CONGRUENCES FOR NEW TYPES OF PARTITIONS

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Consider the number of partitions of a positive integer n into two sets of positive integers satisfying the conditions

$$n = \sum_{k=1}^{n} a_k + \sum_{j=1}^{n} b_j$$

$$a_1 < a_2 < a_3 < \dots a_r; \quad b_1 \leqslant b_2 \leqslant b_3 \dots \leqslant b_s; \quad b_s \leqslant a_r$$

The set 'b' can be empty. Three different types of partitions under certain conditions were discussed by Auluck (1951). We obtain congruent properties for those three types of partition functions.

Generating functions and asymptotic expressions for the number of partitions of a positive integer n into two sets of positive integers satisfying the conditions

$$n = \sum_{k=1}^{r} a_k + \sum_{j=1}^{s} b_j \quad \dots \quad (1)$$

$$a_1 \leqslant a_2 \leqslant a_3 \leqslant \dots a_r; \quad b_1 \leqslant b_2 \leqslant b_3 \leqslant \dots \leqslant b_s; \quad b_s \leqslant a_r \quad \dots \quad (2)$$

were obtained by Auluck (1951).

The set 'b' here can be empty. The following three types of partitions were studied by Auluck.

Type A. In this type, $b_s = a_r - 1$. Every integer up to a_r is taken at least once in the set 'a' and every integer up to b_s is taken at least once in the set 'b'. This type of partition will be denoted by $P(n)$.

Type B. There are no restrictions on the a's and b's except those stated in (2). $Q(n)$ denotes the total number of such partitions of n.

Type C. In this type every integer up to a_r is taken at least once in the set 'a', there being no such restriction on the set 'b'. Such partitions are being denoted by $R(n)$.

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In this paper, we prove that

$$P(2m+1) \equiv p(2m+1) \pmod 2 \quad \ldots \quad (3)$$

$$P(2m) \equiv p(2m) \pmod 2 \text{ if } m \neq \frac{\beta(3\beta\pm1)}{2} \quad \ldots \quad (4)$$

$$P(2m) \equiv p(2m)+1 \pmod 2 \text{ if } m = \frac{\beta(3\beta\pm1)}{2} \quad \ldots \quad (5)$$

$$Q(2n+1) \equiv 1 \pmod 2 \text{ if } 2n+1 = \frac{\beta(3\beta\pm1)}{2} \quad \ldots \quad (6)$$

$$Q(2n+1) \equiv 0 \pmod 2 \text{ if } 2n+1 \neq \frac{\beta(3\beta\pm1)}{2} \quad \ldots \quad (7)$$

and

$$Q(2n) \equiv p(n)+1 \text{ if } 2n = \frac{\beta(3\beta\pm1)}{2} \quad \ldots \quad (8)$$

$$Q(2n) \equiv p(n) \text{ if } 2n \neq \frac{\beta(3\beta\pm1)}{2} \quad \ldots \quad (9)$$

where $p(n)$ denotes the unrestricted partitions of the number n. Very little is known about the arithmetical properties of $p(n)$; we do not know, for example, when it is odd or even. However, MacMahon (1916) proved that

$$p(n) \equiv \sum_t p(t) \pmod 2$$

where t has the integral values given by $t = \dfrac{2n-k(k+1)}{8}$. An elementary proof for this result was given by Gandhi (1963), who had also obtained three more congruences of MacMahon type for $p(n)$ modulus 2.

The above results tell us that the parities of $p(n)$, $P(n)$ and $Q(n)$ are simply related and that if we know the parity of one, the parity of the other can be determined. These results also help us to check the values of P's and Q's. It is interesting to note that we know for certain when $Q(2n+1)$ is odd or even.

