ON ORTHOGONAL ENNUPLES IN A PAIR OF RIEMANNIAN SPACES

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(*Received October 12, 1965*)

In a Riemannian space, the coefficients of rotation of Ricci, their properties and applications are well known. In this paper, a pair of Riemannian spaces have been taken where, besides the two sets of the coefficients of rotations of Ricci, two other analogous sets of quantities, also called by us the coefficients of rotations in the two spaces, have been considered. Certain relations between these four sets of coefficients of rotations, their properties, applications and geometrical interpretations have been given. Finally, we have studied how these relations between the coefficients and their applications behave when the two spaces are related in some special ways.

1. *Notations and Identities*

Let g_{ij} and $\bar{g}_{ij}$ be two non-singular symmetric covariant tensors, and $\left\{ {t \atop ij} \right\}_g$ and $\left\{ {t \atop ij} \right\}_{\bar{g}}$ be the Christoffel symbols formed with them. Let the covariant derivatives with respect to them be denoted respectively by comma and semi-colon. Then we have the identities

$$\left\{ {t \atop ij} \right\}_{\bar{g}} - \left\{ {t \atop ij} \right\}_g = \tfrac{1}{2}\bar{g}^{mt}(\bar{g}_{im,j}+\bar{g}_{jm,i}-\bar{g}_{ij,m})$$

$$= -\tfrac{1}{2}g^{mt}(g_{im;j}+g_{jm;i}-g_{ij;m}). \quad \ldots \quad \ldots \quad (1.1)$$

Let g_{ij} and $\bar{g}_{ij}$ be taken as the fundamental tensors of two Riemannian spaces V_n and $\bar{V}_n$ with arc-elements ds and $d\bar{s}$ respectively, and let two orthogonal ennuples be defined in the spaces by

$$g_{ij} = \sum_t \lambda_{t|i}\lambda_{t|j} \text{ and } \bar{g}_{ij} = \sum_t \bar{\lambda}_{t|i}\bar{\lambda}_{t|j}.$$

Let us form the different sets of scalars in terms of the ennuples and their covariant derivatives in the spaces V_n and $\bar{V}_n$ as shown below:

$$\bar{\gamma}_{pqr} = \bar{\lambda}_{p|i;j}\,\bar{\lambda}^i_{q|}\,\bar{\lambda}^j_{r|} \quad (\bar{\gamma}_{pqr}+\bar{\gamma}_{qpr}=0)$$

$$\gamma_{pqr} = \lambda_{p|i;j}\,\lambda^i_{q|}\,\lambda^j_{r|}$$

$$\beta_{pqr} = \lambda_{p|i,j}\,\lambda^i_{q|}\,\lambda^j_{r|} \quad (\beta_{pqr}+\beta_{qpr}=0)$$

$$\bar{\beta}_{pqr} = \bar{\lambda}_{p|i,j}\,\bar{\lambda}^i_{q|}\,\bar{\lambda}^j_{r|}.$$

Then, by eqn. (1.1), we have the identities

$$\bar{\gamma}_{pqr} = \tfrac{1}{2}(\bar{\beta}_{pqr}+\bar{\beta}_{qrp}+\bar{\beta}_{rqp}-\bar{\beta}_{qpr}-\bar{\beta}_{prq}-\bar{\beta}_{rpq}) \qquad (1.2a)$$

$$\beta_{pqr} = \tfrac{1}{2}(\gamma_{pqr}+\gamma_{qrp}+\gamma_{rqp}-\gamma_{qpr}-\gamma_{prq}-\gamma_{rpq}). \qquad (1.2b)$$

It is well known that the quantities $\bar{\gamma}_{pqr}$ are the coefficients of rotation of Ricci in $\bar{V}_n$ of the orthogonal ennuple $\bar{\lambda}^i_{p|}$ with respect to the Levi-Civita parallelism in that space. Similarly, the quantities β_{pqr} are the coefficients of rotation of Ricci in V_n of the orthogonal ennuple $\lambda^i_{p|}$ with respect to the Levi-Civita parallelism in this space (Eisenhart 1949, § 30). We shall call γ_{pqr} as the coefficients of rotation in V_n of the orthogonal ennuple $\lambda^i_{p|}$ with respect to the Levi-Civita parallelism as obtains in $\bar{V}_n$. Similarly, we shall call $\bar{\beta}_{pqr}$ as the coefficients of rotation in $\bar{V}_n$ of the ennuple $\bar{\lambda}^i_{p|}$ with respect to the Levi-Civita parallelism as obtains in V_n.

Now let $\bar{R}^t_{ijk}$ and R^t_{ijk} be the curvature tensors in $\bar{V}_n$ and V_n respectively. If we write

$$\left.\begin{aligned} \bar{R}^t_{ijk}\,\bar{\lambda}_{l|t}\,\bar{\lambda}^i_{p|}\,\bar{\lambda}^j_{q|}\,\bar{\lambda}^k_{r|} &= \bar{\gamma}_{lpqr} \\ \bar{R}^t_{ijk}\,\lambda_{l|t}\,\lambda^i_{p|}\,\lambda^j_{q|}\,\lambda^k_{r|} &= \gamma_{lpqr} \\ R^t_{ijk}\,\lambda_{l|t}\,\lambda^i_{p|}\,\lambda^j_{q|}\,\lambda^k_{r|} &= \beta_{lpqr} \\ R^t_{ijk}\,\bar{\lambda}_{l|t}\,\bar{\lambda}^i_{p|}\,\lambda^j_{q|}\,\bar{\lambda}^k_{r|} &= \bar{\beta}_{lpqr} \end{aligned}\right\} \qquad (1.2c)$$

then it is known that (Eisenhart 1949, § 30)

$$\bar{\gamma}_{lpqr} = \frac{\partial\bar{\gamma}_{lpq}}{\partial\bar{s}_r} - \frac{\partial\bar{\gamma}_{lpr}}{\partial\bar{s}_q} + \sum_t \{\bar{\gamma}_{lpt}(\bar{\gamma}_{tqr}-\bar{\gamma}_{trq})+\bar{\gamma}_{ltq}\bar{\gamma}_{tpr}-\bar{\gamma}_{ltr}\bar{\gamma}_{tpq}\}$$

$$\beta_{lpqr} = \frac{\partial\beta_{lpq}}{\partial s_r} - \frac{\partial\beta_{lpr}}{\partial s_q} + \sum_t \{\beta_{lpt}(\beta_{tqr}-\beta_{trq})+\beta_{ltq}\beta_{tpr}-\beta_{ltr}\beta_{tpq}\}.$$

