THE PROBLEM OF TWO ROTATING BODIES IN THE RELATIVITY THEORY

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The existence of an Einstein's static space-time has been proved (Racine 1965) in the case of a rotating fluid contained in a massless shell of negligible thickness. It is proved now that in the case of two such bodies and under some restrictions there is no regular static space-time.

In a recent paper, hereafter called paper I, Racine (1965) proved an existence theorem for the relativistic problem of a rotating fluid when it occupies a simply connected domain bounded by a massless shell of negligible thickness. The proof given there applies as well to the case of several bodies; but then the question arises whether the metric obtained is everywhere regular as in the case of a single body.

Solutions of an 'exterior' problem in the case of axial symmetry and of several bodies have been constructed by Bach, Palatini and Chazy (Darmois 1927). But, as I have pointed out in my thesis (Racine 1934), the metric of these authors fails to be regular at points of the segments of the axis of rotation lying between the bodies. Whether there exists the same kind of irregularity under the general conditions stated in paper I referred to above (Racine 1965) remains an open question. However, under some restrictive conditions, the question can be answered in the affirmative and it is the purpose of this paper to show it.

The restrictive conditions are the following: (1) that there be only two bodies and that, in the space sections, there be symmetry with respect to the origin of coordinates; (2) that, in the underlying space of the space sections, the two bodies be contained each in a shell which is a sphere of radius $L/8$, the distance between the shells being $L/4$; (3) that the angular velocity be sufficiently small, along with the quantity ϵL, where $\epsilon = c^{-1}$, c denoting the velocity of light.

But for these new conditions, the assumptions and notations will be the same as in my paper I. The metric of the space-time is of the form:

$$ds^2 = e^{2\epsilon^2\phi}\, dt^2 - e^{2\epsilon^2\alpha}(dr^2+dz^2) - e^{2\epsilon^2\beta}(d\theta+\epsilon^2\bar{\omega}I\, dt)^2, \quad \ldots \quad \ldots \quad (1)$$

where

$$\alpha = \psi-\phi, \; \beta = \log r+h-\phi,$$

and ϕ, h, I are functions of r and z only; r, z, θ stand for the usual cylindrical coordinates in the underlying space of the space sections and t is the time variable. In each space section, the axis of rotation is of equation $r = 0$; $\bar{\omega}$ is the constant angular velocity. The relativistic potentials ϕ, h and I are first determined throughout space and ψ only inside the masses; then ψ is determined in the empty space as follows: Let $\Delta_1(U)$ and $\Delta_1(U, V)$ stand for $\left(\frac{\partial U}{\partial r}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2$ and $\frac{\partial U}{\partial r}\frac{\partial V}{\partial r} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z}$ respectively. Let further

$$L(\epsilon^2 h) = 1 + 2\epsilon^2 r\frac{\partial h}{\partial r} + \epsilon^4 r^2 \Delta_1 h, \quad u = \beta - \phi \qquad (2)$$

and

$$\Omega = \frac{r}{L(\epsilon^2 h)}\Big\{[-\Delta_1\phi + \tfrac{1}{4}\epsilon^2\bar{\omega}^2 e^{2\epsilon^2 u}\Delta_1 I](dr + \epsilon^2 r\, dh)$$
$$+2\left[\frac{\partial \phi}{\partial r} + \epsilon^2 r\Delta_1(\phi, h)\right]d\phi - \frac{\epsilon^2\bar{\omega}^2}{2}e^{2\epsilon^2 u}\left[\frac{\partial I}{\partial r} + \epsilon^2 r\Delta_1(I, h)\right]dI\Big\}. \qquad (3)$$

