ELECTROMAGNETIC BEHAVIOUR IN SPACE-TIMES CONFORMAL TO SOME WELL-KNOWN EMPTY SPACE-TIMES

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Space-times of vanishing scalar curvature $\bar{R}$ conformal to empty spaces are considered. The solutions of the equation $\bar{R} = 0$, which is a necessary condition for the field to be electromagnetic, are discussed in two stages. In a particular case it is shown that the conformal space-time is null electromagnetic if a certain vector associated with it is null and the empty space is self-conjugate or flat. When the vector is time-like the conformal space is one of disordered radiation and the empty space is necessarily flat. The general solution is discussed for some well-known empty fields including flat space. It is shown that electromagnetic fields exist when the empty space is either flat or one of plane gravitational wave. It is further found that space-times conformal to empty spaces of spherical, cylindrical and static axial symmetries do not represent electromagnetic fields when the corresponding symmetry is preserved. An interesting feature emerging from these solutions is that the square-root of the conformal factor propagates with fundamental velocity.

Introduction

The conformal curvature tensor C_{klmn} obtains its importance in general relativity through Pirani's (1962) formulation of gravitational radiation in empty and non-empty space-times. The metric tensors g_{lm} and $\bar{g}_{lm}$ of two conformal space-times V_4 and $\bar{V}_4$ are related by the equation

$$\bar{g}_{lm} = e^{2\sigma} g_{lm}, \qquad \qquad (1)$$

where σ is any scalar function of coordinates. From equation (1) arises the necessary and sufficient condition that (Eisenhart 1949)

$$\bar{C}_{klmn} = C_{klmn}, \qquad \qquad (2)$$

where

$$C_{klmn} = R_{klmn} + \tfrac{1}{2}(g_{km}R_{ln} - g_{kn}R_{lm} + g_{ln}R_{km} - g_{lm}R_{kn}) + \tfrac{1}{6}R(g_{kn}g_{lm} - g_{km}g_{ln}). \quad (3)$$

Because of this identity of the two conformal curvature tensors, it follows that gravitational space-times which are conformal to each other will have similar radiative behaviour. From this point of view a study of Riemannian fourfolds which are conformal to certain well-known gravitational fields will be of special interest.

If V_4 is an empty space, i.e. if $R_{lm} = 0$, we have

$$C_{klmn} = R_{klmn},$$

which implies that in an empty space the gravitational features are completely characterized by the conformal curvature tensor. For $\bar{V}_4$ the following possibilities arise:

(i) $\bar{R}_{klmn} = 0$, (ii) $\bar{R}_{lm} = 0$, (iii) $\bar{R}_{lm} \neq 0$, $\bar{R} = 0$, (iv) $\bar{R}_{lm} \neq 0$, $\bar{R} \neq 0$.

Case (i): This case is of no gravitational consequence.

Case (ii): If $\bar{R}_{lm} = 0$ we get

$$\bar{R}_{klmn} = \bar{C}_{klmn}$$

which leads to

$$\bar{R}_{klmn} = R_{klmn}$$

implying that V_4 and $\bar{V}_4$ are gravitationally identical. Hence if an empty space is conformal to another empty space it is identical with the latter. In particular an empty space cannot be conformal to flat space unless it is itself flat. The value of σ which carries over V_4 into an identical empty space is given by

$$\sigma_{;lm} - \sigma_{;l}\sigma_{;m} + \tfrac{1}{2}g_{lm}g^{rs}\sigma_{;r}\sigma_{;s} = 0. \qquad (4)$$

Here a suffix following a semi-colon indicates covariant differentiation with respect to the metric g_{lm}.

