ON THE ORDER OF THE ERROR FUNCTION OF THE SQUARE-FREE NUMBERS*

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Let $G_k(x)$ be the error function arising in the enumeration of the sum of the kth powers of the square-free integers not exceeding x. Orders of $G_k(x)$ are considered under the assumption of Riemann hypothesis. While elementary methods suffice for $G(x) = G_0(x)$, analytical methods are used for higher values of k and for finding the 'average' order of $G(x)$. It is believed that the result regarding the order of $G(x)$ can be improved.

1. For a positive real number x, let $Q(x)$ denote the number of square-free integers not exceeding x. It can be easily shown (Landau 1953, p. 604) that

$$Q(x) = \sum_{n \leqslant x} \mu^2(n) = \sum_{n \leqslant \sqrt{x}} \mu(n)[x/n^2] \sim x/\zeta(2), \quad \ldots \quad \ldots \quad (1.1)$$

and that

$$G(x) = Q(x) - x/\zeta(2) = O(\sqrt{x}), \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

where $G(x)$ is the error function.

Here μ is the Mobius function and $\zeta(s)$ is the Riemann zeta function of the complex variable $s = \sigma + it$. Also $[y]$ denotes the greatest integer not exceeding y. Further improvements in (1.2) were made by Landau (Landau 1953, p. 606) and Axer (1911). In this paper our object is to investigate the order of $G(x)$ and related error functions on the assumption of the Riemann hypothesis. Throughout the paper it will, therefore, be assumed that the Riemann hypothesis is true. (This assumption is not required for theorems 6, 7, 8 and 11.)

In addition to $G(x)$, we shall also consider the error functions $G_k(x)$ defined for non-negative integral k by

$$G_k(x) = \sum_{n \leqslant x} \mu^2(n) n^k - \frac{1}{(k+1)\zeta(2)} x^{k+1}, \quad \ldots \quad \ldots \quad (1.3)$$

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and the functions $M(x)$ and $N(x)$ defined by

$$M(x) = \sum_{n \leqslant x} \mu(n), \quad N(x) = \sum_{n > x} \mu(n) n^{-2}. \qquad \ldots \quad \ldots \quad (1.4)$$

ϵ will denote any pre-assigned positive number.

The results proved in this paper are:

$$G(x) = O(x^{2/5+\epsilon}). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

For all $k \geqslant 1$,

$$(G - x)^k = O(x^{k+\frac{1}{2}+\epsilon}), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

where on the left side we replace G^r by $G_r(x)$ after the binomial expansion

$$\frac{1}{x}\int_1^x G(t)dt = O(x^{\frac{1}{4}+\epsilon}). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.7)$$

It seems reasonable to conjecture that

$$G(x) = O(x^{\frac{1}{4}+\epsilon}). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.8)$$

2. THEOREM 1 (Titchmarsh 1951).

$$M(x) = O(x^{\frac{1}{2}+\epsilon}).$$

THEOREM 2.

$$N(x) = O(x^{-3/2+\epsilon}).$$

PROOF:

$$N(x) = \sum_{n > x} \frac{\mu(n)}{n^2}$$

$$= \frac{\mu([x+1])}{([x+1])^2} + \frac{\mu([x+2])}{([x+2])^2} + \ldots$$

$$= \frac{M(x+1) - M(x)}{([x+1])^2} + \frac{M(x+2) - M(x+1)}{([x+2])^2} + \ldots$$

$$= \sum_{n=x} M(n)(n^{-2} - (n+1)^{-2}) - M(x)([x+1])^{-2}$$

$$= O(x^{-3/2+\epsilon}), \text{ by Theorem 1.}$$

THEOREM 3.

$$\sum_{n < \sqrt{x}} \mu(n)\{x/n^2\} = O(x^{2/5+\epsilon}),$$

where

$$\{x/n^2\} = x/n^2 - [x/n^2].$$

PROOF:

$$\sum_{n \leqslant \sqrt{x}} \mu(n)\,\{x/n^2\} = \Bigg(\sum_{n \leqslant x^{2/5}} + \sum_{x^{2/5} < n \leqslant \sqrt{x}}\Bigg)\,\mu(n)\,\{x/n^2\}$$

$$= O(x^{2/5}) + \sum_{t < x^{1/5}} \; \sum_{\sqrt{\frac{x}{t+1}} \leqslant n < \sqrt{\frac{x}{t}}} \mu(n) n^{-2}(x - tn^2)$$

$$= O(x^{2/5}) + x\big(N(x^{2/5}) - N(\sqrt{x})\big) - \sum_{t < x^{1/5}} M\big(\sqrt{x/t}\big) - (x^{1/5}-1)M(x^{2/5})$$

$$= O(x^{2/5}) + O(x^{2/5+\epsilon}) + O(x^{2/5+\epsilon})$$

$$= O(x^{2/5+\epsilon}).$$

THEOREM 4.

$$G(x) = -\sum_{n \leqslant \sqrt{x}} \mu(n)\{x/n^2\} + O(x^{\frac{1}{4}+\epsilon}).$$