Then

$$\psi = h + \frac{c^2}{2}\log L(\epsilon^2 h) - \epsilon^2\int_\Gamma \Omega \qquad (4)$$

where Γ is a regular curve lying entirely in the empty space, starting from the point (r, z) and, for instance, asymptotic to the positive part of the axis of rotation. However, it may be any other curve deduced from this particular one by a regular and continuous deformation. No proof of this fact was given in paper I, because it is a direct consequence of the properties of the Newtonian potentials called into play to define ϕ, h and I. However, it may not be quite improper to propose now an explicit proof. It suffices evidently to show that

$$\lim_{l\to\infty}\int_{C_l}\Omega = 0 \qquad (5)$$

where C_l denotes any arc of the curve $r^2+z^2 = l^2$ and Ω, as shown in paper I, is an exact differential. It has also been shown in paper I that ϕ, h and I can be defined as Newtonian potentials in the Euclidean spaces R^3, R^4 and R^5 respectively, these spaces being referred to cylindrical coordinates $r, z, \theta_1, \theta_2, \ldots$ where r and θ_i denote polar coordinates in the subspace $z = 0$. These three Newtonian potentials are due to spreads of densities ρ_3, ρ_4 and ρ_5 respectively. These spreads are functions of r and z only; their absolute values are bounded by constants M_n, $n = 3, 4, 5$. Further for $l > L$, one has

$$\rho_n = \frac{K_n L^{n+1}}{l^{n+1}} + \frac{f_n(r, z)L^{n+2}}{l^{n+2}}, \quad n = 3, 4, 5,$$

where $l^2 = r^2+z^2$ and the $f_n(r, z)$ are continuous functions such that $|f_n(r, z)| \leqslant N_n$, $n = 3, 4, 5$, the N_n, as well as the K_n, being absolute constants. Norms are defined for the densities by setting

$$\|\rho_n\| = M_n + |K_n| + N_n.$$

Let

$$u_n(P) = -\frac{f}{(n-2)\omega_n}\int_{R^n}\frac{\rho_n(Q)\,dQ}{|PQ|^{n-2}},$$

where $\omega_3 = 4\pi$, $\omega_4 = 2\pi^2$, $\omega_5 = \frac{8\pi^2}{3}$ and f is the usual Newtonian gravitational constant. Let Du_n stand for either of the first derivatives of u. At a point P of R^n such that $l > 2L$, let S_n be a sphere of centre P and of radius $l/2$. Then, on the one side, we have throughout space, $|Du_n| \leqslant f.\|\rho_n\|.2L$, as proved in [I]. On the other, we have, for $l > 2L$, the estimate:

$$|Du_n(P)| \leqslant \frac{f}{\omega_n}\left[\min_{Q\in S_n}|\rho_n(Q)|.\int_{S_n}\frac{dQ}{|PQ|^{n-1}}+\frac{2^{n-1}}{l^{n-1}}\int_{CS_n}|\rho_n(Q)|\,dQ\right],$$

CS_n denoting the complement of S_n in R^n. It is easy to deduce that, for $l > 2L$:

$$|Du_n| \leqslant f.\|\rho_n\|\frac{(2L)^n}{l^{n-1}}. \qquad (6)$$

It follows at once that on C_l we have

$$\Omega = O(l^{-2}).d\theta$$

whence

$$\lim_{l\to\infty}\int_{C_l}\Omega = 0.$$

Now the solution given in paper I determines clearly a space-time which is regular everywhere. This is because there is only one rotating body. However, in the case of two rotating bodies it is not difficult to see that the same conclusion may not hold. In fact, let us consider a point P on the segment of the axis of rotation abutting on the two bodies. At such a point we have $L(\epsilon^2 h) = 1$. Hence

$$\psi = h-\epsilon^2\int_\Gamma\Omega,$$

Γ being one of the admissible curves defined in (4). Let it be possible to prove that

$$-\int_\Gamma\Omega = a \neq 0$$

then at P the metric of a space section is

$$ds^2 = e^{2\epsilon^2(h-\psi)}\left[e^{2\epsilon^2 a}(dr^2+dz^2)+r^2\,d\theta^2\right].$$

It is sufficient to express it in Cartesian coordinates to make it plain that it is not regular at P. The necessary and sufficient condition for regularity at this point is obviously: $a = 0$.