Case (iii): When $\bar{R}_{lm} \neq 0$ the material distribution for $\bar{V}_4$ is given by

$$-8\pi\bar{T}_{lm} = 2(\sigma_{;lm} - \sigma_{;l}\sigma_{;m}) - g_{lm}g^{rs}(2\sigma_{;rs} + \sigma_{;r}\sigma_{;s}) \qquad (5)$$

which on contraction leads to

$$4\pi\bar{T} = 3e^{-2\sigma}g^{rs}(\sigma_{;rs} + \sigma_{;r}\sigma_{;s}). \qquad (6)$$

Also, since $\bar{R} = 0$, we have

$$g^{rs}(\sigma_{;rs} + \sigma_{;r}\sigma_{;s}) = 0 \qquad (7)$$

or

$$(g^{rs}e^{\sigma})_{;rs} = 0 \qquad (8)$$

which is the necessary condition that $\bar{V}_4$ be an electromagnetic field. This together with

$$\bar{R}_{lm}\bar{R}^{mn} - \tfrac{1}{4}\bar{R}_{km}\bar{R}^{km}\delta^n_l = 0 \qquad (9)$$

and

$$\alpha_{l;m} - \alpha_{m;l} = 0, \qquad (10)$$

where

$$\alpha_l = \frac{\sqrt{-g}\,\epsilon_{lmnp}g^{nr}R^m_{s;r}R^{sp}}{R^k_h R^h_k} \qquad (11)$$

gives sufficient conditions when the field is non-null (Wheeler 1961). If the field be null electromagnetic we must have

$$\bar{R}_{lm}\bar{R}^{mn} = 0. \qquad \qquad (12)$$

The equation (8) is a wave equation in curved space and has solutions for given boundary conditions. The quantity e^{σ} is associated with the deviation of the new metric from the old one and the condition (8) shows that it is propagated with fundamental velocity.

In the following, we investigate the existence of electromagnetic fields which are conformal to empty space-time fields. As a particular case, electromagnetic fields conformal to flat space-time have been discussed. The solution of equation (7) has been discussed in two stages. In the first instance, when the expression in the parentheses vanishes, we obtain that $\bar{V}_4$ leads to a null electromagnetic field if $\sigma_{;l}$ is a null vector and that it represents a field of disordered radiation if $\sigma_{;l}$ is time-like. The integrability condition requires that V_4 be self-conjugate (Roy and Radhakrishna 1963) when $\bar{V}_4$ is null electromagnetic, and flat when $\bar{V}_4$ is a field of disordered radiation. This result has been further generalized by assuming that

$$\sigma_{;lm} = \rho\sigma_{;l}\sigma_{;m}. \qquad \qquad (13)$$

This at once leads to $\bar{V}_4$ being a null electromagnetic field when $\rho \neq \pm 1$. The general solution of (7) has been discussed when V_4 is—

(i) flat,
(ii) a field of plane gravitational wave,
(iii) a Schwarzschild exterior field,
(iv) a cylindrically symmetric empty field, and
(v) an axially symmetric empty field.

In the last four cases, conformal space-time $\bar{V}_4$ is taken to be having a symmetry same as that of V_4. It is found that in the first two cases conformal electromagnetic fields exist and that in the last three cases no such electromagnetic fields are possible. The solutions (29) and (43) obtained in the sequel have singularities in the finite region of space-time; and as we proceed to infinity the metric shrinks and finally vanishes indicating that there is an impassable barrier at infinity. Hence, if at all such electromagnetic fields physically exist, they will be valid only within a finite region of space-time excluding the singularities; and they will have to be continued with an appropriate field of a different kind. The electromagnetic field tensors F_{ij} have been calculated and it is found that the fields represented by (29) and (43) are uniform electromagnetic fields. These metrics differ from other well-known space-time metrics of electromagnetic or gravitational fields in the sense that, in the latter cases when the sources are annihilated, a flat substratum of space-time remains behind as a residue. But here, in the cases (29) and (43), when the sources are removed the space-time is also annihilated. This is not unexpected, for here we have not formulated our problem from an analogy of the classical electromagnetic theory in which the Minkowskian space exists independently of the sources. The starting point in this investigation is the Rainich Equations of the 'already unified field' which have no classical analogue; and the question under investigation is to find if non-trivial solutions of these equations exist which are conformal to space-times of well-known gravitational behaviour. Here we find the most general solutions of these equations when the symmetries are specified. The significance of such solutions, as has already been pointed out, is that, if such electromagnetic fields are physically possible, their behaviour with regard to gravitational radiation is the same as that of the gravitational field to which they are conformal. Thus the space-time that is conformal to (49), discussed in section IV, represents one of plane gravitational wave. The Rainich geometries given by (29), (43) and (48) escape classification in respect of gravitational radiation according to Pirani criterion, as the Petrov-Pirani matrix vanishes identically. But a common feature of the solutions is that if the original space-time is suddenly deformed into a conformal space-time by certain disturbances, then a certain influence in the field is propagated with the fundamental velocity.