The quantities $\bar{\gamma}_{lpqr}$, β_{lpqr} possess properties of indices of fully covariant curvature tensors. Now it is also seen that

$$\gamma_{lpqr} = \frac{\partial\gamma_{lpq}}{\partial s_r} - \frac{\partial\gamma_{lpr}}{\partial s_q} + \sum_t \{\gamma_{lpt}(\gamma_{tqr}-\gamma_{trq})+\gamma_{ltq}\gamma_{tpr}-\gamma_{ltr}\gamma_{tpq}\}$$

$$\bar{\beta}_{lpqr} = \frac{\partial\bar{\beta}_{lpq}}{\partial\bar{s}_r} - \frac{\partial\bar{\beta}_{lpr}}{\partial\bar{s}_q} + \sum_t \{\bar{\beta}_{lpt}(\bar{\beta}_{tqr}-\bar{\beta}_{trq})+\bar{\beta}_{ltq}\bar{\beta}_{tpr}-\bar{\beta}_{ltr}\bar{\beta}_{tpq}\}.$$

From eqn. (1.1) it follows that

$$\begin{aligned} &\bar{R}^t_{ijk}-\tfrac{1}{2}R^t_{ijk}+\tfrac{1}{2}\bar{g}^{mt}\bar{g}_{ih}R^h_{mjk} \\ &= \tfrac{1}{2}\bar{g}^{mt}(\bar{g}_{km,\,ij}-\bar{g}_{jm,\,ik}+\bar{g}_{ij,\,mk}-\bar{g}_{ik,\,mj}) \\ &\quad+\tfrac{1}{4}\bar{g}^{mt}\bar{g}^{uv}\{(\bar{g}_{jm,\,u}-\bar{g}_{uj,\,m}-\bar{g}_{mu,\,j})(\bar{g}_{iv,\,k}+\bar{g}_{kv,\,i}-\bar{g}_{ik,\,v}) \\ &\quad-(\bar{g}_{km,\,u}-\bar{g}_{uk,\,m}-\bar{g}_{mu,\,k})(\bar{g}_{iv,\,j}+\bar{g}_{jv,\,i}-\bar{g}_{ij,\,v})\} \end{aligned} \qquad (1.3a)$$

and

$$
\begin{aligned}
&R^t_{ijk}-\tfrac{1}{2}\overline{R}^t_{ijk}+\tfrac{1}{2}g^{mt}g_{ih}\overline{R}^t_{mjk}\\
&=\tfrac{1}{2}g^{mt}(g_{km\,;\,ij}-g_{jm\,;\,ik}+g_{ij\,;\,mk}-g_{ik\,;\,mj})\\
&\quad+\tfrac{1}{4}g^{mt}g^{uv}\{(g_{jm\,;\,u}-g_{uj\,;\,m}-g_{mu\,;\,j})(g_{iv\,;\,k}+g_{kv\,;\,i}-g_{ik\,;\,v})\\
&\quad-(g_{km\,;\,u}-g_{uk\,;\,m}-g_{mu\,;\,k})(g_{iv\,;\,j}+g_{jv\,;\,i}-g_{ij\,;\,v})\}. \quad \ldots \quad \ldots \quad (1.3b)
\end{aligned}
$$

Multiply the first equation by $\bar{\lambda}_{l|t}\bar{\lambda}^i_{p|}\bar{\lambda}^j_{q|}\bar{\lambda}^k_{r|}$ and the second equation by $\lambda_{l|t}\lambda^i_{p|}\lambda^j_{q|}\lambda^k_{r|}$ and sum for t, i, j, k. After simplification it is seen that

$$
\begin{aligned}
&\bar{\gamma}_{lpqr}-\tfrac{1}{2}(\bar{\beta}_{lpqr}-\bar{\beta}_{plqr})\\
&=\tfrac{1}{2}\left[\frac{\partial}{\partial\bar{s}_q}\overline{B}_{rlp}-\frac{\partial}{\partial\bar{s}_r}\overline{B}_{qlp}+\sum_t(\bar{\beta}_{trq}-\bar{\beta}_{tqr})\overline{B}_{tlp}\right]\\
&\quad+\tfrac{1}{4}\Big[\sum_t\overline{B}_{rlt}\overline{B}_{qpt}-\overline{B}_{qlt}\overline{B}_{rpt}+\overline{B}_{rlt}(\bar{\beta}_{tpq}-\bar{\beta}_{ptq})\\
&\quad+\overline{B}_{qpt}(\bar{\beta}_{tlr}-\bar{\beta}_{ltr})-\overline{B}_{qlt}(\bar{\beta}_{tpr}-\bar{\beta}_{ptr})-\overline{B}_{rpt}(\bar{\beta}_{tlq}-\bar{\beta}_{ltq})\\
&\quad+(\bar{\beta}_{tlr}+\bar{\beta}_{ltr})(\bar{\beta}_{tpq}+\bar{\beta}_{ptq})-(\bar{\beta}_{tlq}+\bar{\beta}_{ltq})(\bar{\beta}_{tpr}+\bar{\beta}_{ptr})\Big] \quad \ldots \quad \ldots \quad (1.4a)
\end{aligned}
$$

and

$$
\begin{aligned}
&\beta_{lpqr}-\tfrac{1}{2}(\gamma_{lpqr}-\gamma_{plqr})\\
&=\tfrac{1}{2}\left[\frac{\partial}{\partial s_q}C_{rlp}-\frac{\partial}{\partial s_r}C_{qlp}+\sum_t(\gamma_{trq}-\gamma_{tqr})C_{tlp}\right]\\
&\quad+\tfrac{1}{4}\sum_t[C_{rlt}C_{qpt}-C_{qlt}C_{rpt}+C_{rlt}(\gamma_{tpq}-\gamma_{ptq})\\
&\quad+C_{qpt}(\gamma_{tlr}-\gamma_{ltr})-C_{qlt}(\gamma_{tpr}-\gamma_{ptr})-C_{rpt}(\gamma_{tlq}-\gamma_{ltq})\\
&\quad+(\gamma_{tlr}+\gamma_{ltr})(\gamma_{tpq}+\gamma_{ptq})-(\gamma_{tlq}+\gamma_{ltq})(\gamma_{tpr}+\gamma_{ptr})] \quad (1.4b)
\end{aligned}
$$

where we have put

$$\overline{B}_{pqr}=\bar{\beta}_{pqr}+\bar{\beta}_{qpr}-\bar{\beta}_{prq}-\bar{\beta}_{rpq},\ \text{so}\ \overline{B}_{pqr}+\overline{B}_{prq}=0 \quad \ldots \quad (1.5a)$$

$$C_{pqr}=\gamma_{pqr}+\gamma_{qpr}-\gamma_{prq}-\gamma_{rpq},\ \text{so}\ C_{pqr}+C_{prq}=0. \quad \ldots \quad (1.5b)$$

