of explicitly mentioning A_l. From its definition, $\psi_l P_l(\cos \theta | p)$ satisfies $(\nabla^2 + k^2) \psi_l P_l(\cos \theta | p) = 0$ everywhere except at $\vec{r} = 0$. The modification to be affected at $r = 0$ may be found as follows. We find

$$(\nabla^2 + k^2)(r^{-p/2} J_{l+p/2}(kr) P_l(\cos \theta | p)) = 0 \quad \text{(everywhere)}.$$

Let us first consider the case when $p/2$ is non-integral.
We write

$$(\nabla^2 + k^2)[r^{-p/2} J_{-l-p/2}(kr) P_l(\cos \theta | p)] = P_l(\cos \theta | p) F_l(r) \quad \ldots \quad (17)$$

where $F_l(r)$ is easily shown to be given by

$$F_l(r) = \left[\frac{1}{r^{p+1}} \frac{d}{dr}\left(r^{p+1} \frac{d}{dr}\right) + k^2 - \frac{l(l+p)}{r^2}\right] J_{-l-p/2}(kr) r^{-p/2} \quad \ldots \quad (18)$$

which is zero everywhere except at $r = 0$. Multiplying $F_l(r)$ by r^l and integrating over a small '$p+2$' dimensional sphere of infinitesimal radius ϵ about the origin, and leaving terms that vanish as $\epsilon \to 0$ we find using (14b)

$$\int d\vec{r}\, r^l F_l(r) = \int d\vec{r}\, r^l \left[\frac{1}{r^{p+1}} \frac{d}{dr}\left(r^{p+1}\frac{d}{dr}\right) + k^2 - \frac{l(l+p)}{r^2}\right] J_{-l-p/2}(kr) r^{-p/2}$$

$$= -\frac{(l+p)\, 2^{l+p/2} s(p+2)}{\Gamma(-l-p/2+1)\, k^{l+p/2}} \frac{\epsilon^{p+1}\cdot\epsilon^l}{\epsilon^{p+l+1}}$$

$$= \frac{(-1)^{l+1}\Gamma(l+p/2) \sin p\pi/2.\, 2^{l+p/2}}{\pi k^{l+p/2}} s(p+2)(l+p).$$

Hence we conclude,

$$F_l(r) = \frac{(-1)^{l+1}\Gamma(l+p/2)(l+p) \sin p\pi/2}{\pi k^{l+p/2}} 2^{l+p/2} \frac{\delta(r)}{r^{l+p+1}}. \quad \quad .. \quad (19)$$

Carrying out a similar calculation, we find for $p/2$ integral,

$$F_l(r) = \frac{2^{l+p/2}}{k^{l+p/2}} \frac{(l+p)\Gamma(l+p/2)}{\pi} \frac{\delta(r)}{r^{l+p+1}} \quad .. \quad .. \quad .. \quad (20)$$

we have the equation

$$(\nabla^2 + k^2)(\psi_l P_l(\cos\theta \,|\, p)) = -A_l P_l(\cos\theta \,|\, p) F_l(r) \sin\eta_l. \quad .. \quad (21)$$

Substituting the explicit forms of $F_l(r)$ and using (15b) we find finally for the extended wave-function,

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{p\Gamma(p/2+1)\Gamma(p/2)2^p \sin p\pi/2}{p!\ \pi k^p} \tan\eta_0 s(p+2)\delta(\vec{r}) \left(\frac{d}{dr}\right)^p (r^p\psi)$$

$$+ \sum_{l=1}^{\infty} P_l(\cos\theta \,|\, p) \frac{\delta(r)}{r^{l+p+1}} f_l(k) \quad .. \quad .. \quad .. \quad .. \quad (22)$$

where

$$f_l(k) = \frac{(-1)^l (l+p)\Gamma(l+p/2)\Gamma(l+p/2+1) 2^{2l+p} \sin p\pi/2}{\pi(2l+p)!\ k^{2l+p}} \tan\eta_l$$

$$\times \left\{ \left(\frac{d}{dr}\right)^{2l+p} \left[r^{l+p} \int \sin^p\theta P_l(\cos\theta \,|\, p) \psi(\vec{r})\, d\theta \right] \right\}_{r=0} \quad .. \quad (22a)$$

and for $p/2$ integral,

$$(\nabla^2 + k^2)\psi(\vec{r}) = -\frac{p\Gamma(p/2+1)\Gamma(p/2)}{\pi} \frac{2^p}{k^p} \frac{\tan\eta_0}{p!} s(p+2)\delta(\vec{r}) \left(\frac{d}{dr}\right)^p (r^p\psi)$$