Particular Solution of $\bar{R} = 0$

As a particular solution of (7) we consider

$$\sigma_{lm;} + \sigma_{;l}\sigma_{;m} = 0. \qquad (14)$$

In this case

$$\bar{R}_{lp} = -4\sigma_{;l}\sigma_{;p} + g_{lp}g^{rs}\sigma_{;r}\sigma_{;s}. \qquad (15)$$

Substituting (15) in (12) when n is set equal to l we find that for a null electromagnetic field $g^{rs}\sigma_{;r}\sigma_{;s}$ must vanish. In that case $\bar{R}_{lp}$ takes the form

$$\bar{R}_{lp} = -4\sigma_{;l}\sigma_{;p}, \qquad (16)$$

$\sigma_{;l}$ being a null vector. If the field be non-null electromagnetic it is necessary that

$$\bar{R}_{lm}\bar{R}^{lm} \neq 0$$

which requires that

$$k^2 \equiv g^{rs}\sigma_{;r}\sigma_{;s} \neq 0.$$

The condition (9) then requires that

$$\sigma_{;l}\sigma_{;m} = \tfrac{1}{4}g_{lm}g^{rs}\sigma_{;r}\sigma_{;s}$$

which leads to

$$\bar{R}_{lm} = 0.$$

Hence the condition (14) is incompatible with a non-null electromagnetic field. The integrability condition of (14) requires that

$$\sigma_{;m} R^m_{kln} = 0. \qquad \ldots \qquad (17)$$

Hence if the conformal space be null electromagnetic the empty space is necessarily self-conjugate.

If $\sigma_{;l}$ is not a null vector but a time-like vector with magnitude k, viz.

$$\sigma_{;l} = u_l,$$

u_l being a unit vector, then the field is one of disordered radiation with pressure $k^2/8\pi$ and density $3k^2/8\pi$. The condition (17) then requires that R^m_{kln} should vanish, i.e. the empty space is necessarily flat.

The above discussion suggests the consideration of a relation more general than (14), viz.

$$\sigma_{;lm} = \rho\sigma_{;l}\sigma_{;m}, \qquad \ldots \qquad (18)$$

where ρ is a function of the coordinates. The case $\rho = -1$ has been discussed above. If $\rho = 1$ we find that $\bar{R}_{lm}$ vanishes. The integrability condition of (18) requires that

$$\sigma_{;m} R^m_{kln} + \sigma_{;k}(\rho_{;n}\sigma_{;l} - \rho_{;l}\sigma_{;n}) = 0. \qquad \ldots \qquad (19)$$

If ρ be a constant or a function of σ, then (19) reduces to (17). That is to say, the empty space has to be self-conjugate if the conformal space is electromagnetic and (18) satisfied. From (19) we find that

$$g^{lm}\sigma_{;l}\rho_{;m} = 0.$$

If $\sigma_{;l}$ is null $\rho_{;m}$ must be space-like and orthogonal to it. From (19) we also have

$$R_{klmn}R^{klmn} = 0,$$

$$R_{klmn}{}^*R^{klmn} = 0.$$

It may be noted that when $\rho = -2$ and $\sigma_{;l}$ is non-null, the field is one of incoherent matter.