2. *Some Geometrical Interpretations*

The coefficients of rotations $\bar{\gamma}_{pqr}$ and β_{pqr} of Ricci have well-known geometrical interpretations. Analogous interpretations may be given of the coefficients $\bar{\beta}_{pqr}$ and γ_{pqr}. Let us consider the scalar $\bar{f}_{pq}=\bar{\lambda}_{p|i}\bar{\lambda}^i_{q|}$ at a point of $\overline{V}_n$; and let $\bar{\lambda}_{p|i}$ be given local displacement and $\bar{\lambda}^i_{q|}$ be given Levi-Civita parallel displacement as obtains in V_n along a third line $\bar{\lambda}^j_{r|}=\dfrac{dx^j}{d\bar{s}_r}$ of the same ennuple. Then the rate of change of the scalar is given by

$$\frac{\partial \bar{f}_{pq}}{\partial \bar{s}_r} = \bar{\beta}_{pqr}.$$

Similarly for the quantities γ_{pqr}.

The following facts are well known (Eisenhart 1949, §§ 30, 35). In V_n

(i) the curves of the congruence $\lambda_{q|i}$ are geodesics, if and only if

$$\beta_{pqq} = 0 \quad (p = 1, \ldots n);$$

(ii) the curves of the congruence $\lambda_{n|i}$ are normal (*i.e.* they are orthogonal trajectories of a family of hypersurfaces), if and only if

$$\beta_{npq} - \beta_{nqp} = 0;$$

(iii) the commutativity formula for the intrinsic derivative

$$\frac{\partial f}{\partial s_r} = \lambda^i_{r|}\left(\frac{\partial f}{\partial x^i}\right)$$

is given by

$$\frac{\partial}{\partial s_k}\frac{\partial f}{\partial s_h} - \frac{\partial}{\partial s_h}\frac{\partial f}{\partial s_k} = \sum_l (\beta_{hlk} - \beta_{klh})\frac{\partial f}{\partial s_l}.$$

In view of (1.2) we may now say that

(i′) the curves of the congruence $\lambda_{q|i}$ are geodesics, if and only if $\gamma_{qqp} - \gamma_{qpq} = 0$; similarly, in $\bar{V}_n$ the curves of the congruence $\bar{\lambda}_{q|i}$ are geodesics, if and only if $\bar{\beta}_{qqp} - \bar{\beta}_{qpq} = 0$;

(ii′) the curves of the congruence $\lambda_{n|i}$ are normal in V_n, if and only if $\gamma_{npq} - \gamma_{nqp} = 0$; similarly, in $\bar{V}_n$ the curves of the congruence $\bar{\lambda}_{n|i}$ are normal, if and only if $\bar{\beta}_{npq} - \bar{\beta}_{nqp} = 0$;

(iii′) a formula analogous to that given in (iii) can be given in terms of the orthogonal ennuple in $\bar{V}_n(V_n)$ and the Levi-Civita parallelism as obtains in $V_n(\bar{V}_n)$ as follows:

We have

$$\frac{\partial f}{\partial \bar{s}_r} = \bar{\lambda}^i_{r|}\frac{\partial f}{\partial x^i}.$$

So

$$\frac{\partial}{\partial \bar{s}_k}\frac{\partial f}{\partial \bar{s}_h} = \bar{\lambda}^i_{k|}\frac{\partial}{\partial x^i}\left(\bar{\lambda}^j_{h|}\frac{\partial f}{\partial x^j}\right) = \bar{\lambda}^i_{k|}(\bar{\lambda}^j_{h|,i}f_{,j} + \bar{\lambda}^j_{h|}f_{,ji})$$

$$= \sum_l \bar{\lambda}^i_{k|}\left\{(\bar{\lambda}_{h|t,i}\bar{\lambda}^t_{l|} + \bar{g}^{jt}_{,i}\bar{\lambda}_{h|t}\bar{\lambda}_{l|j})\frac{\partial f}{\partial \bar{s}_l} + \bar{\lambda}^j_{h|}f_{,ji}\right\}.$$

Therefore,

$$\frac{\partial}{\partial \bar{s}_k}\frac{\partial f}{\partial \bar{s}_h} - \frac{\partial}{\partial \bar{s}_h}\frac{\partial f}{\partial \bar{s}_k} = \sum_l [(\bar{\beta}_{hlk} - \bar{\beta}_{klh}) + \bar{g}^{jt}_{,i}\bar{\lambda}_{l|j}(\bar{\lambda}_{h|t}\bar{\lambda}^i_{k|} - \bar{\lambda}_{k|t}\bar{\lambda}^i_{h|})]\frac{\partial f}{\partial \bar{s}_l}. \qquad (2.1)$$

This is the form which the condition of integrability of the intrinsic derivative $\frac{\partial f}{\partial \bar{s}_r}$ assumes. The geometrical meaning of the invariant

$$\bar{g}^{jt}_{,i}\bar{\lambda}_{l|j}\bar{\lambda}_{h|t}\bar{\lambda}^{i}_{k|} = -\bar{g}_{jt,i}\bar{\lambda}^{j}_{l|}\bar{\lambda}^{t}_{h|}\bar{\lambda}^{i}_{k|}$$

is easily seen. The expression on the l.h.s. gives the rate of change of the scalar product in $\bar{V}_n$ of the covariant vectors $\bar{\lambda}_{l|}$, $\bar{\lambda}_{h|}$ (in $\bar{V}_n$) when the vectors are given the Levi-Civita parallel transport as obtains in V_n along the direction of $\bar{\lambda}_k$ in $\bar{V}_n$. The right-hand side says that this invariant changes its sign when the covariant vectors are replaced by the associate contravariant vectors.

Similarly we have the following formula in terms of the orthogonal ennuple in V_n and the Levi-Civita parallelism as obtained in $\bar{V}_n$.

$$\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_h} - \frac{\partial}{\partial s_h}\frac{\partial}{\partial s_k} = \sum_l \left\{(\gamma_{hlk}-\gamma_{klh})+g^{jt}_{;i}\lambda_{l|j}(\lambda_{h|t}\lambda^{i}_{k|}-\lambda_{k|t}\lambda^{i}_{h|})\right\}\frac{\partial f}{\partial s_l} \quad .. \quad (2.2)$$

and a similar interpretation may be given.