Firstly, we reproduce some formulae obtained by Auluck (1951)

$$\frac{x-x^3+x^6-x^{10}+\ldots}{(1-x)(1-x^2)(1-x^3)\ldots} = \sum_{n=1}^{\infty} P(n)x^n \quad \ldots \quad (10)$$

$$P(n) = \sum_{r(r+1) < 2n} (-1)^{r-1} p[n-\tfrac{1}{2}r(r+1)] \quad \ldots \quad (11)$$

$$\frac{x-x^3+x^6-x^{10}+\ldots}{[(1-x)(1-x^2)(1-x^3)\ldots]^2} = \sum_{n=1}^{\infty} Q(n)x^n \quad \ldots \quad (12)$$

$$P(n) = Q(n) + \sum_{k} (-1)^k \{Q(n-\tfrac{1}{2}k(3k-1)) + Q(n-\tfrac{1}{2}k(3k+1))\} \quad \ldots \quad (13)$$

$$Q(n) = \sum_{m=1}^{n} P(m)p(n-m) \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

$$\frac{1}{\prod_{n=1}^{\infty}(1-x^n)} \sum_{n=0}^{\infty} \frac{x^{(n+1)(2n+1)}}{(1-x^2)(1-x^4)\ldots(1-x^{2n})} = \sum_{n=1}^{\infty} R(n)x^n \quad \ldots \quad (15)$$

$$R(n) = \sum_{m,\,k} p(n-2k^2-k-2m-1)p_{(k)}(m) \quad \ldots \quad \ldots \quad (16)$$

$$2k^2+k+2m \leqslant n-1$$

where $p_{(k)}(m)$ denotes the number of ways in which m can be represented as the sum of exactly k positive integers. Also

$$R(n) = \sum_{m,\,k} p(n-k^2-2k-2m-1)q_{(k)}(m) \quad \ldots \quad \ldots \quad (17)$$

where $q_{(k)}(m)$ denotes the number of ways in which m can be represented as the sum of exactly k different positive integers.

Now we can write (10) in the form

$$x-x^3+x^6-x^{10}+\ldots = \sum_{n=0}^{\infty} p(n)x^n \prod_{n=0}^{\infty}(1-x^n) \quad \ldots \quad \ldots \quad (18)$$

Using Euler's identity that

$$\prod_{n=0}^{\infty}(1-x^n) = \sum_{\beta \geqslant 0} (-1)^{\beta} x^{\frac{\beta(3\beta\pm 1)}{2}} \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

it is easy to prove from (18) that

$$\sum_{k \geqslant 0} P\left[n-\frac{k}{2}(3k\pm 1)\right](-1)^k = (-1)^{m+1} \text{ or } 0$$

according as

$$n = \frac{m(m+1)}{2} \text{ or } n \neq \frac{m(m+1)}{2} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Define $p_r(n)$ by

$$\{(1-x)(1-x^2)(1-x^3)\ldots\}^{-r} = \sum_{n=0}^{\infty} p_r(n)x^n \quad \ldots \quad \ldots \quad (21)$$

Then from (12) we have

$$[x-x^3+x^6-x^{10}+\ldots] \sum_{n=0}^{\infty} p_2(n)x^n = \sum_{m=1}^{\infty} Q(m)x^m \quad \ldots \quad \ldots \quad (22)$$

Whence equating the coefficients of x^n we get

$$Q(m) = \sum_{k>0} p_2 \left[m - \frac{k(k+1)}{2}\right] \quad \ldots \quad (23)$$

Then we use the above formulae to obtain the various congruences.

It was proved by Gandhi (1963) that

$$\sum_{k \geqslant 0} p\left(2m+1-\frac{k(k+1)}{2}\right) \equiv 0 \pmod 2 \quad \ldots \quad (24)$$

and

$$\sum_{k \geqslant 0} p\left(2m-\frac{k(k+1)}{2}\right) \equiv 1 \text{ or } 0 \pmod 2$$

according as

$$m = \frac{\beta(3\beta \pm 1)}{2} \text{ or } m \neq \frac{\beta(3\beta \pm 1)}{2} \quad \ldots \quad (25)$$

Using (24), (25), from (11) we get congruences (3) to (5).