Now our restrictive assumptions permit us to choose as a curve Γ on which to integrate Ω, the curve defined in a space section by $z = 0$, $\theta = 0$.

It is evidently an admissible curve. On it we have, on account of the symmetry:

$$\frac{\partial \phi}{\partial z} = \frac{\partial h}{\partial z} = \frac{\partial I}{\partial z} = 0$$

and also

$$L(\epsilon^2 h) = \left(1+\epsilon^2 r \frac{\partial h}{\partial r}\right)^2.$$

Hence, on Γ:

$$\Omega = \left(1+\epsilon^2 r \frac{\partial h}{\partial r}\right)^{-1} \left[\left(\frac{\partial \phi}{\partial r}\right)^2 - \frac{\bar{\omega}^2 \epsilon^2}{4} e^{2\epsilon^2 u} \left(\frac{\partial I}{\partial r}\right)^2\right] r\,dr. \quad .. \quad .. \quad (7)$$

Under the general assumptions made in [I] we may write

$$\phi = \phi_0 + \epsilon^2 \phi_1$$

where ϕ_0 is a Newtonian potential in R^3 due to a spread of constant density ρ in the two shells mentioned above. As in [I], ρ will be taken equal to 10; ϕ_1 is also a Newtonian potential due to a spread whose density has a norm denoted by $f^{-1} \| A \|$ in [I] and at the most equal to $4 . L^2$. From (6) we get, when $l > 2 . L$:

$$\epsilon^2 \left| \frac{\partial \phi_1}{\partial r} \right| \leqslant 4\epsilon^2 f L^2 \frac{(2L)^3}{l^2}.$$

Further, for $z = 0$:

$$\frac{\partial \phi_0}{\partial r} = \frac{8\pi\rho}{3} \left(\frac{L}{8}\right)^3 \frac{r}{(r^2+d^2)^{\frac{3}{2}}},$$

where $d = L/4$. Provided ϵL is sufficiently small we have, for $z = 0$:

$$\epsilon^2 \left| \frac{\partial \phi_1}{\partial r} \right| < \frac{1}{2} \left| \frac{\partial \phi_0}{\partial r} \right|, \quad \left| \frac{\partial \phi}{\partial r} \right| > \frac{1}{2} \left| \frac{\partial \phi_0}{\partial r} \right|.$$

An easy computation shows that it is sufficient to have $\epsilon^2 L^2 < 1/500$. We shall assume here that $\epsilon L < 1/25$.

Now h is defined as a Newtonian potential in R^4 whose density has a norm denoted by $f^{-1} \| B \|$ in [I]; its upper bound is shown there to be $3\epsilon^2 L^2$. Therefore, as $f = 10^{-7}$:

$$1 - 10^{-10} < 1 + \epsilon^2 r \frac{\partial h}{\partial r} < 1 + 10^{-10}$$

and

$$\int_\Gamma \Omega > \frac{1}{1+10^{-10}} \int_{2L}^{\infty} \frac{1}{4} \left(\frac{\partial \phi_0}{\partial r}\right)^2 r\,dr - \max \frac{1}{1-10^{-10}} \int_0^{\infty} \frac{\epsilon^2 \bar{\omega}^2}{4} e^{2\epsilon^2 u} \left(\frac{\partial I}{\partial r}\right)^2 r\,dr.$$

The estimates for ϕ and h yield the bound

$$e^{2\epsilon^2 (h - 2\phi)} < \tfrac{3}{2}.$$

From (5) we deduce that, when $l > 2 . L$:

$$\left| \frac{\partial I}{\partial r} \right| \leqslant f \| \rho_5 \| \frac{(2L)^5}{l^4}.$$

However, when $l \leqslant 2 \,.\, L$:

$$\left|\frac{\partial I}{\partial r}\right| \leqslant f \| \rho_5 \| \,.\, 2L.$$