On the r.h.s. of eqn. (1.2*a*) there appear certain combinations of coefficients of rotations for which geometrical interpretations may be given. The combinations are

$$\bar{\beta}_{pqr}+\bar{\beta}_{qpr},\ \bar{\beta}_{pqr}-\bar{\beta}_{qpr},\ \bar{\beta}_{pqr}+\bar{\beta}_{qrp}+\bar{\beta}_{rqp}$$

and

$$\bar{\beta}_{pqr}-\bar{\beta}_{prq}.$$

In $\bar{V}_n$ we have

$$\bar{\lambda}_{p|i}\bar{\lambda}^{i}_{q|} = \delta^{p}_{q}.$$

Taking the covariant derivative of this with respect to the Levi-Civita parallelism as obtains in V_n

$$\bar{\lambda}_{p|i,j}\bar{\lambda}^{i}_{q|}+\bar{\lambda}_{p|i}\bar{\lambda}^{i}_{q|,j} = \bar{\lambda}_{p|i,j}\bar{\lambda}^{i}_{q|}+\bar{\lambda}_{q|t,j}\bar{\lambda}^{t}_{p|}+\bar{g}^{tm}_{,j}\bar{\lambda}_{p|t}\bar{\lambda}_{q|m} = 0.$$

Multiplying by $\bar{\lambda}^{j}_{r|}$ and summing for j, we may write

$$\bar{a}_{pqr} = \bar{a}_{qpr} = \bar{\beta}_{pqr}+\bar{\beta}_{qpr} = \bar{g}_{st,j}\bar{\lambda}^{s}_{p|}\bar{\lambda}^{t}_{q|}\bar{\lambda}^{j}_{r|}. \quad .. \quad .. \quad (2.3)$$

The expression on the r.h.s. gives the rate of change of the scalar product in $\bar{V}_n$ of the vectors $\bar{\lambda}^{s}_{p|}, \bar{\lambda}^{t}_{q|}$ due to the Levi-Civita parallel transport as obtains in V_n along $\bar{\lambda}^{j}_{r|}$. Hence the interpretation of $\bar{a}_{pqr}$.

Next, the covariant derivative of $\bar{\lambda}_{p|i}$ with respect to the parallelism

$$dV^l+\left(\left\{ {l \atop ij} \right\}_g -\bar{g}^{lt}\bar{g}_{ij,t}\right)V^i dx^j = 0 \quad .. \quad .. \quad .. \quad (2.4)$$

is

$$\bar{\lambda}_{p|i,j}+\sum_s \bar{\lambda}^{t}_{p|}(\bar{\lambda}_{s|j,t}\bar{\lambda}_{s|i}+\bar{\lambda}_{s|i,t}\bar{\lambda}_{s|j}).$$

Multiply by $\bar{\lambda}^i_{q|}\bar{\lambda}^j_{r|}$ and sum for i, j. Then we may write

$$\bar{b}_{pqr} = \bar{\beta}_{pqr}+\bar{\beta}_{qrp}+\bar{\beta}_{rqp} = \bar{\beta}_{pqr}+\bar{a}_{qrp}.$$

The quantities $\bar{b}_{pqr}$ may, therefore, be called the coefficients of rotations of the ennuple $\bar{\lambda}_{p|i}$ with respect to the parallelism (2.4).

Again, the covariant derivative of $\bar{\lambda}_{p|i}$ with respect to the parallelism

$$dV^l+\left[\left\{ {l \atop ij} \right\}_g +g^{lt}\sum_s \bar{\lambda}_{s|i}\bar{\lambda}_{s|t,j}\right]V^i dx^j = 0 \qquad \ldots \qquad \ldots \qquad (2.5)$$

is

$$\bar{\lambda}_{p|i,j}-\bar{\lambda}^t_{p|}\sum_s \bar{\lambda}_{s|t,j}\bar{\lambda}_{s|i}.$$

Multiply by $\bar{\lambda}^i_{q|}\bar{\lambda}^j_{r|}$ and sum for i, j. Then we may write

$$\bar{c}_{pqr} = -\bar{c}_{qpr} = \bar{\beta}_{pqr}-\bar{\beta}_{qpr}.$$

The quantities $\bar{c}_{pqr}$ are, therefore, the coefficients of rotation of the ennuple $\bar{\lambda}_{p|i}$ with respect to the parallelism (2.5).

Lastly, the covariant derivative of $\bar{\lambda}_{p|i}$ with respect to the parallelism

$$dV^l+\left[\left\{ {l \atop ij} \right\}_g +\sum_s \bar{\lambda}^l_{s|}\bar{\lambda}_{s|j,i}\right]V^i dx^j = 0 \quad \ldots \qquad \ldots \qquad \ldots \qquad (2.6)$$

is

$$\bar{\lambda}_{p|i,j}-\bar{\lambda}_{p|j,i}.$$

Multiply by $\bar{\lambda}^i_{q|}\bar{\lambda}^j_{r|}$ and sum for i, j. Then we may write

$$\bar{d}_{pqr} = -\bar{d}_{prq} = \bar{\beta}_{pqr}-\bar{\beta}_{prq}.$$

The quantities $\bar{d}_{pqr}$ are the coefficients of rotation of the ennuple $\bar{\lambda}_{p|i}$ with respect to the parallelism (2.6).

In view of the above, the expression on the r.h.s. of eqn. (1.2*a*) can be written as

$$2\bar{\gamma}_{pqr} = \bar{b}_{pqr}-\bar{b}_{qpr} = \bar{d}_{pqr}+\bar{d}_{qrp}+\bar{d}_{rqp} = \bar{c}_{pqr}+\bar{a}_{qrp}-\bar{a}_{prq}.$$

And eqn. (1.5*a*) can be written as

$$\bar{B}_{pqr} = \bar{a}_{pqr}-\bar{a}_{prq}.$$

Similarly, interpretations of similar combinations of γ_{pqr} can be given.

