GROUND-STATE OF INTERACTING BOSONS

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We have derived a general expression for the ground-state energy of an assembly of N interacting Bose particles enclosed in a large volume V, the two-particle interaction being of repulsive type. We then give three particular cases in which the energy can be evaluated in a finite form. We then extend the treatment, following Huang, to the case when the two-particle interaction consists of a repulsive part and an attractive part.

1. The problem of the ground-state of an interacting Boson system has been discussed by several authors. (See, for example, Auluck and Kothari (1961), Lee and Feenberg (1965) and Hugenholtz and Pines (1959)).

In this paper we deal with some elementary aspects of the problem of the ground-state of an assembly of N interacting Bose particles enclosed in a (large) volume V, the two-particle interaction consisting of a repulsive part and an attractive part. In Sec. 2 we derive a general expression for the energy of the ground-state and in Sec. 3 we obtain by a straightforward extension of the well-known Bogoliubov (1947) transformation the result (Eqn. 56) derived by Huang (1959). We begin with a simple discussion.

If $\vec{k}$ denotes the propagation vector in vacuum and $\vec{k}_1$ in a medium, then we have the relation (Hamilton 1959)

$$k_1^2 = k^2 + 4\pi\rho f(\theta) \quad \ldots \quad (1)$$

or

$$\mu = 1 + \frac{2\pi\rho}{k^2} f(\theta). \quad \ldots \quad (2)$$

Here ρ is the particle density and $f(\theta)$ is the 'scattering amplitude' defined by (in the usual notation)

$$\psi = e^{ikz} + \frac{e^{ikr}}{r} f(\theta), \quad \ldots \quad (3)$$

where $\frac{e^{ikr}}{r} f(\theta)$ is the outgoing scattered spherical wave corresponding to the incident plane wave e^{ikz}. The effective scattering cross-section is

$$d\sigma = 2\pi \sin\theta \, d\theta \, |f(\theta)|^2, \quad \ldots \quad (4)$$

which for 'slow' particles reduces to

$$\sigma = 4\pi |f(0)|^2. \quad \ldots \quad (5)$$

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In the Born approximation for slow collisions between two particles of mass m the scattering amplitude is given by

$$f(\theta) = \frac{m}{4\pi\hbar^2}\int u\,d\tau, \qquad (6)$$

where $\int u\,d\tau$ is the integration over volume of the two-particle interaction potential $u(r)$. For a hard sphere interaction, the scattering cross-section is readily shown to be $4\pi a^2$ (that is $f(\theta) = a$ where a is the diameter of the hard sphere); and hence we can take the potential to be equivalent to $u(r) = c\delta(r-a)$, where r is the interparticle distance and c is given by

$$a = \frac{m}{4\pi\hbar^2}\int c\delta(r-a)4\pi r^2\,dr$$

or

$$c = \frac{\hbar^2}{ma}$$

and hence

$$u(r) = \frac{\hbar^2}{ma}\delta(r-a)$$

or

$$u(r) = \frac{2}{a}\delta(r-a) \qquad (7)$$

in units of $\hbar = 1$, $2m = 1$. The ground-state energy per particle for a hard sphere gas is* substituting in Eqn. (1) $f(\theta) = a$ and $k = 0$

$$E_0 = \frac{\hbar^2}{2m}k_1^2 = \frac{\hbar^2}{2m}4\pi\rho a. \qquad (8)$$

Alternatively, for a hard-sphere interaction, we can take $u(r)$ to be a step function

$$u(r) = u_0 \text{ for } 0 < r < a$$

and

$$u(r) = 0 \qquad r \geqslant a$$

with u_0 given by

$$f(\theta) = a = \frac{m}{4\pi\hbar^2}\frac{4\pi}{3}u_0a^3$$

or

$$u_0 = \frac{3\hbar^2}{ma^2}.$$

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* For the relativistic case one may take from (1)

$$E_0 = ck = c(4\pi\rho a)^{\frac{1}{2}}$$

which for a photon gas gives

$$\frac{E_0}{\rho\hbar\omega} = \frac{c}{\hbar\omega}\left(\frac{4\pi a}{\rho}\right)^{\frac{1}{2}}$$

where a, as regards order of magnitude, is given by the Heisenberg-Euler formula

$$a = \beta\frac{r_0}{(4\pi)^{\frac{1}{2}}}\frac{\mu}{\hbar\omega} \qquad \hbar\omega \gg \mu$$

r_0 is the classical electronic radius, μ is the rest energy of an electron and β is a small numerical constant.

This averaged (smoothed out) over the volume gives

$$\bar{u} = \frac{1}{2}\,\frac{N}{V}\,\frac{4\pi}{3}\,a^3u_0 = \frac{\hbar^2}{2m}\,4\pi\rho a,$$

each particle being taken twice. This is the same as the ground-state energy (8).