$$+ \sum_{l=1}^{\infty} f_l(k) P_l(\cos\theta \,|\, p) \frac{\delta(r)}{r^{l+p+1}} \quad .. \quad .. \quad .. \quad (23)$$

where

$$f_l(k) = \frac{-(l+p)\Gamma(l+p/2+1)\Gamma(l+p/2)}{\pi}\frac{2^{2l+p}}{k^{2l+p}}\frac{\tan\eta_l}{(2l+p)!}$$
$$\times\left\{\left(\frac{d}{dr}\right)^{2l+p}[r^{l+p}\int \sin^p\theta P_l(\cos\theta\,|\,p)\psi(\vec{r})d\theta\right\}_{r=0}. \quad .. \ (23a)$$

The terms on the right-hand sides of (22) and (23) contain the pseudopotentials, representing S, P, D... effects. The S-wave pseudopotential (exact) for $p/2$ non-integral is from (22),

$$\frac{p\Gamma(p/2+1)\Gamma(p/2)2^p\sin p\pi/2}{p!\,\pi k^p}\tan\eta_0 s(p+2)\delta(\vec{r})\left(\frac{d}{dr}\right)^p(r^p\psi). \quad .. \ (24)$$

From (13a)

$$\tan\eta_0 = \frac{J_{p/2}(ka)}{J_{-p/2}(ka)} = \frac{(ka)^p\Gamma(-p/2+1)}{2^p\Gamma(p/2+1)}[1+\ldots(ka)^2+\ldots]. \quad .. \ (25)$$

Hence an inspection of (24) and (25) shows that (7) is correct up to order a^{p+1}. Extension to the N-body problem proceeds exactly as in the three-dimensional case, except that we have $(p+2)N$ dimensional configuration space. We can also show from dimensionality arguments that the three-body pseudopotential is of order a^{p+3}, 4-body pseudopotential is of order a^{2p+5}, and so on. These may be ignored if we are interested in an accuracy up to order a^{p+1}.

III. Now we shall calculate the thermodynamic functions such as pressure, free energy, etc. We take the effective Hamiltonian to be

$$H = -\frac{\hbar^2}{2m}[\nabla_1^2+\ldots+\nabla_N^2]+s(p+2)pa^p\frac{\hbar^2}{m}\sum_{i<j}\delta^{p+2}(\vec{r}_i-\vec{r}_j) \quad .. \ (26)$$

the factor $\left[\left(\frac{d}{dr}\right)^p(r^p\psi_{ex})\right]_{r=0}$ giving $p!\,\psi_{ex}(0)$ since $\psi_{ex}(\vec{r})$ is well-behaved for $r=0$. (The differential operator here acts on unperturbed wave functions which are well-behaved as in the three-dimensional case.) Let the unperturbed wave functions be Φ_n labelled by the occupation numbers $\{\ldots n_p\ldots\}$ where n_p is the number of bosons with momentum p. The energy levels are given by Huang (1963)

$$E_n = \sum_p (p^2/2m)n_p+\frac{s(p+2)pa^p}{mV}\hbar^2(N^2-\tfrac{1}{2}n_0^2) \quad .. \quad .. \ (27)$$

where V is the $(p+2)$ dimensional volume and we have assumed that the temperature is so low that only a few particles are excited. The partition function is

$$Q_N = \sum_{\{n_p\}}\exp\left\{-\beta\left[\sum_p\frac{p^2}{2m}n_p+\frac{s(p+2)pa^p}{mV}\hbar^2(N^2-\tfrac{1}{2}n_0^2)\right]\right\}$$

or

$$Q_N = e^{-\frac{2NA\lambda^2}{v}} \sum_{\xi=0}^{1} e^{\frac{A\lambda^2\xi^2 N}{v}} Q(\xi) \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

where

$$\xi = \frac{n_0}{N}\left(\xi = 0, \frac{1}{N}, \ldots 1\right), \; v = \frac{V}{N} \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

$$A = \frac{s(p+2)pa^p}{4\pi}, \; \lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}, \; \beta = \frac{1}{kT}. \quad \ldots \quad \ldots \quad (30)$$

It can be shown that

$$\frac{1}{N}\log Q_N = -\frac{2A\lambda^2}{v} + f(\bar{\xi}) \quad \ldots \quad \ldots \quad \ldots \quad (30a)$$

where

$$f(\bar{\xi}) = \text{Max}\,.\,f(\xi) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

$$f(\xi) = \frac{A\lambda^2}{v}\xi^2 + \frac{1}{N}\log Q(\xi). \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