3. *Application to Conformal Spaces*

Let

$$\bar{g}_{ij} = e^{2\sigma}g_{ij}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3.1)$$

Then eqn. (1.1) reduces, as usual, to

$$\left\{ {t \atop ij} \right\}_{\bar{g}} = \left\{ {t \atop ij} \right\}_{g} + \left(\delta^t_i\sigma_j + \delta^t_j\sigma_i - g^{mt}g_{ij}\sigma_m\right) \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

where

$$\sigma_i = \frac{\partial\sigma}{\partial x^i}.$$

Differentiating both sides of eqn. (3.1) covariantly with respect to $\left\{ {t \atop ij} \right\}_{\bar{g}}$ or directly from eqn. (3.2) we find

$$g_{ij\,;\,k} + 2g_{ij}\sigma_k = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

The eqn. (3.3) shows that $\left\{ {t \atop ij} \right\}_{\bar{g}}$ are the coefficients of a Weyl's parallelism. Thus Weyl's parallelism is Levi-Civita parallelism in a conformal space, as is to be expected. In $\overline{V}_n$, Weyl's characteristic eqn. (3.3) can be written as

$$\bar{g}_{ij,\,k} - 2\bar{g}_{ij}\sigma_k = 0.$$

Further from eqn. (1.1) we have

$$\bar{g}^{mt}\left(\bar{g}_{im,\,j} + \bar{g}_{jm,\,i} - \bar{g}_{ij,\,m}\right) = -g^{mt}\left(g_{im\,;\,j} + g_{jm\,;\,i} - g_{ij\,;\,m}\right). \quad \ldots \quad (3.4)$$

From eqns. (3.1) and (3.3.) we get

$$\bar{g}^{mt}\bar{g}_{ij,\,m} = 2g^{mt}g_{ij}\sigma_m = -g^{mt}g_{ij\,;\,m}$$

$$\bar{g}^{mt}\bar{g}_{im,\,j} = 2\delta^t_i\sigma_j = -g^{mt}g_{im\,;\,j}.$$

Hence the equality (3.4) is a term-by-term equality when eqn. (3.1) holds.

Also eqn. (1.3) easily reduces to the well-known result (Eisenhart 1949, § 28)

$$\bar{R}^t_{ijk} = R^t_{ijk} + \delta^t_k\sigma_{ji} - \delta^t_j\sigma_{ik} + g^{mt}\left(g_{ij}\sigma_{mk} - g_{ik}\sigma_{mj}\right)$$
$$+ \left(\delta^t_k g_{ij} - \delta^t_j g_{ik}\right)\Delta_1\sigma, \quad \left(\sigma_{ij} = \sigma_{i,\,j} - \sigma_i\sigma_j\right).$$

Now, when in the relations (3.1)

$$g_{ij} = \sum_t \lambda_{t|i}\lambda_{t|j}, \text{ and } \bar{g}_{ij} = \sum_t \bar{\lambda}_{t|i}\bar{\lambda}_{t|j}$$

hold, let

$$\bar{\lambda}_{t|i} = e^{\sigma}\lambda_{t|i}, \text{ so } \bar{\lambda}^j_{t|} = e^{-\sigma}\lambda^j_{t|}. \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

Then

$$\gamma_{pqr} = \lambda_{p|i\,;\,j}\lambda^i_{q|}\lambda^j_{r|} = \left(e^{-\sigma}\bar{\lambda}_{p|i}\right)_{;\,j} e^{2\sigma}\bar{\lambda}^i_{q|}\bar{\lambda}^j_{r|}.$$

Therefore

similarly

$$\left.\begin{aligned} \gamma_{pqr} &= e^{\sigma}\left(\bar{\gamma}_{pqr} - \delta_{pq}\bar{\lambda}^j_{r|}\sigma_j\right) \\ \bar{\beta}_{pqr} &= e^{-\sigma}\left(\beta_{pqr} + \delta_{pq}\lambda^j_{r|}\sigma_j\right) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

where δ_{pq} is the Kronecker delta. These relations express γ_{pqr} and $\bar{\beta}_{pqr}$ in terms of the coefficients of rotations of Ricci in the two spaces. Hence the eqns. (1.2*a*), (1.2*b*) reduce to

$$\left.\begin{aligned}\bar{\gamma}_{pqr} &= e^{-\sigma}\left\{\beta_{pqr}+(\delta_{qr}\lambda^{j}_{p|}-\delta_{pr}\lambda^{j}_{q|})\sigma_j\right\}\\ \beta_{pqr} &= e^{\sigma}\left\{\bar{\gamma}_{pqr}+(\delta_{rp}\bar{\lambda}^{j}_{q|}-\delta_{rq}\bar{\lambda}^{j}_{p|})\sigma_j\right\}\end{aligned}\right\}. \qquad \ldots \quad \ldots \quad (3.7)$$

These equations connect the coefficients of rotations of Ricci in the two spaces. Also we have

$$\frac{\partial f}{\partial \bar{s}_r} = \bar{\lambda}^{k}_{r|}\frac{\partial f}{\partial x^k} = e^{-\sigma}\lambda^{k}_{r|}\frac{\partial f}{\partial x^k} = e^{-\sigma}\frac{\partial f}{\partial s_r}.$$

It is easily seen from eqns. (3.5) and (1.2*c*) that

$$\left.\begin{aligned}\bar{\beta}_{lpqr} &= e^{-2\sigma}\beta_{lpqr}\\ \gamma_{lpqr} &= e^{2\sigma}\bar{\gamma}_{lpqr}\end{aligned}\right\}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

Also it is easily seen from eqn. (3.7) or directly that

$$e^{2\sigma}\,\bar{\gamma}_{lpqr} = \beta_{lpqr}+\delta_{lr}f_{pq}+\delta_{pq}f_{lr}-\delta_{lq}f_{pr}-\delta_{pr}f_{lq} +(\delta_{lr}\delta_{pq}-\delta_{lq}\delta_{pr})\Delta_1\sigma,$$

where we have put

$$(\sigma_{i,j}-\sigma_i\sigma_j)\lambda^{i}_{p|}\lambda^{j}_{q|} = f_{pq}.$$

Put $q = l$ and $r = p$ and sum for p; then

$$e^{2\sigma}\sum_p \bar{r}_{lp} = \sum_p (r_{lp}-f_{pp})+f_{ll}. \qquad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

This is the relation between the mean curvature of $\bar{V}_n$ for the direction $\bar{\lambda}^{i}_{l|}$ and the mean curvature of V_n for the direction $\lambda^{i}_{l|}$.

Finally, let Γ^{t}_{ij} be the coefficients of an affine connection (not necessarily symmetric) with respect to which the covariant derivative of g_{ij} vanishes. Then

$$\bar{\Gamma}^{t}_{ij} = \Gamma^{t}_{ij}+\delta^{t}_{i}\sigma_j$$

are the coefficients of an affine connection with respect to which the covariant derivative of $\bar{g}_{ij}$ vanishes. Therefore, if ordinary and square brackets followed by indices denote covariant derivatives with respect to Γ^{t}_{ij} and $\bar{\Gamma}^{t}_{ij}$, respectively, then we have, as in Weyl's connection,

$$[g_{ij}]_k+2g_{ij}\sigma_k = 0 \text{ and } (\bar{g}_{ij})_k-2\bar{g}_{ij}\sigma_k = 0\,;$$

and if Γ^{t}_{ijk} and $\bar{\Gamma}^{t}_{ijk}$ denote curvature tensors formed with respect to Γ^{t}_{ij} and $\bar{\Gamma}^{t}_{ij}$, respectively, then

$$\bar{\Gamma}^{t}_{ijk} = \Gamma^{t}_{ijk}, \text{ so } e^{-2\sigma}\bar{\Gamma}_{hijk} = \Gamma_{hijk}, \quad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

where, by (3.8),

$$e^{2\sigma} = \beta_{lpqr}/\bar{\beta}_{lpqr} = \gamma_{lpqr}/\bar{\gamma}_{lpqr}.$$