It follows that

$$\int_0^\infty \frac{\epsilon^2 \bar{\omega}^2}{4} e^{2\epsilon^2 u} \,.\, \left(\frac{\partial I}{\partial r}\right)^2 r\, dr \leqslant \frac{18}{(25)^2} \bar{\omega}^2 f^2 \| \rho_5 \|^2 \,.\, L^4.$$

Now we have for $z = 0$

$$\tfrac{1}{4} \int_{2L}^\infty \left(\frac{\partial \phi_0}{\partial r}\right)^2 r\, dr \geqslant \frac{f^2 \rho^2 L^4}{2 \,.\, 8^6} [1 - (8 \,.\, 16)^{-2}].$$

In order that the integrand in (7) be positive, we must have, as can be easily concluded from a simple computation:

$$\bar{\omega}^2 \leqslant \frac{(25)^2}{18} \frac{1}{2 \,.\, 8^6} \frac{\rho^2}{\| \rho_5 \|^2}.$$

From the value of $\frac{\rho}{\| \rho_5 \|}$ given in [I] one gets $\frac{\rho}{\| \rho_5 \|} < \frac{1}{17\pi}$. It follows that one may take $\bar{\omega} < \frac{1}{5 \,.\, 10^6}$. With the rough bounds just obtained for $\bar{\omega}$ and ϵL it is seen that the metric (1) has the irregularity mentioned above at the origin of coordinates of a space section. The same kind of irregularity exists, however, at every point of the segment of the axis of rotation lying between the two bodies. In fact, since Ω is an exact differential and since it vanishes on the axis, we have

$$\int_\Gamma \Omega = \int_{\Gamma_1} \Omega$$

where Γ_1 denotes any curve starting from a point of the above-mentioned segment of the axis and on which z is constant.

Now the same estimates for $\bar{\omega}$ and for ϵL hold *a fortiori* if the fluids are contained within two non-overlapping shells which are: (1) symmetric with regard to the origin, in the sense already explained; (2) contained in the sphere of centre the origin and of radius L; (3) containing the two spheres of radius $L/8$ considered above.

In particular let us assume that the fluids are in a position of equilibrium; this is to say that the pressure, denoted p in [I] is vanishing on the boundary of the volumes occupied by the fluids. There is therefore no need any more of massless and thin shells to contain these fluids. Under such conditions, Ω is continuous throughout space in the space sections. Further we have

$$\int_{\Gamma_2} \Omega = 0$$

where Γ_2 is the curve formed by the portion of the axis on which $z > 0$, and exterior to the bodies, as well as by the curve which in the plane $\theta = 0$ generates the boundary of the volume containing the fluid for which $z > 0$. Let γ denote this curve. We have a unique value for the integral, whether it is calculated on one, or on the other, side of γ. In my paper [I] it is shown that this is the case only when $p = 0$ on γ. We have, therefore, in the present case

$$\int_{\gamma} \Omega = \int \Omega,$$

where the last integral is taken on the segment of the axis interior to the fluid for which $z > 0$. But this last integral obviously vanishes since $\Omega = 0$ on the axis.

Yet we have that

$$\int_{\Gamma_1} \Omega \neq 0, \int_{\Gamma_1} \Omega = \int_{\Gamma_2} \Omega.$$

We arrive, therefore, at a contradiction and this shows that it is impossible to have the two rotating fluids in a state of equilibrium.

It is likely that the same kind of irregularity exists when the conditions are nearly the same as those considered in this paper and it is a reasonable conjecture to think that the same kind of irregularity persists when there are two or several bodies in any case of static axial symmetry. But it seems difficult to prove it with great generality; this would certainly require much more refined estimates.

Naturally there is a case in which it is easy to prove the irregularity of the metric on the same segment of the axis: it is when the terms in ϵ^2 and in $\bar{\omega}$ in the expressions of ϕ, h and I are negligible compared to the remaining terms. One may then take for Γ_1 the curve defined by the equation $\dfrac{\partial \phi}{\partial z} = 0$, $\theta = 0$. However, these conditions are too restrictive to be of any interest in problems of Celestial Dynamics.

References

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