Also

$$f(\xi) = \frac{A\lambda^2}{v}\xi^2 + \frac{v}{\lambda^{p+2}} g_{p/2+2}(z) - (1-\xi)\log z \quad \ldots \quad \ldots \quad (32a)$$

$$\frac{\lambda^{p+2}}{v}(1-\xi) = g_{p/2+1}(z) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32b)$$

where

$$g_n(z) = \sum_{\nu=1}^{\infty} \frac{z^\nu}{\nu^n}\,;\; z = e^{\beta\mu}. \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

To find $f(\bar{\xi})$ we proceed as in the three-dimensional case:

1. We plot ξ against z for given $\frac{\lambda^{p+2}}{v}$ from (32*b*).
2. For fixed $\frac{\lambda^{p+2}}{v}$ we find $\bar{\xi}$ by maximizing $f(\xi)$ along the appropriate ξ *vs* z curve.

To illustrate the method tables are drawn for the case of four dimensions ($p = 2$). For details reference may be made to Appendix B. We have

$$\frac{df}{d\xi} = \frac{A\lambda^2}{v}\,.\,2\xi + \log z$$

$$\frac{d^2f}{d\xi^2} = -\frac{\lambda^{p+2}}{v}\left[\frac{1}{g_{p/2}(z)} - \frac{2A}{\lambda_p}\right] < 0$$

(except when $z \approx 0$).

The last statement holds if $\frac{2A}{\lambda^p} < z$. Since $\frac{a}{\lambda} \ll 1$ this may be considered always fulfilled. Hence

$$\bar{\xi} = 0, \quad \frac{\lambda^{p+2}}{v} < g_{p/2+1}(1) \qquad (34a)$$

$$= 1 - \frac{v g_{p/2+1}(z_0)}{\lambda^{p+2}}, \quad \frac{\lambda^{p+2}}{v} > g_{p/2+1}(1) \qquad (34b)$$

where z_0 is obtained from the equation,

$$-\log z_0 = \frac{2A\lambda^2}{v}\left[1 - \frac{v g_{p/2+1}(z_0)}{\lambda^{p+2}}\right] \qquad (35)$$

$\frac{\lambda^{p+2}}{v} = g_{p/2+1}(1)$ gives the transition line of the Bose-Einstein condensation. The transition temperature T_c and the transition volume v_c are, therefore, as in the ideal Bose gas case,

$$kT_c = \frac{2\pi\hbar^2}{m} \frac{1}{[v g_{p/2+1}(1)]^{(2/p+2)}} \qquad (36)$$

$$v_c = \frac{\lambda^{p+2}}{g_{p/2+1}(1)}. \qquad (37)$$

From (35) we have

$$z_0 \approx 1 - \frac{2A\lambda^2}{v}\left(1 - \frac{v}{v_c}\right), \quad \frac{A\lambda^2}{v} \ll 1 \qquad (37a)$$

$$z_0 \approx e^{-\frac{2A\lambda^2}{v}} \qquad \frac{A\lambda^2}{v} \gg 1. \qquad (37b)$$

We shall consider (37*a*) only as in the three-dimensional case. Hence

$$\bar{\xi} = 0 \qquad v > v_c \qquad T > T_c \qquad (38)$$

$$= 1 - v/v_c \qquad v < v_c \qquad T < T_c \qquad (38a)$$

and

$$\frac{1}{N}\log Q_N = -\frac{2A\lambda^2}{v} - \log z + \frac{v}{\lambda^{p+2}} g_{p/2+2}(z) \; (v > v_c,\; T > T_c) \qquad (39)$$

$$= \frac{v}{\lambda^{p+2}} g_{p/2+2}(1) - \frac{2A\lambda^2}{v}\left[1 - \tfrac{1}{2}(1 - v/v_c)^2\right] \; (v < v_c,\; T < T_c). \qquad (39a)$$

We proceed to calculate the various thermodynamic parameters. For the free energy, we have

$$\frac{F}{NkT} = -\frac{1}{N}\log Q_N = \frac{2A\lambda^2}{v} + \log z - \frac{v}{\lambda^{p+2}} g_{p/2+2}(z) \; (v > v_c,\; T > T_c)$$

$$= \frac{2A\lambda^2}{v}\left[1 - \tfrac{1}{2}(1 - v/v_c)^2\right] - \frac{v}{\lambda^{p+2}} g_{p/2+2}(1) \; (v < v_c,\; T < T_c).$$