If we superimpose an attractive potential on hard-sphere potential and write

$$u(r) = \frac{\hbar^2}{ma}\,\delta(r-a) - A\epsilon(r-b). \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

where $\epsilon(r-b) = 1$ for $r \leqslant b$ and zero otherwise. Here b is greater than a and A is a positive quantity. This gives for the ground-state energy per particle the expression

$$E_0 = \tfrac{1}{2}\rho \int u\,d\vec{\tau}$$

$$= \frac{\hbar^2}{2m}\,4\pi\rho a - \frac{2\pi}{3}\,A\rho b^3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

which can be negative for suitably large values of Ab^3. This is further discussed in Sec. 3, where a rather rigorous derivation is given of Eqn. (10). We note that for a two-particle bound state to occur, dimensional argument gives $A \gg \frac{\hbar^2}{mb^2}$ whereas for the occurrence of an N-body bound state, according to (10), $A \gg \frac{\hbar^2}{mb^2}\left(\frac{a}{b}\right)$. Thus an N-body bound state can occur for a two-particle potential which is not strong enough to lead to a two-particle bound state.

Huang (1959) has considered an interparticle central pseudopotential given by

$$u(r) = 8\pi a\,\delta(r)\,\frac{\partial}{\partial r}\,r - 32\pi^2(a+d)k_0^3\,\frac{j_1(k_0r)}{k_0r}. \qquad \ldots \quad \ldots \quad (11)$$

where $j_1(k_0r)$ is a spherical Bessel function.

This consists of a hard-sphere potential for $r \leqslant a$. The second terms in Eqn. (11) oscillate about zero for large values of r and is negative for small values of r. The oscillations, however, are quite small because of the smallness of k_0. It may be noticed that the Fourier transform $u(k)$ of $u(r)$ is*

$$u(k) = \frac{8\pi a}{V} \qquad k > k_0$$

$$= -\frac{8\pi d}{V} \quad k < k_0. \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

* In this paper we choose units such that $\hbar = 1$ and $2m = 1$. V is the volume of normalization.

Notice that we can take $d = 0$ in Eqn. (11) without essentially altering the nature of the potential. However, it is easily seen that the ground-state energy per particle to the first approximation is

$$E_0 = \tfrac{1}{2}\int \rho u \, d\tau = -4\pi\rho d \qquad \text{(12a)}$$

which shows that a N-body bound state is possible only for $d > 0$. In this paper we shall work with the potential (9) which is the counterpart of (11) in space and is in a sense more realistic.

2. As usual the Hamiltonian of a system of N interacting Bose particles is

$$H = H_0 + H', \qquad \text{(13)}$$

where H_0 is the kinetic energy

$$H_0 = \sum_p p^2 \xi_p^* \xi_p \qquad \text{(14)}$$

and the interaction energy H' is

$$H' = \tfrac{1}{2} \sum_{p_1' p_2' p_1 p_2} u_{p_1' p_2', p_1 p_2} \xi_{p_1'}^* \xi_{p_2'}^* \xi_{p_1} \xi_{p_2}. \qquad \text{(15)}$$

Here ξ_p^* and ξ_p are the creation and annihilation operators of a free particle with momentum $\vec{p}$ in the state described by

$$\psi_p(\vec{r}) = \frac{1}{V^{\frac{1}{2}}} e^{i\vec{p}\cdot\vec{r}}, \qquad \text{(16)}$$

V being the volume of enclosure in the form of a cube (L^3). The usual commutation relations are

$$[\xi_p, \xi_{p'}] = [\xi_p^*, \xi_{p'}^*] = 0 [\xi_p, \xi_{p'}^*] = \delta_{pp'}. \qquad \text{(17)}$$

The matrix elements $u_{p_1' p_2', p_1 p_2}$ are defined with respect to a set of plane wave functions

$$u_{p_1' p_2', p_1 p_2} = \delta_{p_1' p_2', p_1 p_2} \frac{1}{V} \int e^{i\vec{p}\cdot\vec{r}} u(\vec{r}) \, d\tau, \qquad \text{(17a)}$$

where $\vec{p} = \vec{p}_1' - \vec{p}_1 = \vec{p}_2 - \vec{p}_2'$ is the change in the momentum of a particle; $\delta_{p_1' p_2', p_1 p_2}$ expresses the conservation of total momentum. The interaction $u(\vec{r})$ is a two-body interaction. Assuming further that the inter-particle potential is central, we obtain for the Fourier transform of the potential the expression

$$u(p) = \frac{1}{V} \int e^{i\vec{p}\cdot\vec{r}} u(\vec{r}) \, d\tau$$

$$= \frac{4\pi}{Vp} \int_0^\infty r u(r) \sin pr \, dr \qquad \text{(18)}$$

and

$$u(-p) = u(p).$$

Since $n_p = \xi_p^* \xi_p$ is the occupation number of the state for momentum $\vec{p}$, we have