4. *Application to Associate Spaces*

Let a_{ij} be a non-singular symmetric covariant tensor. Suppose that given g_{ij} and a_{ij}, the tensor $\bar{g}_{ij}$ is defined by

$$\bar{g}_{ij} = a_{ik}a_{jl}g^{kl}. \quad \ldots \quad (4.1)$$

It follows that

and, therefore,

$$\left.\begin{aligned} \bar{g}^{ij} &= a^{ik}a^{jl}g_{kl} \\ g_{ij} &= a_{ik}a_{jl}\bar{g}^{kl} \end{aligned}\right\}. \quad \ldots \quad (4.2)$$

Thus by the use of a_{ij}, g_{ij} is obtained from $\bar{g}_{ij}$ in the same way as $\bar{g}_{ij}$ is obtained from g_{ij}. Two spaces whose fundamental tensors g_{ij} and $\bar{g}_{ij}$ are connected by eqn. (4.1) and, therefore, by eqn. (4.2) shall be called *associate* with respect to a_{ij} (Sen 1962). As in previous sections, orthogonal ennuples are introduced in the spaces. Given the ennuple $\lambda_{t|i}$ we shall choose the ennuple $\bar{\lambda}_{t|i}$ in the following manner:

From eqn. (4.1) we have

$$\sum_t \bar{\lambda}_{t|i}\bar{\lambda}_{t|j} = \sum_t a_{ik}a_{jl}\lambda^k_{t|}\lambda^l_{t|}.$$

So we can take

$$\left.\begin{aligned} a_{ik}\lambda^k_{p|} &= \bar{\lambda}_{p|i}, \; a^{ik}\lambda_{p|k} = \bar{\lambda}^i_{p|} \\ a_{ik}\bar{\lambda}^k_{p|} &= \lambda_{p|i}, \; a^{ik}\bar{\lambda}_{p|k} = \lambda^i_{p|} \end{aligned}\right\}. \quad \ldots \quad (4.3)$$

It follows that

$$a_{ij} = \sum_t \lambda_{t|i}\bar{\lambda}_{t|j} = \sum_t \bar{\lambda}_{t|i}\lambda_{t|j}. \quad \ldots \quad (4.4)$$

Also

$$\left.\begin{aligned} \lambda^i_{p|}\bar{\lambda}_{q|i} &= a_{kl}\lambda^k_{p|}\lambda^l_{q|} = a^{kl}\bar{\lambda}_{p|k}\bar{\lambda}_{q|l} \\ \lambda_{p|i}\bar{\lambda}^i_{q|} &= a_{kl}\bar{\lambda}^k_{p|}\bar{\lambda}^l_{q|} = a^{kl}\lambda_{p|k}\lambda_{q|l} \end{aligned}\right\}. \quad \ldots \quad (4.5)$$

These are scalars symmetric in p, q. We have also the scalars

$$\left.\begin{aligned} \sum_t \lambda^i_{t|}\bar{\lambda}_{t|i} &= a_{ik}g^{ik} = a^{ik}\bar{g}_{ik} \\ \sum_t \lambda_{t|i}\bar{\lambda}^i_{t|} &= a_{ik}\bar{g}^{ik} = a^{ik}g_{ik} \end{aligned}\right\}.$$

Let the covariant derivative with respect to the Christoffel symbols formed from the a_{ij}'s be denoted by an ordinary bracket followed by indices and denote the coefficients of rotations of the two ennuples with respect to these symbols by

$$\alpha_{pqr} = (\lambda_{p|i})_j\lambda^i_{q|}\lambda^j_{r|}, \; \bar{\alpha}_{pqr} = (\bar{\lambda}_{p|i})_j\bar{\lambda}^i_{q|}\bar{\lambda}^j_{r|}.$$

Then from the identities

$$\begin{Bmatrix} t \\ ij \end{Bmatrix}_g = \begin{Bmatrix} t \\ ij \end{Bmatrix}_a + \tfrac{1}{2}g^{mt}\{(g_{im})_j+(g_{jm})_i-(g_{ij})_m\},$$

$$\begin{Bmatrix} t \\ ij \end{Bmatrix}_{\bar{g}} = \begin{Bmatrix} t \\ ij \end{Bmatrix}_a + \tfrac{1}{2}\bar{g}^{mt}\{(\bar{g}_{im})_j+(\bar{g}_{jm})_i-(\bar{g}_{ij})_m\}$$

we derive respectively the following identical relations connecting the coefficients of rotations:

$$\beta_{pqr} = \tfrac{1}{2}(\alpha_{pqr}+\alpha_{qrp}+\alpha_{rqp}-\alpha_{qpr}-\alpha_{prq}-\alpha_{rpq}) \quad \cdots \quad (4.6a)$$

$$\bar{\gamma}_{pqr} = \tfrac{1}{2}(\bar{\alpha}_{pqr}+\bar{\alpha}_{qrp}+\bar{\alpha}_{rqp}-\bar{\alpha}_{qpr}-\bar{\alpha}_{prq}-\bar{\alpha}_{rpq}) \quad \cdots \quad (4.6b)$$

where, as in §1, β_{pqr} and $\bar{\gamma}_{pqr}$ are Ricci's coefficients of rotations in the spaces V_n and $\bar{V}_n$, respectively.

Comparing the two eqns. (4.6*a*) and (4.6*b*) with the two eqns. (1.2*a*) and (1.2*b*), let us enquire what connections exist between γ_{pqr}, $\bar{\beta}_{pqr}$, α_{pqr} and $\bar{\alpha}_{pqr}$. For this purpose we have to take into consideration the relation (4.1) or (4.4). From the identity

$$\begin{Bmatrix} t \\ ij \end{Bmatrix}_{\bar{g}} = \begin{Bmatrix} t \\ ij \end{Bmatrix}_a + \tfrac{1}{2}\bar{g}^{mt}\left\{(\bar{g}_{im})_j+(\bar{g}_{jm})_i-(\bar{g}_{ij})_m\right\}$$

it follows that

$$\gamma_{pqr} = \alpha_{pqr} - \tfrac{1}{2}\lambda_{p|l}\sum_{s,t}\bar{\lambda}^l_{s|}\bar{\lambda}^m_{s|}\left\{\bar{\lambda}_{t|m}(\bar{\lambda}_{t|i})_j+\bar{\lambda}_{t|i}(\bar{\lambda}_{t|m})_j\right.$$
$$\left.+\bar{\lambda}_{t|m}(\bar{\lambda}_{t|j})_i+\bar{\lambda}_{t|j}(\bar{\lambda}_{t|m})_i-\bar{\lambda}_{t|i}(\bar{\lambda}_{t|j})_m-\bar{\lambda}_{t|j}(\bar{\lambda}_{t|i})_m\right\}\lambda^i_{q|}\lambda^j_{r|}. \quad \cdots \quad (4.7)$$

In order to reduce this we notice that $(a_{ij})_k = 0$.