Using, $P = -\left(\frac{\partial F}{\partial V}\right)_T$ (Landau and Lifshitz 1959) we have

$$P = \frac{kT}{\lambda^{p+2}} g_{p/2+2}(z) + \frac{4\pi A\hbar^2}{mv^2}$$

$$= \mathcal{P}^{(0)} + \frac{4\pi A\hbar^2}{mv^2}, \; v > v_c \quad \ldots \quad (40)$$

$$P = \mathcal{P}^{(0)} + \frac{2\pi A\hbar^2}{m}\left(\frac{1}{v^2} + \frac{1}{v_c^2}\right), \; v < v_c \quad \ldots \quad (40a)$$

where $\mathcal{P}^{(0)}$ is the pressure of the ideal Bose gas. We proceed to calculate the first two virial coefficients. We have in the gas phase, from (40),

$$\frac{Pv}{kT} = \frac{v}{\lambda^{p+2}} g_{p/2+2}(z) + \frac{2A\lambda^2}{v} = \frac{g_{p/2+2}(z)}{g_{p/2+1}(z)} + \frac{2A\lambda^2}{v}.$$

Let

$$y = g_{p/2+2}(z) = z + \frac{z^2}{2^{p/2+2}} + \frac{z^3}{3^{p/2+2}} + \ldots$$

$$x = g_{p/2+1}(z) = z + \frac{z^2}{2^{p/2+1}} + \frac{z^3}{3^{p/2+1}} + \ldots$$

To eliminate z, we follow London (1954) and obtain

$$y = x + x^2\left[-\frac{1}{2^{p/2+2}}\right] + x^3\left[\frac{1}{2^{p+2}} - \frac{2}{3^{p/2+2}}\right] + \ldots$$

$$\therefore \quad \frac{Pv}{kT} = 1 + \left[-\frac{1}{2^{p/2+2}} + \frac{2^p s(p+2)}{4\pi}\left(\frac{a}{\lambda}\right)^p\right]\left(\frac{\lambda^{p+2}}{v}\right) + \left[\frac{1}{2^{p+2}} - \frac{2}{3^{p/2+2}}\right]\left(\frac{\lambda^{p+2}}{v}\right)^2 + \ldots$$

Hence higher virial coefficients, if they depend on a, must involve orders of a^{p+1} or higher.

For three dimensions, $p = 1$,

$$\frac{Pv}{kT} = 1 + \left[-\frac{1}{2^{5/2}} + 2\left(\frac{a}{\lambda}\right)\right]\left(\frac{\lambda^3}{v}\right) + \left[\frac{1}{2^3} - \frac{2}{3^{5/2}}\right]\left(\frac{\lambda^3}{v}\right)^2 + \ldots$$ higher coefficients involve

a^2 or higher powers of a.

For four dimensions, $p = 2$,

$$\frac{Pv}{kT} = 1 + \left[-\frac{1}{2^3} + 2\pi\left(\frac{a}{\lambda}\right)^2\right]\left(\frac{\lambda^4}{v}\right) + \left[\frac{1}{2^4} - \frac{2}{3^3}\right]\left(\frac{\lambda^4}{v}\right)^2 + \ldots$$ higher coefficients involve

a^3 or higher powers of a.

For five dimensions, $p = 3$,

$$\frac{Pv}{kT} = 1 + \left[-\frac{1}{2^{7/2}} + 4\pi\left(\frac{a}{\lambda}\right)^3\right]\left(\frac{\lambda^5}{v}\right) + \left[\frac{1}{2^5} - \frac{2}{3^{7/2}}\right]\left(\frac{\lambda^5}{v}\right)^2 + \ldots$$ higher coefficients in-

volve a^4 or higher powers of a.

We now proceed to calculate c_V. Using

$$E = -T^2\left(\frac{\partial}{\partial T}\frac{F}{T}\right)_V$$

we get

$$E_+ = (p/2+1)RT\,\frac{g_{p/2+2}(z)}{g_{p/2+1}(z)} + \frac{4\pi A N\hbar^2}{mv}$$

where positive refers to $T > T_c$.

$$\therefore \quad (C_V)_+ = \left(\frac{\partial E_+}{\partial T}\right)_V = (p/2+1)\left[(p/2+2)\frac{g_{p/2+2}(z)}{g_{p/2+1}(z)} - (p/2+1)\frac{g_{p/2+1}(z)}{g_{p/2}(z)}\right] R$$

which is the same as for an ideal gas. Similarly,

$$E_- = (p/2+1)\frac{Nkv}{\lambda^{p+2}}\,T g_{p/2+2}(1) + \frac{4\pi A N\hbar^2}{mv} - \frac{2\pi A\hbar^2 N}{mv}\left\{1-\left(\frac{T}{T_c}\right)^{p/2+1}\right\}^2$$

$$-4(p/2+1)\frac{\pi A\hbar^2 N}{mv}\left(\frac{T}{T_c}\right)^{p/2+1}\left\{1-\left(\frac{T}{T_c}\right)^{p/2+1}\right\}$$