$$N = n_0 + \sum_{p \neq 0} n_p, \quad \ldots \quad (19)$$

where $n_0 = \zeta N$ and $n_p = 0(N)$ for all $p \neq 0$. ζ is a proper fraction. It was shown by Bogoliubov (1947) that the interaction energy term (15) can be simplified to a great extent, so that it is possible to diagonalize the Hamiltonian and obtain the energy spectrum. Using (19) and (13) we obtain, retaining only terms that are significant,

$$H = \sum_{p \neq 0} p^2 \xi_p^* \xi_p + \tfrac{1}{2}\zeta^2 N^2 u(o) + \tfrac{1}{2}\zeta N \sum_{p \neq 0} u(p)\left(\xi_p \xi_{-p} + \xi_p^* \xi_{-p}^* + 2\xi_p^* \xi_p\right). \quad (20)$$

In order to diagonalize this Hamiltonian we make use of Bogoliubov's linear transformation

$$\xi_p = \frac{\eta_p + b_p \eta_{-p}^*}{(1-b_p^2)^{\frac{1}{2}}}, \quad \ldots \quad (21)$$

where the C-numbers b_p are determined so that the non-diagonal terms vanish. We obtain

$$b_p = -\left(1 + \frac{p^2}{\zeta N u(p)}\right) + \left\{\frac{p^4}{\zeta^2 N^2 u^2(p)} + \frac{2p^2}{\zeta N u(p)}\right\} \quad \ldots \quad (22)$$

and thus we obtain for the Hamiltonian the expression

$$H = \tfrac{1}{2}u(o)\zeta^2 N^2 + \tfrac{1}{2}\zeta N \sum_{p \neq 0} u(p) b_p + \sum_{p \neq 0} \eta_p^* \eta_p E(p), \quad \ldots \quad (23)$$

where

$$E(p) = \left\{p^4 + 2\zeta N u(p)^2 p\right\}^{\frac{1}{2}}. \quad \ldots \quad (24)$$

Since the second term in (23) diverges we use perturbation theory up to second order (Landau and Lifshitz 1959) and replace $u(o)$ by

$$u(o) + \tfrac{1}{2} \sum_{p \neq 0} \frac{1}{p^2} u^2(p) \quad (2m = 1) \quad \ldots \quad (25)$$

and thus we obtain

$$H = \tfrac{1}{2}u(o)\zeta^2 N^2 + \tfrac{1}{2}\zeta N \sum_{p \neq 0} u(p)\left\{b_p + \frac{\zeta N u(p)}{2p^2}\right\} + \sum_{p \neq 0} \eta_p^* \eta_p E(p) \quad \ldots \quad (26)$$

$$= E_0 + \sum_{p \neq 0} \eta_p^* \eta_p E(p),$$

where

$$E_0 = \tfrac{1}{2}u(o)\zeta^2N^2+\tfrac{1}{2}\zeta N\sum u(p)\left[\left(\frac{p^4}{\zeta^2N^2u^2(p)}+\frac{2p^2}{\zeta Nu(p)}\right)^{\frac{1}{2}}-\left(1+\frac{p^2}{\zeta Nu(p)}\right)+\frac{\zeta Nu(p)}{2p^2}\right].$$

Replacing $\sum\limits_p$ by $\dfrac{V}{(2\pi)^3}\displaystyle\int d^3p$ we have

$$E_0 = \tfrac{1}{2}u(o)\zeta^2N^2+\frac{V}{4\pi^2}\int_0^\infty\left[p(p^2+2\zeta Nu(p))^{\frac{1}{2}}-p^2-\zeta Nu(p)+\frac{\zeta^2N^2u^2(p)}{2p^2}\right]p^2\,dp, \quad (27)$$

E_0 is the ground-state energy of the system, we shall calculate it in some simple cases. Assume $u(p)$ to be positive for all values of p. We can then write the ground-state energy E_0 as

$$E_0 = \tfrac{1}{2}u(o)\zeta^2N^2+\frac{V}{4\sqrt{2}}\frac{(\zeta N)^{\frac{5}{2}}}{\pi^2}\int_0^1\frac{u^3(p)z(1+z^2)(2-z^2)\,dz}{2zu^{\frac{1}{2}}-(1-z^2)\left(\dfrac{\zeta N}{2}\right)^{\frac{1}{2}}\dfrac{du}{dp}}, \quad \ldots \quad (28)$$

where

$$\frac{1}{z}-z = p\left[\frac{2}{\zeta Nu(p)}\right]^{\frac{1}{2}}.$$

In the particular case when $u(p)$ is constant, *i.e.* $u(p) = u(o)$ for all values of the momenta, we have

$$E_0 = \tfrac{1}{2}\zeta^2N^2u(o)\left[1+\frac{4}{15\pi^2}V(2\zeta Nu^3(o))^{\frac{1}{2}}\right]. \quad \ldots \quad \ldots \quad (29)$$

For Lee and Yang's hard-sphere pseudopotential

$$u(o) = \frac{8\pi a}{V} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

and therefore

$$E_0 = 4\pi a\rho N\zeta^2\left[1+\frac{128}{15}\left(\frac{\rho a^3\zeta}{\pi}\right)^{\frac{1}{2}}\right]. \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

For weak fields $\zeta = 1$ we have Lee and Yang's result for the hard-sphere potential. The same result was obtained by Beliaev (1958*a*, *b*) using the Green's function method.