Space-times Conformal to Flat Metric

When V_4 is flat equation (8) reduces to

$$\eta^{lm}(e^\sigma)_{,lm} = 0, \qquad \ldots \qquad (20)$$

where η_{lm} is the metric tensor of Minkowski space. Here a suffix succeeding a comma indicates ordinary partial differentiation. We consider the following cases:

Case (i):

$$e^\sigma = F(lx+my+nz-t) \text{ and } l^2+m^2+n^2 = 1. \qquad \ldots \qquad (21)$$

This leads to

$$\bar{R}_{pq} = 2\left\{\frac{d^2f}{d\tau^2} - \left(\frac{df}{d\tau}\right)^2\right\} l_p l_q, \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

where $l_p = (l, m, n, -1)$, $\tau = lx+my+nz-t$, and $f = \log F$.
The conformal space represents a null electromagnetic field unless $(d^2f/d\tau^2) -(df/d\tau)^2$ vanishes, in which case it becomes flat.

Case (ii): Let $\sigma = \sigma(r, t)$. Then

$$e^{\sigma} = \frac{1}{r}f(r-t)+\frac{1}{r}F(r+t). \qquad \ldots \quad \ldots \quad \ldots \quad (23)$$

Using spherical polar coordinates,

$$\bar{R}^1_1 = e^{-2\sigma}\left[\frac{-2(f_{uu}+F_{vv})}{f+F}+\frac{4\{(f_u)^2+f_uF_v+(F_v)^2\}}{(f+F)^2}-\frac{2}{r}\frac{f_u+F_v}{f+F}-\frac{1}{r^2}\right],$$

$$\bar{R}^4_1 = e^{-2\sigma}\left[\frac{-2(f_{uv}-F_{uv})}{f+F}+\frac{4\{(f_u)^2-(F_v)^2\}}{(f+F)^2}-\frac{2}{r}\frac{f_u-F_v}{f+F}\right],$$

$$\bar{R}^2_2 = \bar{R}^3_3 = e^{-2\sigma}\left[\frac{-4f_uF_v}{(f+F)^2}+\frac{1}{r^2}\right],$$

$$\bar{R}^4_4 = e^{2-\sigma}\left[\frac{2(f_{uu}+F_{vv})}{f+F}-\frac{4\{(f_u)^2-f_uF_v+(F_v)^2\}}{(f+F)^2}+\frac{2}{r}\frac{f_u+F_v}{f+F}-\frac{1}{r^2}\right], \qquad (24)$$

where $u = r-t$, $v = r+t$, $fu = df/du$, $fuu = d^2f/du^2$, etc.
For an electromagnetic field we have, from (9), $\bar{R}^l_1\bar{R}^4_l = 0$ which is equivalent to $(\bar{R}^1_1+\bar{R}^4_4)\bar{R}^4_1 = 0$.
Thus either

(*a*) $\bar{R}^4_1 \neq 0$, $\bar{R}^1_1+\bar{R}^4_4 = 0$,

or (*b*) $\bar{R}^4_1 = 0$.

In case of (*a*), we have by virtue of (24)

$$\frac{4f_uF_v}{(f+F)^2} = \frac{1}{r^2}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

This shows that the eigenvalues of $\bar{R}^m_l$ are zero. Equation (9) further requires that

$$\bar{R}^1_1\bar{R}^4_4 - \bar{R}^4_1\bar{R}^1_4 = 0$$

which leads to

$$\left[f_{uu}+\frac{1}{r}f_u-\frac{2}{f+F}(f_u)^2\right]\left[F_{vv}+\frac{1}{r}F_v-\frac{2}{f+F}(F_v)^2\right] = 0. \qquad \ldots \quad (26)$$

Equations (25) and (26) are not compatible.