$-ve$ sign refers to $T < T_c$

$$\therefore \quad (c_V)_- = \left(\frac{\partial E_-}{\partial T}\right)_V = (p/2+1)(p/2+2)\frac{g_{p/2+2}(1)}{g_{p/2+1}(1)}\,R$$

$$+4(p/2+1)^2\frac{\pi A\hbar^2 N}{mv}\,\frac{T^{p+1}}{T_c^{p+2}} + \text{two terms involving}\left\{1-\left(\frac{T}{T_c}\right)^{p/2+1}\right\}.$$

Hence the specific heat decreases across the transition point by the amount,

$$\frac{\Delta c_v}{Nk} = \frac{4A}{2}\,\frac{(p/2+1)^2}{\lambda_c^p}\,g_{p/2+1}(1) + \frac{(p/2+1)^2}{Nk}\,\underset{\substack{T\to T_c\\ Z\to 1}}{\mathrm{Lt}}\,\frac{g_{p/2+1}(z)}{g_{p/2}(z)}$$

where

$$\lambda_c^{p+2} = v g_{p/2+1}(1).$$

For three dimensions, $p = 1$. The second term vanishes

$$\therefore \quad \frac{\Delta c_v}{Nk} = \frac{9a}{2\lambda_c}\,g_{3/2}(1).$$

For four dimensions, $p = 2$

$$\frac{\Delta c_v}{Nk} = \frac{8a^2\pi}{\lambda_c^2}\,g_2(1) = \frac{8a^2\pi}{\lambda_c^2}\,(1{\cdot}645).$$

For five dimensions, $p = 3$

$$\frac{\Delta c_v}{Nk} = 25\pi\left(\frac{a}{\lambda_c}\right)^3(1{\cdot}341) + \frac{25}{4Nk}\,\frac{1{\cdot}341}{2{\cdot}612}\,.$$

To calculate entropy we use

$$F = E - TS$$

and get

$$\frac{S_+}{Nk} = (p/2+2)\frac{v}{\lambda^{p+2}}g_{p/2+2}(z) - \log z$$

which is the same as in the ideal gas case, while

$$\frac{S_-}{Nk} = (p/2+2)\frac{v}{\lambda^{p+2}}g_{p/2+2}(1) - (p+2)\frac{A\lambda^2}{v}\left\{\left(\frac{T}{T_c}\right)^{p/2+1} - \left(\frac{T}{T_c}\right)^{p+2}\right\}.$$

IV. We now take up the two-dimensional case which we deliberately relegated to the end because of its singular character. As already stated in the introduction, there is no condensation in this case. We again start from the Schrödinger equation in relative coordinates,

$$\left.\begin{aligned}(\nabla^2+k^2)\psi(\vec{r}) &= 0 \quad r > a\\ \psi(\vec{r}) &= 0 \quad r \leqslant a\end{aligned}\right\}. \qquad \ldots \quad \ldots \quad \ldots \quad (41)$$

We write the solution for $r \geqslant a$ in the following form,

$$\psi(\vec{r}) = \sum_l \psi_l(r)e^{il\theta} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

where

$$\psi_l(r) = A_l[\cos\eta_l J_l(kr) - \sin\eta_l N_l(kr)] \qquad \ldots \quad \ldots \quad \ldots \quad (42a)$$

which can be easily verified.

Using (14) we find

$$\psi_l(r) \underset{r\to 0}{\to} \frac{A_l\cos\eta_l}{\Gamma(l+1)}\left[1+\tan\eta_l\frac{\Gamma(l)}{\pi}\Gamma(l+1)\left(\frac{2}{kr}\right)^{2l}\right]\left(\frac{kr}{2}\right)^l. \qquad \ldots \quad (43)$$

Setting

$$B_l = \frac{A_l\cos\eta_l}{\Gamma(l+1)}\left(\frac{k}{2}\right)^l \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43a)$$

$$\psi_l(r) \underset{i\to 0}{\to} B_l r^l\left[1+\frac{\tan\eta_l}{\pi}\Gamma(l)\Gamma(l+1)\left(\frac{2}{kr}\right)^{2l}\right]. \qquad \ldots \quad \ldots \quad (44)$$

It is convenient to express B_l as

$$B_l = \frac{1}{2l!}\left[\left(\frac{d}{dr}\right)^{2l}(r^l\psi_l(r)\right]_{r=0} \qquad \ldots \quad \ldots \quad \ldots \quad (45)$$

which follows directly from (44). $\psi_l e^{il\theta}$ satisfies the equation $(\nabla^2+k^2)\psi_l e^{il\theta} = 0$ everywhere except at $\vec{r} = 0$.