The energy spectrum of elementary excitations is given by (24) which for small momenta and the potential (30) becomes

$$E(p) \approx \sqrt{\frac{16\pi aN}{V}}\,p. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31a)$$

Using (31) we find that the velocity of sound equals

$$u = \sqrt{\frac{2V^2}{N}\frac{\partial^2E}{\partial V^2}} \quad (2m = 1)$$

$$= \sqrt{\frac{16\pi aN}{V}}.$$

As must be the case, this expression coincides with the coefficient of p in (31a) for the phonon part of the spectrum. If we take the hard-sphere potential in the form

$$u(r) = \frac{2}{a}\,\delta(r-a) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

then

$$u(p) = \frac{8\pi a}{V}\,\frac{\sin pa}{pa}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

It may be noticed that this value of $u(p)$ reduces to the value $\frac{8\pi a}{V}$ only for $pa \ll 1$. For the case of hard-sphere pseudopotential, $u(p)$ has the same value $\frac{8\pi a}{V}$ for all values of p, but for the delta-type hard-sphere potential, the Fourier transform $u(p)$ has the value $\frac{8\pi a}{V}$ only for $p = 0$, and for large values of the momenta it oscillates between positive and negative values, its amplitude diminishing to zero as $1/p$. The integral (28) cannot be evaluated in a finite form for this potential. However, we can consider other forms of the potential for which the ground-state energy can be evaluated in a finite form.

Consider now the potential $u(p)$ which satisfies the equation

$$\frac{u(o)}{u(p)} - \frac{u(p)}{u(o)} = p\left(\frac{2}{\zeta N u(p)}\right)^{\frac{1}{2}} \qquad \ldots \quad \ldots \quad \ldots \quad (34)$$

and which takes the positve value $u(o)$ for $p = 0$. It decreases to zero uniformly as $p \to \infty$. Also $\frac{du}{dp}$ is negative and $\frac{d^2u}{dp^2}$ is positive for all values of p. The integral in (28) gives $\frac{104}{385}[u(o)]^{\frac{5}{2}}$, so that

$$E_0 = \tfrac{1}{2}\zeta^2 N^2 u(o)\left[1 + \frac{26}{385\pi^2}\,V(2\zeta N u^3(o))^{\frac{1}{2}}\right]. \qquad \ldots \quad \ldots \quad (35)$$

The factor $\frac{4}{15}$ in the second term in (29) is replaced by a smaller number $\frac{26}{385}$. We may also approximate to the repulsive potential by the form

$$u(p) = u(o)\left[1 - \left(\frac{2}{\zeta N u(o)}\right)^{\frac{1}{2}} p\right]. \qquad \ldots \quad \ldots \quad \ldots \quad (36)$$

It can be shown that in this case

$$u(p) = u(o) z^2. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

The potential decreases from the value $u(o)$ at $p = 0$ $(z = 1)$ to zero at $p = \left(\frac{N\zeta u(o)}{2}\right)^{\frac{1}{2}}$ corresponding to $z = 0$.

The integral in (28) on evaluation in this case gives $\frac{6}{40}[u(o)]^{\frac{5}{2}}$, so that the ground-state energy is given by

$$E_0 = \tfrac{1}{2}\zeta^2 N^2 u(o)\left\{1+\frac{3}{80\pi^2}V(2\zeta N u^3(o))^{\frac{1}{2}}\right\} \quad \ldots \quad (38)$$

which differs from the value given by (29), in that the factor $\frac{4}{15}$ is replaced by the still smaller factor $\frac{3}{80}$. The first term in all these cases is the same because its value depends only upon the value of the potential $u(p)$ at $\vec{p}=0$, which is positive for repulsive potentials.