Therefore, from eqn. (4.4) we have

$$\sum_t \left\{\lambda_{t|i}(\bar{\lambda}_{t|j})_k+\bar{\lambda}_{t|j}(\lambda_{t|i})_k\right\} = 0.$$

Multiplying by suitable λ's and $\bar{\lambda}$'s and summing for i, j, k we get the following results:

$$\left.\begin{aligned}
&(\bar{\lambda}_{t|i})_j\bar{\lambda}^i_{s|}\lambda^j_{r|}+\alpha_{str} = 0\\
&(\bar{\lambda}_{s|i})_j\lambda^i_{q|}\lambda^j_{r|}+\sum_u(\lambda^j_{q|}\bar{\lambda}_{u|j})\alpha_{usr} = 0\\
&(\lambda_{u|i})_j\lambda^i_{t|}\bar{\lambda}^j_{s|}+\bar{\alpha}_{tus} = 0\\
&(\bar{\lambda}_{t|i})_j\lambda^i_{r|}\bar{\lambda}^j_{s|}-\sum_u(\lambda^j_{r|}\bar{\lambda}_{u|j})\bar{\alpha}_{tus} = 0
\end{aligned}\right\}. \quad \cdots \quad (4.8)$$

Applying these results we get finally

$$\gamma_{pqr}-\alpha_{pqr} = \tfrac{1}{2}\sum_{s,t,u}(\lambda_{p|i}\bar{\lambda}^i_{s|})\left\{(\lambda^j_{q|}\bar{\lambda}_{t|j})(\alpha_{str}+\alpha_{tsr})\right.$$
$$\left.+(\lambda^j_{r|}\bar{\lambda}_{t|j})(\alpha_{stq}+\alpha_{tsq})+(\lambda^j_{q|}\bar{\lambda}_{t|j})(\lambda^k_{r|}\bar{\lambda}_{u|k})(\bar{\alpha}_{tus}+\bar{\alpha}_{uts})\right\}. \quad (4.9a)$$

Similarly,

$$\bar{\beta}_{pqr}-\bar{\alpha}_{pqr}=\tfrac{1}{2}\sum_{s,t,u}(\bar{\lambda}_{p|i}\lambda^{i}_{s|})\{(\bar{\lambda}^{j}_{q|}\lambda_{t|j})(\bar{\alpha}_{str}+\bar{\alpha}_{tsr})$$
$$+(\bar{\lambda}^{j}_{r|}\lambda_{t|j})(\bar{\alpha}_{stq}+\bar{\alpha}_{tsq})+(\bar{\lambda}^{j}_{q|}\lambda_{t|j})(\bar{\lambda}^{k}_{r|}\lambda_{u|k})(\alpha_{tus}+\alpha_{uts})\}. \qquad (4.9b)$$

These are the required equations; they involve the scalars as given in eqn. (4.5). Finally, to express the coefficients of rotations of Ricci, the expression on the l.h.s. of eqn. (4.9a) can be written by the help of eqn. (1.2b) or (4.6a), either as

$$\beta_{pqr}-\alpha_{pqr}+\tfrac{1}{2}(\gamma_{pqr}+\gamma_{qpr}+\gamma_{prq}+\gamma_{rpq}-\gamma_{qrp}-\gamma_{rqp})$$

or as

$$-\beta_{pqr}+\gamma_{pqr}-\tfrac{1}{2}(\alpha_{pqr}+\alpha_{qpr}+\alpha_{prq}+\alpha_{rpq}-\alpha_{qrp}-\alpha_{rqp})$$

and the expression on the l.h.s. of eqn. (4.9b) can be written by using eqn. (1.2a) or (4.6b), either as

$$\bar{\gamma}_{pqr}-\bar{\alpha}_{pqr}+\tfrac{1}{2}(\bar{\beta}_{pqr}+\bar{\beta}_{qpr}+\bar{\beta}_{prq}+\bar{\beta}_{rpq}-\bar{\beta}_{qrp}-\bar{\beta}_{rqp})$$

or as

$$-\bar{\gamma}_{pqr}+\bar{\beta}_{pqr}-\tfrac{1}{2}(\bar{\alpha}_{pqr}+\bar{\alpha}_{qpr}+\bar{\alpha}_{prq}+\bar{\alpha}_{rpq}-\bar{\alpha}_{qrp}-\bar{\alpha}_{rqp}).$$

When the expression on the l.h.s. of the above equations are replaced by these expressions, they express the coefficients of rotations of Ricci in terms of other coefficients.

The expressions for β_{lpqr} and $\bar{\gamma}_{lpqr}$ in the case of associate spaces are obtained from eqns. (1.4a) and (1.4b) when γ_{pqr} and $\bar{\beta}_{pqr}$ are replaced by their values from eqns. (4.9a) and (4.9b). The results become too big for incorporation here.

As in § 2, we can give geometrical interpretations of certain combinations of the α's. The combinations are $\alpha_{pqr}+\alpha_{qpr}$, $\alpha_{pqr}-\alpha_{qrp}$, $\alpha_{pqr}+\alpha_{qrp}+\alpha_{rpq}$ and $\alpha_{pqr}-\alpha_{prq}$. As in the article mentioned, the quantities α_{pqr} are the coefficients of rotation of the ennuple $\lambda_{p|i}$ with respect to the Levi-Civita parallelism as obtains in a space V'_n, of which a_{ij} is the fundamental tensor. Similarly, $\bar{\alpha}_{pqr}$ are the coefficients of rotations of the ennuple $\bar{\lambda}_{p|i}$ with respect to the Levi-Civita parallelism as obtains in V'_n. It can be shown that $\alpha_{pqr}+\alpha_{qrp}$ is the change in the scalar product in V_n of the contravariant vectors $\lambda_{p|}$, $\lambda_{q|}$ due to the Levi-Civita parallel transport as obtains in V'_n along $\lambda_{r|}$, that $\alpha_{pqr}-\alpha_{qpr}$ are the coefficients of rotation of the ennuple $\lambda_{p|i}$ with respect to the parallelism $\left\{ {t \atop ij} \right\}_a+\sum_s g^{tl}\lambda_{s|i}(\lambda_{s|l})_j$, that $\alpha_{pqr}+\alpha_{qrp}+\alpha_{rpq}$ are the coefficients of rotation of $\lambda_{p|i}$ with respect to the parallelism $\left\{ {t \atop ij} \right\}_a-g^{tl}(g_{ij})_l$ and that $\alpha_{pqr}-\alpha_{prq}$ are the coefficients of rotation of $\lambda_{p|i}$ with respect to the parallelism $\left\{ {t \atop ij} \right\}_a+\sum_s \lambda^{t}_{s|}(\lambda_{s|j})_i$. Geometrical interpretations of similar combinations of the $\bar{\alpha}$'s can also be given.