The modification needed at $\vec{r} = 0$ may be found by considering only the second term in (3) since $(\nabla^2+k^2)J_p(kx)e^{ip\theta} = 0$ (everywhere).

We write

$$(\nabla^2+k^2)[N_l(kr)e^{il\theta}] = F_l(r)e^{il\theta}$$

where $F_l(r)$ is obviously given by

$$F_l(r) = \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)+k^2-\frac{l^2}{r^2}\right]N_l(kr) \quad \ldots \quad \ldots \quad \ldots \quad (46)$$

which is zero everywhere except at $\vec{r} = 0$. Multiplying (46) by r^l and integrating over a circle of infinitesimal radius ϵ about the origin, we get (noting 14*b*) and using Green's theorem,

$$I = \int d^2r r^l F_l(r) = \frac{\pi\Gamma(l)}{\pi}\, 4l\left(\frac{2}{k}\right)^l$$

so that

$$F_l(r) = \frac{2l}{\pi}\,\Gamma(l)\left(\frac{2}{k}\right)^l\frac{\delta(r)}{r^{l+1}}. \quad \ldots \quad \ldots \quad \ldots \quad (47)$$

We have the equation

$$(\nabla^2+k^2)(\psi_l(r)\exp^{il\theta})$$
$$= -\exp^{il\theta}A_l\sin\eta_l F_l(r).$$

Using (42), (45), (43) and (47) we finally obtain

$$(\nabla^2+k^2)\psi(\vec{r}) = -4\tan\eta_0\delta(\vec{r})\psi(\vec{r})$$
$$+\sum_{l=1}^{\infty}\frac{l!\,(2l+1)\,.\,2^{l+1}}{(2l+1)!}\,\frac{1}{k^{2l}}\,\frac{\tan\eta_l}{\pi}\exp^{il}$$
$$\times\frac{\delta(r)}{r^{l+1}}\left[\left(\frac{d}{dr}\right)^{2l}(r^l\psi_l)\right]_{r=0}.$$

As in the other cases, we shall consider only the *s*-wave pseudopotential. Following Feshbach and Morse (1953), we use for $ka \ll 1$ (as in the other cases) $\eta_0 \to \frac{\pi}{2}\, l_n\frac{1}{ka}$, so that our *s*-wave pseudopotential becomes

$$2\pi l_n(ka)\delta(\vec{r}).$$

We take the energy levels to be

$$E_n = \sum_p \frac{p^2}{2m}n_p + \frac{2\pi l_n(ka)\hbar^2}{mV}N^2.$$

Since $\frac{<n_0>}{V}$, $\frac{<n_1>}{V}\ldots$ all approach zero as $V\to\infty$ (*see* Huang 1963)

$$\therefore\quad Q_n = \sum_{\{n_p\}}\exp\left\{-\beta\sum_p\frac{p^2}{2m}n_p - \frac{2\pi l_n(ka)}{mV}\hbar^2N^2\beta\right\}$$
$$= \exp\left\{-\frac{\lambda^2 l_n(ka)N}{v}\right\}\sum_{\{n_p\}}\exp\left\{-\beta\sum_p\frac{p^2}{2m}n_p\right\}$$

$$\therefore\quad \frac{1}{N}\log Q_N = -\frac{\lambda^2 l_n(ka)}{v}+\frac{1}{N}\log Q_N^{(0)}$$

where $Q_N^{(0)}$ is the partition function of the ideal two-dimensional Bose gas. It is well known that

$$\left.\begin{aligned}\frac{1}{N}\log Q_N^{(0)} &= \frac{v}{\lambda^2}\,g_2(z)\\ \frac{\lambda^2}{v} &= g_1(z)\end{aligned}\right\},$$

$$\therefore\quad \frac{1}{N}\log Q_N = -\frac{\lambda^2 l_n(ka)}{v}+\frac{v}{\lambda^2}\,g_2(z)$$

$$\therefore\quad F = -kT\log Q_N = \frac{\lambda^2 l_n(ka)}{v}-\frac{v}{\lambda^2}\,g_2(z)$$

$$\therefore\quad P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{kT}{\lambda^2}\,g_2(z)+\frac{2\pi\hbar^2}{mv^2}\,l_n(ka)$$

$$\text{or } \frac{Pv}{kT} = 1+\left[-1/2^2+l_n(ka)\right]\left(\frac{\lambda^2}{v}\right)+\frac{1}{36}\left(\frac{\lambda^2}{v}\right)^2+\ldots$$

C_V is, as in the ideal gas, given by

$$C_V = \left[2\frac{g_2(z)}{g_1(z)}-\frac{g_1(z)}{g_0(z)}\right]$$

where $z = e^{\frac{\mu}{kT}}$, μ being the chemical potential.