In all the three cases considered above $u(p)$ is positive for all values of the momentum $\vec{p}$. We can extend the expression (28) for the ground-state energy to the case when $u(p)$ may also become negative for some values of $\vec{p}$, provided $E^2(p) = p^2(p^2+2\zeta N u(p))$ does not become negative, that is the energy does not become imaginary for any value of the momentum $\vec{p}$. We divide the whole range $(0, \infty)$ of $\vec{p}$ into intervals (p_r, p_{r+1}), in each of which $u(p)$ keeps the same sign, positive or negative. The ground-state energy of the assembly is, in this case

$$E_0 = \tfrac{1}{2}\zeta^2 N^2 u(o) + \sum_{r=1}^{\infty} I_r. \quad \ldots \quad (39)$$

Here

$$I_r = \frac{V}{4\sqrt{2}}\,\frac{(\zeta N)^{\frac{5}{2}}}{\pi^2}\int_{p_r}^{p_{r+1}} \frac{u^3(p)z(1+z^2)(2-z^2)\,dz}{2zu^{\frac{1}{2}}-(1-z^2)\left(\frac{\zeta N}{2}\right)^{\frac{1}{2}}\frac{du}{dp}}, \quad \ldots \quad (40)$$

where

$$p = \left(\frac{\zeta N u(p)}{2}\right)^{\frac{1}{2}}\frac{1-z^2}{z} \quad \ldots \quad (41)$$

if $u(p)$ is positive throughout the range (p_r, p_{r+1}); and

$$I_r = \frac{V}{4\sqrt{2}}\,\frac{(\zeta N)^{\frac{5}{2}}}{\pi^2}\int_{p_r}^{p_{r+1}} \frac{u^3(p)z(1-z^2)(2+z^2)\,dz}{2z|u|^{\frac{1}{2}}+(1+z^2)\left(\frac{\zeta N}{2}\right)^{\frac{1}{2}}\frac{du}{dp}}, \quad \ldots \quad (42)$$

where

$$p = \left[\frac{\zeta N|u(p)|}{2}\right]^{\frac{1}{2}}\frac{1+z^2}{z} \quad \ldots \quad (43)$$

if $u(p)$ is negative throughout the range (p_r, p_{r+1}). Let $|u(p)|$ denote the absolute value of $u(p)$. The Fourier transform $u(p)$ of the hard-sphere potential given by (7) is

$$u(p) = \frac{8\pi a}{V}\,\frac{\sin pa}{pa} \quad \ldots \quad (44)$$

which takes positive and negative values periodically with period π/a. The ground-state energy E_0 could be evaluated with the help of the result (39), by using the integrals (40) and (42).

3. We shall now extend the above formulae to the case when $E(p)$ becomes imaginary for some values of the momentum p. The Bogoliubov linear transformation loses its meaning, because b_p then becomes a complex quantity, and $b_p\bar{b}_p = 1$. The transformation (21) must be replaced by the transformation

$$\xi_p = \frac{\eta_p + b_p\eta^*_{-p}}{(1-b_p\bar{b}_p)^{\frac{1}{2}}}$$

$$\xi^*_p = \frac{\eta^*_p + \bar{b}_p\eta_{-p}}{(1-b_p\bar{b}_p)^{\frac{1}{2}}} \qquad \ldots \quad (45)$$

and this gives, for such values of p, $\eta_p = 0$, $\eta^*_p = 0$ and therefore $\eta^*_p\eta_p = 0$. The occupation number of these quasi-particles vanishes. This is due to the approximation made in writing the Hamiltonian H in the form (20) where we have retained only those terms in which at least two of the momenta have the value zero. We should retain the higher order terms also. This will effect the value of $E(p)$ not only for those values of p for which $(p^2+2\zeta Nu(p)<0)$ but also for the higher values of the momenta. We accordingly divide all momenta into two classes, and denote by λ those momenta for which this effect is dominant and by k those momenta for which this effect is negligible. There is no sharp line of demarcation of these momenta, but let us assume that momentum k_0 separates the two ranges of momenta. Following Huang (1959), we write the Hamiltonian in the form (retaining only those terms which are significant)

$$H = \tfrac{1}{2}\zeta^2N^2u(o) + \sum_k \left[(k^2+\zeta Nu(k))\xi^*_k\xi_k + \tfrac{1}{2}\zeta Nu(k)(\xi_k\xi_{\underline{k}} + \xi^*_k\xi^*_{\underline{k}})\right]$$

$$+\sum_\lambda \left[\left\{\lambda^2+\zeta Nu(\lambda) - \tfrac{1}{2}\sum_k u(k)\left(\xi^*_k\xi^*_{\underline{k}} + \xi_{\underline{k}}\xi_k\right)\right\}\xi^*_\lambda\xi_\lambda + \tfrac{1}{2}\zeta Nu(\lambda)\left(\xi^*_\lambda\xi^*_{\underline{\lambda}} + \xi_\lambda\xi_{\underline{\lambda}}\right)\right]$$

$$+(\zeta N)^{\frac{1}{2}}\sum_k u(k)\sum_{p\lambda}\left\{[\delta(p+k-\lambda)\xi^*_p\xi^*_k\xi_\lambda + \delta(p-k-\lambda)\xi^*_p\xi_k\xi_\lambda] + \text{H.C.}\right\}$$

$$+\tfrac{1}{2}\sum_k u(k)\sum_\lambda(\xi^*_k\xi^*_{\underline{k}}\xi_\lambda\xi_{\underline{\lambda}} + \xi_k\xi_{\underline{k}}\xi^*_\lambda\xi^*_{\underline{\lambda}}) \cdot \qquad \ldots \quad (46)$$