Again we have

$$\sum_p \bar{\lambda}^t_{p|} \frac{\partial \bar{\lambda}_{p|i}}{\partial x^j} = a^{tm} \frac{\partial a_{im}}{\partial x^j} + a^{tl} a_{im} \sum_p \lambda_{p|l} \frac{\partial \lambda^m_{p|}}{\partial x^j}.$$

If we write

$$\bar{\Gamma}^t_{ij} = \sum_p \bar{\lambda}^t_{p|} \frac{\partial \bar{\lambda}_{p|i}}{\partial x^j}, \quad \Gamma^t_{ij} = \sum_p \lambda^t_{p|} \frac{\partial \lambda_{p|i}}{\partial x^j},$$

then the above result gives

$$\bar{\Gamma}^t_{ij} = a^{tm} \frac{\partial a_{im}}{\partial x^j} - a^{tm} a_{il} \Gamma^l_{mj}.$$

Similarly,

$$\Gamma^t_{ij} = a^{tm} \frac{\partial a_{im}}{\partial x^j} - a^{tm} a_{il} \bar{\Gamma}^l_{mj}.$$

Subtracting, we get

$$a_{tj}(\bar{\Gamma}^t_{ik} - \Gamma^t_{ik}) = a_{ti}(\bar{\Gamma}^t_{jk} - \Gamma^t_{jk}). \quad \dots \quad \dots \quad \dots \quad (4.10)$$

Also

$$\bar{\Gamma}^t_{ij} + \Gamma^t_{ji} = a^{tm}\left(\frac{\partial a_{im}}{\partial x^j} + \frac{\partial a_{jm}}{\partial x^i}\right) - a^{tm}(a_{il}\Gamma^l_{mj} + a_{jl}\bar{\Gamma}^l_{mi}).$$

But,

$$\frac{\partial a_{ij}}{\partial x^m} = \frac{\partial}{\partial x^m} \sum_p (\lambda_{p|i} \bar{\lambda}_{p|j}) = a_{is}\bar{\Gamma}^s_{jm} + a_{js}\Gamma^s_{im} = a_{is}\Gamma^s_{jm} + a_{js}\bar{\Gamma}^s_{im}.$$

Therefore,

$$\tfrac{1}{2}(\bar{\Gamma}^t_{ij} + \Gamma^t_{ji}) = \left\{ {t \atop ij} \right\}_a + \tfrac{1}{2} a^{tm} \left\{ a_{is}(\Gamma^s_{jm} - \Gamma^s_{mj}) + a_{js}(\bar{\Gamma}^s_{im} - \bar{\Gamma}^s_{mi}) \right\}. \quad \dots \quad (4.11a)$$

Similarly,

$$\tfrac{1}{2}(\Gamma^t_{ij} + \bar{\Gamma}^t_{ji}) = \left\{ {t \atop ij} \right\}_a + a^{tm} \left\{ a_{is}(\bar{\Gamma}^s_{jm} - \bar{\Gamma}^s_{mj}) + a_{js}(\Gamma^s_{im} - \Gamma^s_{mi}) \right\}. \quad \dots \quad (4.11b)$$

Let us now consider a special case of associate spaces by choosing the ennuple $\lambda^i_{p|}$ in a special manner. We know that if all the roots of the determinantal equation $|a_{ij} - \rho g_{ij}| = 0$ are real (non-zero) and simple, then corresponding to each root ρ_h the equations

$$(a_{ij} - \rho_h g_{ij})\lambda^i_{h|} = 0$$

define a vector $\lambda^i_{h|}$ which may be taken as a unit vector. These vectors, therefore, define an orthogonal ennuple in V_n which give the principal directions determined by the tensor a_{ij}. In this case

$$a_{ij} = \sum_t \rho_t \lambda_{t|i} \lambda_{t|j} \text{ and so } a_{ij}\lambda^j_{h|} = \rho_h \lambda_{h|i}.$$

We now choose

$$\bar{\lambda}_{h|i} = \rho_h \lambda_{h|i}, \text{ so } \bar{\lambda}^i_{h|} = \frac{1}{\rho_h} \lambda^i_{h|}.$$

Then $\bar{\alpha}_{pqr}$, $\bar{\beta}_{pqr}$, $\bar{\gamma}_{pqr}$ can be expressed in terms of α_{pqr}, β_{pqr}, γ_{pqr} and other scalars involving ρ_h and their derivatives.

The eqns. (4.5) now reduce to

$$\lambda^{i}_{p|}\bar{\lambda}_{q|i} = \rho_q \delta_{pq} = \rho_p \delta_{pq} \quad \text{(no summation)}$$

$$\lambda_{p|i}\bar{\lambda}^{i}_{q|} = \frac{1}{\rho_q}\delta_{pq} = \frac{1}{\rho_p}\delta_{pq} \quad \text{(no summation)}.$$

Therefore, the eqns. (4.9*a*) and (4.9*b*) reduce to

$$\gamma_{pqr} - \alpha_{pqr} = \tfrac{1}{2}\frac{1}{\rho_p}\left\{\rho_q(\alpha_{pqr}+\alpha_{qpr}) + \rho_r(\alpha_{prq}+\alpha_{rpq}) + \rho_q\rho_r(\bar{\alpha}_{qrp}+\bar{\alpha}_{rqp})\right\} \qquad (4.12a)$$

$$\bar{\beta}_{pqr} - \bar{\alpha}_{pqr} = \tfrac{1}{2}\rho_p\left\{\frac{1}{\rho_q}(\bar{\alpha}_{pqr}+\bar{\alpha}_{qpr}) + \frac{1}{\rho_r}(\bar{\alpha}_{prq}+\bar{\alpha}_{rpq}) + \frac{1}{\rho_q\rho_r}(\alpha_{qrp}+\alpha_{rqp})\right\}. \qquad (4.12b)$$

References

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