Now using congruences (3) to (5), from (14) we get

$$Q(n) \equiv \sum_{m=1}^{n} p(m)p(n-m) + \sum_{\beta \geqslant 0} p[n-\beta(3\beta \pm 1)] - p(n) \pmod 2$$

or

$$Q(n) \equiv \sum_{m=0}^{n} p(m)p(n-m) + \sum_{\beta \geqslant 0} p[n-\beta(3\beta \pm 1] \pmod 2$$

or

$$Q(n) \equiv p_2(n) + \sum_{\beta \geqslant 0} p[n-\beta(3\beta \pm 1)] \pmod 2 \quad \ldots \quad (26)$$

It was proved by Gandhi (1963) that

$$\left.\begin{aligned} p_2(2m) &\equiv p(m) \pmod 2 \\ p_2(2m+1) &\equiv 0 \pmod 2 \end{aligned}\right\} \quad \ldots \quad (27)$$

and

$$\sum_{\beta \geqslant 0} p[m-\beta(3\beta \pm 1)] \equiv 1 \text{ or } 0 \pmod 2$$

according as

$$m = \frac{\beta(3\beta \pm 1)}{2} \text{ or } m \neq \frac{\beta(3\beta \pm 1)}{2} \quad \ldots \quad (28)$$

Using (27) and (28) from (26) we get the congruences (6) to (9).

It may be of some interest to give an alternative proof for the congruences (6) to (9).

Now

$$\sum_{n=0}^{\infty} p_{-1}(n)x^n = \sum_{n=0}^{\infty} p_{-3}(n)x^n \sum_{n=0}^{\infty} p_2(n)x^n \quad \ldots \quad (29)$$

In view of Jacobi's identity that

$$\sum_{n=0}^{\infty} p_{-3}(n)x^n = \sum_{k\geqslant 0} (-1)^k(2k+1)x^{\frac{k(k+1)}{2}} \quad \dots \quad \dots \quad (30)$$

we get from (29) that

$$p_{-1}(n) \equiv \sum_{k\geqslant 0} p_2\left[n-\frac{k(k+1)}{2}\right] \pmod 2 \quad \dots \quad \dots \quad (31)$$

In view of (31), (23) becomes

$$Q(m) \equiv p_{-1}(m)+p_2(m) \quad \dots \quad \dots \quad \dots \quad \dots \quad (32)$$

Since from (19) we have $p_{-1}(n) = 1$ or 0 according as

$$n = \frac{\beta(3\beta\pm 1)}{2} \text{ or } n \neq \frac{\beta(3\beta\pm 1)}{2}$$

hence using (27) we get congruences (6) to (9).

In the end we remark that from (15) it is easy to prove that

$$\sum_{\beta} R\left[n-\frac{\beta(3\beta\pm 1)}{2}\right] = \sum_{k,\, m} p_{(k)}(m) \text{ or } 0 \quad \dots \quad \dots \quad \dots \quad (33)$$

according as n is of the form $2k^2+k+2m+1$ or is not of that form. The summation on the right-hand side (1) is to be carried for values of k and m such that $n = 2k^2+k+2m+1$.

Also

$$\sum_{\beta} R\left[n-\frac{\beta(3\beta\pm 1)}{2}\right] = \sum_{k,\, m} q_{(k)}(m) \text{ or } 0 \quad \dots \quad \dots \quad \dots \quad (34)$$

according as n is of the form $k^2+2k+2m+1$ or is not of that form.

Recurrences (33) and (34) give a convenient method to calculate the values of R's.

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References

Auluck, F. C. (1951). On some new types of partitions associated with generalized Ferrer's graphs. *Proc. Camb. phil. Soc. math. phys. Sci.*, **47**, 679.

Gandhi, J. M. (1963). Congruences for $p_r(n)$ and Ramanujan's function. *Am. math. Mon.*, **70**, 265.

MacMahon. P. S. (1916). Combinatory Analysis, Vols. I and II. Cambridge University Press.