We first apply Bogoliubov's transformation to the k-particles. Take the ground-state to be the state in which the occupation numbers of the quasi-k-particles are zero. We then obtain for this state the Hamiltonian

$$<0|H|0>$$

$$= \tfrac{1}{2}\zeta^2 N^2 u(o) + \sum_k \left[(k^2+\zeta N u(k)) \frac{b_k^2}{1-b_k^2} + \zeta N u(k) \frac{b_k}{1-b_k^2} \right]$$

$$+ \sum_\lambda \left[\left\{ \lambda^2 + \zeta N u(\lambda) - \sum_k u(k) \frac{b_k}{1-b_k^2} \right\} \xi_\lambda^* \xi_\lambda + \tfrac{1}{2}\zeta N u(\lambda) \left(\xi_\lambda^* \xi_{-\lambda}^* + \xi_\lambda \xi_{-\lambda} \right) \right]$$

$$+ \tfrac{1}{2} \sum_\lambda \sum_k \frac{b_k}{1-b_k^2} u(k) \left(\xi_\lambda^* \xi_{-\lambda}^* + \xi_\lambda \xi_{-\lambda} \right) + \zeta N \sum_\lambda \left[\left\{ -\sum_k \frac{(1+b_k^2)}{(1-b_k)^2} \frac{u^2(k)}{E(k)} \right\} \xi_\lambda^* \xi_\lambda \right.$$

$$\left. - \left\{ \sum_k \frac{b_k}{(1-b_k)^2} \frac{u^2(k)}{E(k)} \left(\xi_\lambda^* \xi_{-\lambda}^* + \xi_\lambda \xi_{-\lambda} \right) \right\} \right], \quad \ldots \quad (47)$$

where $E(k) = k(k^2+2\zeta N u(k))^{\frac{1}{2}}$.

After simplifying the above expression, we obtain

$$<0|H|0>$$

$$= E_0 + \sum_\lambda \left[(\lambda^2 + \zeta N u(\lambda) + X) \xi_\lambda^* \xi_\lambda + \tfrac{1}{2}(\zeta N u(\lambda) + X)(\xi_\lambda^* \xi_{-\lambda}^* + \xi_\lambda \xi_{-\lambda}) \right], \quad \ldots \quad (48)$$

where

$$X = -\tfrac{1}{2}\zeta N \sum_k \frac{u^2(k)}{E(k)} \left(\frac{1+b_k}{1-b_k} \right)^2$$

$$= -\tfrac{1}{2}\zeta N \sum_k \frac{u^2(k)}{E^3(k)} k^4 \quad \ldots \quad (49)$$

and

$$E_0 = \tfrac{1}{2}\zeta^2 N^2 u(o) + \sum_k \left[(k^2+\zeta N u(k)) \frac{b_k^2}{1-b_k^2} + \zeta N u(k) \frac{b_k}{1-b_k^2} \right]. \quad \ldots \quad (50)$$

b_k is given by the expression

$$b_k = -\left(1 + \frac{k^2}{\zeta N u(k)} \right) + \left\{ \frac{k^4}{\zeta^2 N^2 u^2(k)} + \frac{2k^2}{\zeta N u(k)} \right\}^{\frac{1}{2}}. \quad \ldots \quad (51)$$

It may be noticed that

$$b_k + \frac{1}{b_k} = -2\left(1 + \frac{k^2}{\zeta N u(k)} \right)$$

$$b_k - \frac{1}{b_k} = \frac{2E(k)}{\zeta N u(k)}. \quad \ldots \quad (52)$$

X is a divergent series, and we may, by using the usual subtraction procedure, replace it by X' where

$$X' = \tfrac{1}{2}\zeta N \sum_k u^2(k) \left[\frac{1}{k^2} - \frac{k^4}{E^3(k)} \right]. \quad \ldots \quad (53)$$

We assume that the potential $u(p)$ is such that

$$u'(\lambda) = u(\lambda)+\frac{X'}{\zeta N} \qquad \ldots \quad (54)$$

is positive for all values of λ. This imposes some restrictions on the parameters, defining the potential. We now apply Bogoliubov's transformation to the λ-particle operators and we obtain for the ground-state energy of the system

$$E_0' = \tfrac{1}{2}\zeta^2N^2u(o)+\frac{V}{4\pi^2}\int_0^{k_0}\left\{\lambda(\lambda^2+2\zeta Nu'(\lambda))^{\frac{1}{2}}-\lambda^2-\zeta Nu'(\lambda)+\frac{\zeta^2N^2u'^2(\lambda)}{2\lambda^2}\right\}\lambda^2\,d\lambda$$

$$+\frac{V}{4\pi^2}\int_{k_0}^{\infty}\{k^3(k^2+2\zeta Nu(k))^{\frac{1}{2}}-k^4-\zeta Nu(k)k^2+\tfrac{1}{2}\zeta^2N^2u^2(k)\}\,dk. \qquad \ldots \quad (55)$$