The authors wish to express their thanks to Professor D. S. Kothari, F.N.I., for his kind interest in this work.

Appendix A

Plotting of E–z curves

We have the relation (eq. 32*b*)

$$\frac{\lambda^{p+2}}{v}(1-\xi) = g_{p/2+1}(z) \qquad \text{(A1)}$$

with

$$g_\sigma(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^\sigma}. \qquad \text{(A2)}$$

We have the function

$$F_\sigma(\alpha) = \sum_{n=1}^{\infty} n^{-\sigma}e^{-n\alpha} \text{ (London 1954)} \qquad \text{(A3)}$$

which for σ integral ($m \geqslant 2$) can be written as

$$F_m(\alpha) = \frac{(-\alpha)^{m-1}}{(m-1)!}\left(1+\tfrac{1}{2}+\ldots+\frac{1}{m-1}-l_n\alpha\right)$$

$$+\sum_{\substack{n=0\\ \neq m-1}}^{\infty}(-\alpha)^n\frac{\zeta(m-n)}{n!}. \qquad \text{(A4)}$$

The functions (B2) and (B3) are the same if we identify z with $e^{-\alpha}$. Hence for computing the values of the function (B2) we first construct Table I:

TABLE I

α	$z = e^{-\alpha}$
0	1
0·005	0·994
0·010	0·990
0·020	0·980
0·100	0·904
0·200	0·818
0·400	0·670
0·500	0·606
1·000	0·367
1·500	0·223
2·000	0·135

For the case of four dimensions (we take this for illustration, for other cases the same procedure can be applied), $p = 2$.

Hence (A1) becomes

$$\frac{\lambda^4}{v}(1-\xi) = g_2(z). \qquad \text{(A5)}$$

Putting $m = 2$ in (A4) and using the well-known values of the ζ-function (Emde, Jahnke and Lösch 1960), we get

$$F_2(\alpha) = -\alpha(1-l_n\alpha)+(1{\cdot}654)+\frac{\alpha^2}{2}(-{\cdot}5000)-\frac{\alpha^3}{6}(-0{\cdot}0833)$$
$$-\frac{\alpha^5}{5!}(0{\cdot}0083) \qquad \text{(A6)}$$

Hence, we construct Table II:

TABLE II

z	$g_2(z)$
1	1·645
0·994	1·613
0·990	1·589
0·980	1·546
0·904	1·312
0·812	1·113
0·670	0·838
0·606	0·736
0·367	0·409
0·223	0·237
0·135	0·140
0	0

Appendix B

Derivation of the scattering cross-section in $(p+2)$ dimensions: ($p = 1$, for three dimensions; $p = 2$, for four dimensions). We shall start from (14). The results obtained also hold for $p/2$ non-integral. We use (14) merely for convenience. The asymptotic form of (14) is (following Jackson 1962)

$$\psi_l(r) \underset{r\to\infty}{\to} r^{-p/2} A_l \sqrt{\frac{2}{\pi kr}} \cos\left\{kr - \frac{\pi}{2}(l+p/2) - \frac{\pi}{4} + \eta_l\right\}. \quad \text{(B1)}$$

Also

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} \psi_l(r) P_l \cos(\theta | p). \quad \text{(B2)}$$

We proceed exactly as in the three-dimensional case (Schiff 1955). We take the asymptotic wave-function to be

$$\psi(\vec{r}) = e^{ikr\cos\theta} + f(\theta)(\tfrac{1}{2}\pi k)^{\frac{1}{2}} r^{-p/2} H^{(1)}_{p/2}(kr) \quad \text{(B3)}$$

where $r^{-p/2} H^{(1)}_{p/2}(kr)$ (Sommerfeld 1949) is the term appropriate for outgoing waves. We take the representation of the plane wave in many-dimensional space from Sommerfeld (1949)

$$e^{ikr\cos\theta} = 2^{p/2}\Gamma(p/2) \sum_{l=0}^{\infty} (l+p/2) e^{il\pi/2} P_l(\cos\theta | p) \frac{J_{l+p/2}(kr)}{(kr)^{p/2}}. \quad \text{(B4)}$$