PROOF: From (1.1) and (1.2),

$$G(x) = \sum_{n \leqslant \sqrt{x}} \mu(n)[x/n^2] - x/\zeta(2)$$

$$= x \sum_{n \leqslant \sqrt{x}} \mu(n)/n^2 - \sum_{n \leqslant \sqrt{x}} \mu(n)\{x/n^2\} - x/\zeta(2)$$

$$= -xN(\sqrt{x}) - \sum_{n \leqslant \sqrt{x}} \mu(n)\{x/n^2\}$$

$$= -\sum_{n \leqslant \sqrt{x}} \mu(n)\{x/n^2\} + O(x^{\frac{1}{4}+\epsilon}), \text{ by Theorem 2.}$$

THEOREM 5. We have

$$G(x) = O(x^{2/5+\epsilon}).$$

PROOF: Immediate from Theorems 3 and 4.

3. THEOREM 6 (Landau 1949, *satz* 209).

$$\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{y^{s+k}}{s(s+1)\dots(s+k)}\,ds = \begin{cases} \dfrac{1}{k!}(y-1)^k, & \text{if } y \geqslant 1, \\ 0, & \text{if } 0 < y \leqslant 1. \end{cases}$$

THEOREM 7 (Landau 1953, p. 605). For $\sigma > 1$,

$$\sum_{n=1}^{\infty} \mu^2(n) n^{-s} = \zeta(s)/\zeta(2s).$$

THEOREM 8 (Titchmarsh 1951, p. 82). For $0 < \sigma < 1$,

$$\zeta(s) = O\Big(|t|^{\frac{1-\sigma}{2}}\Big).$$

THEOREM 9 (Titchmarsh 1951, p. 283). If $\delta > 0$, then for $\sigma \geqslant \frac{1}{2}+\delta$,

$$\frac{1}{\zeta(s)} = O(|t|^{\epsilon}).$$

THEOREM 10. For $k \geqslant 1$,

$$(G-x)^k = O(x^{k+\frac{1}{2}+\epsilon}),$$

where the left side has the meaning assigned to it in (1.6).

PROOF: From Theorem 6, we have

$$\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} \frac{x^s}{s(s+1)\dots(s+k)}\,ds = \begin{cases} \dfrac{1}{k!}\left(1-\dfrac{1}{x}\right)^k, & \text{if } x \geqslant 1, \\ 0, & \text{if } 0 < x \leqslant 1. \end{cases}$$

In this result, replace x by $\dfrac{x}{n}$, multiply by $\mu^2(n)$ and add for $n \leqslant \infty$; notice that for $n > x$, the right side is zero.

We get

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{2-\infty i}^{2+\infty i} \frac{\mu^2(n)n^{-s}x^s}{s(s+1)\dots(s+k)}\,ds = \frac{x^{-k}}{k!}\sum_{n \leqslant x}\mu^2(n)(x-n)^k.$$

Changing the orders of summation and integration (which is justified because $k \geqslant 1$) and using Theorem 7, we have

$$\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} x^s \cdot \frac{\zeta(s)}{\zeta(2s)} \cdot \frac{ds}{s(s+1)\dots(s+k)} = \frac{x^{-k}}{k!}\sum_{n \leqslant x}\mu^2(n)(x-n)^k. \quad .. \quad (3.1)$$

Now

$$\sum_{n \leqslant x}\mu^2(n)(x-n)^k = (-1)^k(G-x)^k + \frac{x^{k+1}}{\zeta(2)}(-1)^k\left(\frac{1}{k+1} - \binom{k}{1}\frac{1}{k} + \binom{k}{2}\frac{1}{k-1} + - \dots + (-1)^k\binom{k}{k}\frac{1}{1}\right)$$

$$= (-1)^k(G-x)^k + \frac{x^{k+1}}{k+1}\cdot\frac{(-1)^k}{\zeta(2)}\left(1 - \binom{k+1}{1} + \binom{k+1}{2} - + \dots + (-1)^k\binom{k+1}{k}\right)$$

$$= (-1)^k(G-x)^k + \frac{1}{\zeta(2)}\cdot\frac{x^{k+1}}{k+1}.$$

Thus the right side of (3.1) is

$$\frac{x^{-k}}{k!}\sum_{n \leqslant x}\mu^2(n)(x-n)^k = \frac{(-1)^k x^{-k}}{k!}\cdot(G-x)^k + \frac{1}{\zeta(2)}\cdot\frac{x}{(k+1)!}. \quad .. \quad (3.2)$$

Let $R(s)$ denote the integrand on the left side of (3.1), then

$$\int_{2-\infty i}^{2+\infty i} R(s)\,ds = \lim_{T\to\infty}\int_{2-Ti}^{2+Ti} R(s)\,ds.$$

Now under the Riemann hypothesis, $R(s)$ has only one singularity in the half-plane $\sigma > \frac{1}{4}$, this being $s = 1$ where $\zeta(s)$ has a simple pole with residue 1.