Thus the Hamiltonian can be written as

$$H = E_0'+\sum_{\lambda}\lambda\{\lambda^2+2\zeta Nu'(\lambda)\}^{\frac{1}{2}}n_\lambda+\sum_{k}k\{k^2+2\zeta Nu(k)\}^{\frac{1}{2}}n_k. \qquad \ldots \quad (56)$$

ζ is a function of the temperature T and as $T\to 0$, $\zeta\to 1$. Therefore, the ground-state energy per unit volume is given by

$$U = \frac{E_0'}{V} = \tfrac{1}{2}(u(o)V)\rho^2+f(\rho). \qquad \ldots \quad (57)$$

where $f(\rho)$ denotes a function of the particle density ρ. Moreover $u(o)V$ is independent of the volume of enclosure V of the assembly. For potentials (9) and (12), $u(o)V$ is negative. Thus for sufficiently low densities, U is positive, and for sufficiently high densities U is negative, because the function $f(\rho)$ is proportional to $\rho^{\frac{5}{2}}$ as is evident from the expression (55). This indicates the possibility of the formation of a N-body system even for potentials which are weak enough to form a two-body bound state. Consider, in particular, the potential given by the expression (9)

$$u(r) = \frac{2}{a}\delta(r-a)-A\epsilon(r-b).$$

The Fourier transform $u(p)$ of $u(r)$ is

$$u(p) = \frac{8\pi a}{V}\frac{\sin pa}{pa}-\frac{4\pi Ab^3}{V}\frac{\sin pb-pb\cos pb}{(pb)^3}$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{\frac{8\pi a}{V}\frac{J_{\frac{1}{2}}(pa)}{(pa)^{\frac{1}{2}}}-\frac{4\pi Ab^3}{V}\frac{J_{\frac{3}{2}}(pb)}{(pb)^{\frac{3}{2}}}\right\}. \qquad \ldots \quad (58)$$

The value of $u(p)$ at $p = 0$ is

$$u(o) = \frac{8\pi a}{V}-\frac{4\pi}{3}\frac{Ab^3}{V}. \qquad \ldots \quad (59)$$

The potential $u(p)$ is negative (we assume that $Ab^3 \gg 6a$) for small values of p and it fluctuates between positive and negative values. When we apply the above transformation we obtain

$$u'(p) = u(p) + \frac{X'}{\zeta N}, \quad \ldots \quad (60)$$

where

$$X' = \tfrac{1}{2}\zeta N \sum_{p > p_0} u^2(p) \left[\frac{1}{p^2} - \frac{p^4}{E^3(p)}\right]$$

$$= \frac{\zeta N V}{4\pi^2} \int_{p_0}^{\infty} u^2(p) \left[\frac{1}{p^2} - \frac{p}{(p^2+2\zeta N u(p))^{\frac{3}{2}}}\right] p^2\, dp. \quad \ldots \quad (61)$$

Here p_0 is such that for $p > p_0$, $p^2+2\zeta N u(p)$ is not only positive but much greater than X'. This fixes the order of magnitude of p_0. The energy spectrum of the quasi-particles is then given by

$$E(p) = \begin{cases} p(p^2+2\zeta N u'(p))^{\frac{1}{2}} & p < p_0 \\ p(p^2+2\zeta N u(p))^{\frac{1}{2}} & p > p_0 \end{cases} \quad \ldots \quad (62)$$

we assume the parameters to be such that $u'(p)$ is positive for $p < p_0$. The ground-state energy is given by

$$\frac{E_0'}{N} = 4\pi a\rho\zeta^2 - \frac{2\pi}{3} Ab^3\rho\zeta^2$$

$$+\frac{1}{4\pi^2\rho}\int_0^{p_0} \{\lambda^3(\lambda^2+2\rho\zeta V u'(\lambda))^{\frac{1}{2}} - \lambda^4 - \rho\zeta V u'(\lambda)\lambda^2 + \tfrac{1}{2}\rho^2\zeta^2 V^2 u'^2(\lambda)\}\, d\lambda$$

$$+\frac{1}{4\pi^2\rho}\int_{p_0}^{\infty} \{p^3(p^2+2\rho\zeta V u(p))^{\frac{1}{2}} - p^4 - \rho\zeta V u(p)\, p^2 + \tfrac{1}{2}\rho^2\zeta^2 V^2 u^2(p)\}\, dp. \quad (63)$$

