STEADY MOTION OF A VISCOUS FLUID DUE TO SLOWLY ROTATING SPHERICAL CAPS

by Prem Narain, *Department of Mathematics, Ramjas College, Delhi 6*

(Communicated by R. S. Varma, F.N.I.)

(*Received February 19, 1965*)

The present work deals with the steady motion of a viscous fluid due to slowly rotating spherical caps. The problem is formulated in terms of a pair of dual series relations. Use has been made of a technique developed recently by Srivastav (1964) for solving the dual series relations. Closed form expressions for the skin friction on the surface of the spherical cap have been obtained.

Introduction

Applications of dual series relations involving Legendre Polynomials and Associated Legendre Polynomials have been investigated widely. Recently, Collins (1963) showed that the stream function for the axisymmetric stokes flow of a viscous fluid past a spherical cap is governed by dual series relations involving Associated Legendre Polynomials. In this note the steady, slow motion of a viscous fluid due to spherical caps occupying the regions ($r = 1, 0 \leqslant \theta < \alpha$) and ($r = 1, \pi - \alpha < \theta \leqslant \pi$) and rotating slowly in opposite directions with equal angular velocities is discussed. Dual series relations have been obtained which determine the velocity field and the skin friction. For the solution of these series equations a technique due to Srivastav (1964) has been used.

Basic Equations of Motion.—For the slow and steady motion of a viscous fluid, for which $u_r = 0$, $u_\theta = 0$ and $u_\phi = u_\phi(r, \theta)$ the equations of motion are given by

$$\frac{\partial p}{\partial \theta} + \frac{2\mu}{r^2}\frac{\partial u_\phi}{\partial \theta} + \frac{2\mu}{r^2}\cot\theta u_\phi = 0 \quad .. \quad .. \quad .. \quad (1.1)$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 u_\phi - \frac{1}{r^2 \sin\theta} u_\phi\right] = 0 \quad .. \quad .. \quad .. \quad (1.2)$$

$$\frac{\partial p}{\partial \theta} = 0 \quad .. \quad .. \quad .. \quad (1.3)$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta}$$

and where p and μ denote respectively the pressure distribution function and the coefficient of viscosity.

Let

$$u_\phi = r \sin \theta \, \Omega(r, \theta) \quad \ldots \quad (1.4)$$

where Ω is the angular velocity of the fluid.

By eliminating p and u_ϕ from (1.1), (1.2), (1.4), it is easily seen that

$$D^2\Omega = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right)\Omega = 0.$$

Formulation of the Problem.—The problem considered in this note is of finding two functions, $\Omega_1(r, \theta)$ with $r > 1$ and $\Omega_2(r, \theta)$ with $r < 1$, satisfying the following boundary conditions :

(I) $D^2\Omega_1 = D^2\Omega_2 = 0$.

(II) $\Omega_1 \to 0$ as $r \to \infty$, $\Omega_2 \to 0$ as $r \to \infty$,

$\Omega_1(1, \theta) = \Omega_2(1, \theta)$, $0 < \theta < \pi$.

(III) $\Omega_1(1, \theta) = \Omega_2(1, \theta) = f(\theta)$, $0 < \theta < \alpha$

$= -f(\pi - \theta)$, $\pi - \alpha < \theta < \pi$.

Since there is no boundary in the region ($r = 1$, $\alpha < \theta < \pi - \alpha$) from physical conditions it follows that

$$\frac{\partial}{\partial r}\Omega_1(r, \theta) = \frac{\partial}{\partial r}\Omega_2(r, \theta).$$

The above boundary value problem can be easily reduced to the following mixed boundary value problem :

(IV) $\Omega_1(1, \theta) = \Omega_2(1, \theta)$, $0 < \theta < \frac{\pi}{2}$.

(V) $\Omega_1(1, \theta) = \Omega_2(1, \theta)$, $0 < \theta < \alpha$.

(VI) $\Omega_1\left(r, \frac{\pi}{2}\right) = \Omega_2\left(r, \frac{\pi}{2}\right)$, $\alpha < \theta < \frac{\pi}{2}$.

(VII) $\frac{\partial}{\partial r}\Omega_1(1, \theta) = \frac{\partial}{\partial r}\Omega_2(1, \theta)$, $\alpha < \theta < \frac{\pi}{2}$.

Formulation of Dual Series Relations.—The expressions for $\Omega_1 \to 0$ as $r \to \infty$ and for $\Omega_2 \to 0$ as $r \to 0$ and satisfying the boundary conditions (I), (IV) and (VI) are given by

$$\Omega_1 = \sum_{n=0}^{n=\infty} a_{2n+1} r^{-2n-1} P_{2n+1}(\cos\theta), \; r > 1 \quad \ldots \quad (3.1)$$

$$\Omega_2 = \sum_{n=0}^{n=\infty} a_{2n+1} r^{2n+2} P_{2n+1}(\cos\theta), \; r < 1 \quad \ldots \quad (3.2)$$

where the coefficients a_{2n+1}, $n = 0, 1, 2, \ldots \infty$ are unknown.
The boundary conditions (V) and (VII) are satisfied if

$$\sum_{n=0}^{\infty} a_{2n+1} P_{2n+1}(\cos\theta) = f(\theta),\ 0 < \theta < \alpha. \qquad \ldots \quad \ldots \quad (3.3)$$

$$\sum_{n=0}^{\infty} (4n+3) a_{2n+1} P_{2n+1}(\cos\theta) = 0,\ \alpha < \theta < \frac{\pi}{2}. \qquad \ldots \quad \ldots \quad (3.4)$$