For arbitrary $\epsilon > 0$ consider the rectangle Γ whose vertices are $2-Ti$, $\frac{1}{4}+\epsilon-Ti$, $\frac{1}{4}+\epsilon+Ti$, $2+Ti$. $R(s)$ is regular everywhere in Γ except at $s = 1$. Hence by Cauchy's theorem

$$\frac{1}{2\pi i}\int_{\Gamma} R(s)\,ds = \text{Residue of } R(s) \text{ at } s = 1$$
$$= \frac{1}{\zeta(2)} \cdot \frac{x}{(k+1)\,!}.$$

On the other hand,

$$\int_{\Gamma} = \int_{2-Ti}^{\frac{1}{4}+\epsilon-Ti} + \int_{\frac{1}{4}+\epsilon-Ti}^{\frac{1}{4}+\epsilon+Ti} + \int_{\frac{1}{4}+\epsilon+Ti}^{2+Ti} - \int_{2-Ti}^{2+Ti}$$
$$= I_1+I_2+I_3-\int_{2-Ti}^{2+Ti}, \text{ say.}$$

Now by Theorems 8 and 9,

$$I_1 = \int_{2-Ti}^{\frac{1}{4}+\epsilon-Ti} x^s \cdot \frac{\zeta(s)}{\zeta(2s)} \cdot \frac{ds}{s(s+1)\ldots(s+k)}$$
$$= O(x^2 \,.\, T^{3/8-k-1+\epsilon})$$
$$= o(1), \text{ as is seen by taking } T = x^2.$$

Similarly $I_3 = o(1)$.
Finally

$$I_2 = x^{\frac{1}{4}+\epsilon}\left|\int_{-T}^{+T} \frac{\zeta(\frac{1}{4}+\epsilon+ti)}{\zeta(\frac{1}{2}+2\epsilon+2ti)} \cdot \frac{dt}{(\frac{1}{4}+\epsilon+ti)\ldots(\frac{1}{4}+k+\epsilon+ti)}\right|$$
$$= O\left(x^{\frac{1}{4}+\epsilon} \,.\, \lim_{\delta\to 0}\left(\int_{-T}^{-\delta}+\int_{\delta}^{T}\right)|t|^{3/8-k-1+\epsilon}\,dt\right)$$
$$= O(x^{\frac{1}{4}+\epsilon}), \text{ since the integrals converge.}$$

Collecting these results together and letting $T \to \infty$, we have (since $k \geqslant 1$), for arbitrary $\epsilon > 0$,

$$\frac{1}{2\pi i}\int_{2-\infty i}^{2+\infty i} R(s)\,ds = \frac{1}{\zeta(2)} \cdot \frac{x}{(k+1)\,!} + O(x^{\frac{1}{4}+\epsilon}).$$

Thus from (3.1) and (3.2),

$$\frac{(-1)^k}{k\,!} \cdot x^{-k}(G-x)^k + \frac{x}{\zeta(2)\,.\,(k+1)\,!} = \frac{x}{\zeta(2)\,.\,(k+1)\,!} + O(x^{\frac{1}{4}+\epsilon}),$$

whence finally

$$(G-x)^k = O(x^{k+\frac{1}{4}+\epsilon}).$$

This completes the proof of the theorem.
As a special case of the theorem, we note that

$$G_1(x) - xG(x) = O(x^{1+\frac{1}{4}+\epsilon}). \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

4. THEOREM 11. For an integer x,

$$\sum_{n<x} G(n) = -(G_1(x)-xG(x))+G(x)+x/(2\zeta(2)).$$

PROOF:

$$\sum_{n<x} G(n) = \sum_{n<x}\sum_{m\leqslant n} \mu^2(m)-x(x+1)/(2\zeta(2))$$

$$= \sum_{n<x} (x-n+1)\mu^2(n)-x(x+1)/(2\zeta(2))$$

$$= (x+1)(G(x)+x/\zeta(2))-x(x+1)/(2\zeta(2))-(G_1(x)+x^2/(2\zeta(2))$$

$$= (x+1)G(x)-G_1(x)+x/(2\zeta(2)).$$

THEOREM 12. For an integer x,

$$\frac{1}{x}\int_1^x G(t)dt = O(x^{\frac{1}{4}+\epsilon}).$$

PROOF: By partial summation,

$$\sum_{n<x} n\mu^2(n) = \sum_{n\leqslant x-1} Q(n)(n-(n-1))+xQ(x)$$

$$= -\int_1^x Q(t)dt+x^2/\zeta(2)+xG(x)$$

$$= -\frac{1}{\zeta(2)}\int_1^x t\,dt-\int_1^x G(t)\,dt+x^2/\zeta(2)+xG(x)$$

$$= -\frac{x^2}{2\zeta(2)}+O(1)-\int_1^x G(t)\,dt+x^2/\zeta(2)+xG(x)$$

$$= \frac{x^2}{2\zeta(2)}+xG(x)-\int_1^x G(t)dt+O(1)$$

i.e.

$$G_1(x)-xG(x) = -\int_1^x G(t)dt+O(1).$$

The theorem now follows from (3.3).

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