As a first approximation we can replace $Vu(\lambda)$ by $\left(8\pi a - \frac{4\pi}{3} Ab^3\right)$ in the first integral and $Vu(p)$ by its greatest value (corresponding to the first maximum) and associate p_0 to the value where this value is attained. Denoting the minimum value of $Vu(p)$ by $-8\pi\beta$ and maximum value by $8\pi\alpha$ (*see* Appendix) we obtain

$$\frac{E_0'}{N} = 4\pi\rho\zeta^2 \left[-\beta + \frac{4\alpha}{15\sqrt{\pi}} (\rho\zeta\alpha^3)^{\frac{1}{2}} (30z+5z^3-3z^5)\right.$$

$$\left. + \frac{4\alpha'}{15\sqrt{\pi}} (\rho\zeta\alpha'^3)^{\frac{1}{2}} (32-30z'-5z'^3+3z'^5)\right], \quad \ldots \quad (64)$$

where

$$z = (1+\xi^2)^{\frac{1}{2}}-\xi$$

$$z' = (1+\xi'^2)^{\frac{1}{2}}-\xi'$$

$$\xi = \frac{p_0}{(16\pi\alpha\rho\zeta)^{\frac{1}{2}}}$$

$$\xi' = \frac{p_0}{(16\pi\alpha'\rho\zeta)^{\frac{1}{2}}}$$

$$\alpha' = -\beta+8\alpha\left(\frac{\rho\zeta\alpha^3}{\pi}\right)^{\frac{1}{2}}\left[\frac{1}{(1+\xi^2)^{\frac{1}{2}}}+\frac{1}{\xi+(1+\xi^2)^{\frac{1}{2}}}\right]$$

and

$$\beta = -a+\tfrac{1}{6}Ab^3.$$

$\zeta = 1$ at the absolute zero, and decreases at higher temperatures. The effect of attraction is introduced through A.

Appendix A

We shall give a brief derivation of (64). The minimum value of $Vu(p)$ is $8\pi a-\frac{4\pi}{3}Ab^3$

$$\therefore \quad \beta = -a+\tfrac{1}{6}Ab^3. \qquad \text{(A1)}$$

In the second integral in (63) we substitute

$$p = \frac{1-x^2}{x}\left(\frac{\zeta Nu(p)}{2}\right)^{\frac{1}{2}}. \qquad \text{(A2)}$$

Now $Vu(p) = 8\pi\alpha$ for $p = p_0$, so that

$$x^2+2x\xi-1 = 0 \qquad \text{(A3)}$$

where

$$\xi = \frac{p_0}{(16\pi\alpha\rho\zeta)^{\frac{1}{2}}}.$$

$$\therefore \quad x = -\xi+(1+\xi^2)^{\frac{1}{2}}$$

$$= z \text{ (say)}. \qquad \text{(A4)}$$

The integral gives, apart from constant terms,

$$\tfrac{1}{15}[30z+5z^3-3z^5].$$

Similarly, in the first integral we put

$$\lambda = \left(\frac{1-x^2}{x}\right)\left(\frac{\zeta Nu'(\lambda)}{2}\right)^{\frac{1}{2}}, \qquad \text{(A5)}$$

where

$$u'(\lambda) = u(\lambda)+\frac{X'}{\zeta N}.$$

Using (61) and putting $u'(\lambda) = \frac{8\pi\alpha'}{V}$ we have

$$\alpha' = -\beta + \frac{V}{8\pi}\frac{V}{4\pi^2}\int_{p_0}^{\infty} u^2(p)\left[\frac{1}{p^2} - \frac{p}{(p^2+2\zeta Nu(p))^{\frac{3}{2}}}\right]p^2\,dp. \quad .. \quad (A6)$$

Upon evaluation the first integral in (63) gives (apart from constant terms)

$$\tfrac{1}{15}[32-30z'-5z'^3+3z'^5]$$

with

$$z' = (1+\xi'^2)^{\frac{1}{2}}-\xi'$$

$$\xi' = \frac{p_0}{(16\pi\alpha'\rho\zeta)^{\frac{1}{2}}}.$$

Finally to calculate α' we substitute (A2) in (A6).
The integral reduces to

$$\int_0^z \frac{x^4+3}{x^4+2x^2+1}\,dx$$

which can be easily evaluated to give $\left[z+\frac{2z}{1+z^2}\right]$. Using (A4) for z and (A6) we finally get

$$\alpha' = -\beta+8\alpha\left(\frac{\rho\zeta\alpha^3}{\pi}\right)^{\frac{1}{2}}\left[\frac{1}{(1+\xi^2)^{\frac{1}{2}}}+\frac{1}{\xi+(1+\xi^2)^{\frac{1}{2}}}\right].$$

This completes the derivation of (64).

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