Substituting the asymptotic form of (B4) into (B3) and equating this to the asymptotic form of (B2), we obtain

$$\begin{aligned}
\sum_{l=0}^{\infty} A_l r^{-p/2} \sqrt{\frac{2}{\pi kr}} \cos\left\{kr - \frac{\pi}{2}(l+p/2) - \frac{\pi}{4} + \eta_l\right\} P_l(\cos\theta | p) \\
= 2^{p/2}\Gamma(p/2) \sum_{l=0}^{\infty} (l+p/2) \exp(il\pi/2) \frac{P_l(\cos\theta | p)}{(kr)^{p/2}} \\
\times \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{\pi}{2}(l+p/2) - \frac{\pi}{4}\right) \\
+ f(\theta) r^{-\left(\frac{p+1}{2}\right)} \exp i\left[kr - (p+1)\frac{\pi}{4}\right]. \quad \text{(B5)}
\end{aligned}$$

Writing the cosines in terms of exponentials and equating coefficients of $\exp(ikr)$ and $\exp(-ikr)$ on the two sides of this equation, we get

$$\begin{aligned}
\sum_{l=0}^{\infty} A_l r^{-p/2} \sqrt{\frac{2}{\pi kr}} \exp\left\{-i\frac{\pi}{2}(l+p/2) - i\frac{\pi}{4}\right\} \exp(-i\eta l) P_l(\cos\theta | p) \\
= 2^{p/2}\Gamma(p/2) \sum_{l=0}^{\infty} (l+p/2) \exp(il\pi/2) \frac{P_l(\cos\theta | p)}{(kr)^{p/2}} \sqrt{\frac{2}{\pi kr}} \\
\times \exp\left\{-\frac{i\pi}{2}(l+p/2) - \frac{i\pi}{4}\right\} + f(\theta) r^{\left(-\frac{p+1}{2}\right)} \exp\{-i(p+1)\pi/4\} \quad \text{(B6)}
\end{aligned}$$

and

$$\sum_{l=0}^{\infty} A_l r^{-p/2} \sqrt{\frac{2}{\pi kr}} \frac{\exp\{i\pi/2(l+p/2)+i\pi/4\}}{2} \exp(-i\eta l)\, P_l(\cos\theta|p)$$

$$= 2^{p/2}\Gamma(p/2) \sum_{l=0}^{\infty} \exp(il\pi/2) \frac{P_l(\cos\theta|p)}{(kr)^{p/2}} \sqrt{\frac{2}{\pi kr}}$$

$$\times \frac{\exp\{i\pi/2(l+p/2)+i\pi/4\}}{2} \times (l+p/2). \quad \ldots \quad \ldots \quad \ldots \quad \text{(B6')}$$

Since these are true for all θ and $P_l(\cos\theta|p)$ are orthogonal to each other, (B6′) becomes

$$A_l = 2^{p/2}\Gamma(p/2)(l+p/2)\exp(il\pi/2)k^{-p/2}\exp^{i\eta l}. \quad \ldots \quad \ldots \quad \text{(B7)}$$

Substituting this into (B6) we get the scattering amplitude and hence the differential cross-section

$$\sigma(\theta) = \frac{2^p\Gamma(p/2)\Gamma(p/2)}{2\pi k^{p+1}} \left[\left\{ \sum_l (l+p/2)P_l(\cos\theta|p)(\cos 2\eta_l - 1) \right\}^2 + \left\{ \sum_l (l+p/2)P_l(\cos\theta|p)\sin 2\eta_l \right\}^2 \right].$$

Using the orthogonality of the P_l's, we have for the total cross-section

$$\sigma = \int |f(\theta)|^2 d\Omega = \frac{2\pi^{\frac{p+1}{2}}}{\Gamma\left(\frac{p+1}{2}\right)} \int_0^{\pi} |f(\theta)|^2 \sin^p\theta\, d\theta$$

$$= \frac{4\pi^{(p+1)/2}}{\Gamma\left(\frac{p+1}{2}\right)} \frac{1}{k^{p+1}} \sum_l \frac{\Gamma(l+p)}{\Gamma(l+1)} (2l+p)\sin^2\eta_l. \quad \ldots \quad \ldots \quad \text{(B8)}$$

Putting $p = 1$, we get the well-known result for three dimensions

$$(\sigma)_3 = \frac{4\pi}{k^2} \sum_l (2l+1)\sin^2\eta_l.$$

Similarly,

$$(\sigma)_4 = \frac{8\pi}{k^3} \sum_l (l+1)(2l+1)\sin^2\eta_l.$$

For two dimensions if we proceed in the same manner, starting from (B4) we get (Fetter 1964),

$$\sigma = 4k^{-1} \sum_l \sin^2\eta_l.$$

(It has the dimensions of length.)

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