Following Srivastav (1964), it is suggested that for $0 < \theta < \alpha$

$$\sum_{n=0}^{n=\infty} (4n+3) a_{2n+1} P_{2n+1}(\cos\theta) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\int_0^\alpha \frac{g(t)\,dt}{\sqrt{(\cos\theta - \cos t)}} \qquad \ldots \quad (3.5)$$

where $g(t)$ is unknown.
Using Legendre Polynomials, it is easily found from (3.4) and (3.5) that

$$a_{2n+1} = \int_0^\alpha P_{2n+1}(\cos\theta)\frac{\partial}{\partial\theta}\left(\int_\theta^\alpha \frac{g(t)\,dt}{\sqrt{(\cos\theta - \cos t)}}\right) d\theta. \qquad \ldots \quad (3.6)$$

Integrating by parts the integral on the right-hand side of (3.6) and interchanging the order of integration we obtain

$$a_{2n+1} = \int_0^\alpha \frac{g(t)\,dt}{\sqrt{(1-\cos t)}} - \int_0^\alpha g(t)\left(\int_0^t \frac{\frac{\partial}{\partial\theta}P_{2n+1}(\cos\theta)}{\sqrt{\cos\theta - \cos t}}\,d\theta\right)dt. \qquad \ldots \quad (3.7)$$

From Mehler's integral, it is easily found that

$$\int_0^t \frac{\frac{\partial}{\partial\theta}P_{2n+1}(\cos\theta)}{\sqrt{(\cos\theta - \cos t)}}\,d\theta = \frac{\sqrt{2}}{\sin t}\left[\cos\left(\frac{4n+3}{2}\right)t - \cos\frac{t}{2}\right]. \qquad \ldots \quad (3.8)$$

Combing (3.7) and (3.8), we have

$$a_{2n+1} = -\sqrt{2}\int_0^\alpha g(t)\ \mathrm{cosec}\, t\ \cos\left(\frac{4n+3}{2}\right)t. \qquad \ldots \quad \ldots \quad (3.9)$$

Substituting for a_{2n+1} from (3.9) in (3.3), we get after interchanging the order of integration and summation,

$$\int_0^\alpha g(t)\ \mathrm{cosec}\, t\left(\sqrt{2}\sum_{n=0}^{\infty}\cos\left(\frac{4n+3}{2}\right)t\ P_{2n+1}(\cos\theta)\,dt = f(\theta),\ 0 < \theta < \alpha.\right.$$

Since,

$$\sqrt{2}\sum_{n=0}^{\infty}\cos\left(\frac{4n+3}{2}\right)t\ P_{2n+1}(\cos\theta) = \frac{H(\theta - t)}{\sqrt{\cos t - \cos\theta}}$$

where $H(x)$ is the Heaviside unit function.
Therefore

$$\int_0^\theta \frac{g(t)\ \mathrm{cosec}\, t}{\sqrt{\cos t - \cos\theta}}\,dt = f(\theta). \qquad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

Equation (3.10) is an integral equation of Abel's type and its solution is given by

$$g(t) = -\frac{\sin t}{\pi}\frac{d}{dt}\int_0^t \frac{\sin\theta f(\theta)\,d\theta}{\sqrt{(\cos\theta-\cos t)}}.$$

A Particular Case. Let us suppose that $f(\theta)=1$. Then from (3.10) it follows that $g(t)=\sqrt{2}\sin t\cos\frac{t}{2}\Big/\pi$. Substituting for $g(t)$ in (3.9), we have

$$a_{2n+1} = \frac{1}{\pi}\left[\frac{\sin(2n+2)\alpha}{2n+2}+\frac{\sin(2n+1)\alpha}{(2n+1)}\right].$$

Hence the expressions for the angular velocities are easily found to be

$$\Omega_1 = \frac{1}{\pi}\sum_{n=0}^{n=\infty}\left[\frac{\sin(2n+2)\alpha}{(2n+2)}+\frac{\sin(2n+1)\alpha}{(2n+1)}\right]r^{-(2n+1)}P_{2n+1}(\cos\theta)$$

$$= \frac{1}{\pi}I_m\int_0^{r^{-1}e^{i\alpha}}\frac{(rz+1)dz}{z\sqrt{(1-2z\cos\theta+z^2)}},\ \text{for } r>1.$$

$$\Omega_2 = \frac{1}{\pi}\sum_{n=0}^{\infty}\left[\frac{\sin(2n+2)\alpha}{2n+2}+\frac{\sin(2n+1)\alpha}{2n+1}\right]r^{(2n+2)}P_{2n+1}(\cos\theta)$$

$$= \frac{1}{\pi}I_m\int_0^{re^{i\alpha}}\frac{(r+z)\,dz}{z\sqrt{(1-2z\cos\theta+z^2)}},\ \text{for } r<1.$$

A physical quantity of interest is the skin friction on the inner and the outer surfaces of the caps. The expressions for the skin friction on the outer and the inner surfaces of the cap are respectively

$$\tau_{r\theta} = \mp\frac{\mu\sin\theta}{\pi}I_m\int_0^{e^{i\alpha}}\frac{dz}{z\sqrt{(1-2z\cos\theta+z^2)}},\ 0<\theta<\alpha.$$

Acknowledgements

I take this opportunity to thank Professors J. N. Kapoor and Dr. R. P. Srivastav for many helpful discussions in connection with the problem considered in this paper. My thanks are also due to Prof. R. S. Varma for communicating the paper.

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