In case of (*b*), since $\bar{R}^2_2 = \bar{R}^3_3$ we have $\bar{R}^1_1 = \bar{R}^4_4$ which gives

$$f_{uu}+F_{vv}-\frac{2}{f+F}\left\{(f_u)^2-(F_v)^2\right\}+\frac{1}{r}(f_u+F_v) = 0. \quad \ldots \quad \ldots \quad (27)$$

Also $\bar{R}^4_1 = 0$ leads to

$$f_{uu}-F_{vv}-\frac{2}{f+F}\left\{(f_u)^2-(F_v)^2\right\}+\frac{1}{r}(f_u-F_v) = 0. \quad \ldots \quad \ldots \quad (28)$$

Equations (27) and (28) have a non-trivial solution only when f and F are constants. The metric of $\bar{V}_4$ in that case has the form

$$ds^2 = \frac{a^2}{r^2}\left(-dr^2-r^2d\theta^2-r^2\sin^2\theta\, d\phi^2+dt^2\right). \quad \ldots \quad \ldots \quad (29)$$

The electromagnetic field tensor F_{ij} corresponding to the metric (29) has the non-vanishing components given by

$$F_{14} = \frac{A}{r^2}, \;\; F_{23} = B\sin\theta,$$

where $A^2+B^2 = \dfrac{a^2}{4\pi}$.

The electromagnetic field thus reduces to a wrench in the given coordinate system in the terminology of flat space-time. It has been verified that the tensor F_{ij} is a covariant constant, which implies that the electromagnetic field is uniform.

Case (iii): $\sigma = \sigma(\rho, t)$. Using cylindrical polar coordinates,

$$\bar{R}^1_1 = -e^{-2\sigma}\left\{2\frac{\partial^2\sigma}{\partial\rho^2}-\left(\frac{\partial\sigma}{\partial\rho}\right)^2-\left(\frac{\partial\sigma}{\partial t}\right)^2\right\},$$

$$\bar{R}^2_2 = -e^{-2\sigma}\left\{\left(\frac{\partial\sigma}{\partial\rho}\right)^2-\left(\frac{\partial\sigma}{\partial t}\right)^2\right\},$$

$$\bar{R}^3_3 = -e^{-2\sigma}\left\{\frac{2}{\rho}\frac{\partial\sigma}{\partial\rho}+\left(\frac{\partial\sigma}{\partial\rho}\right)^2-\left(\frac{\partial\sigma}{\partial t}\right)^2\right\},$$

$$\bar{R}^4_4 = -e^{-2\sigma}\left\{2\left(\frac{\partial^2\sigma}{\partial t^2}\right)-\left(\frac{\partial\sigma}{\partial\rho}\right)^2-\left(\frac{\partial\sigma}{\partial t}\right)^2\right\},$$

$$\bar{R}^1_4 = -2e^{-2\sigma}\left\{\frac{\partial^2\sigma}{\partial\rho\partial t}-\frac{\partial\sigma}{\partial\rho}\frac{\partial\sigma}{\partial t}\right\},$$

where

$$(\rho, z, \phi, t) \equiv (x^1, x^2, x^3, x^4).$$

Now

$$\bar{R}^k_1\bar{R}^4_k = 0, \text{ i.e. } \bar{R}^4_1(\bar{R}^1_1+\bar{R}^4_4) = 0.$$

Hence either

$$\bar{R}_1^4 = 0 \quad \text{or} \quad \bar{R}_1^4 \neq 0,\ \bar{R}_1^1 + \bar{R}_4^4 = 0.$$

If $\bar{R}_1^4 = 0$ we have

$$\frac{\partial^2 \sigma}{\partial \rho \partial t} = \frac{\partial \sigma}{\partial \rho}\frac{\partial \sigma}{\partial t} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

so that

$$e^{-\sigma} = f(\rho) + F(t). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

Hence

$$\bar{R}_1^1 = 2(f+F)f_{\rho\rho} - (f_\rho)^2 + (F_t)^2,$$

$$\bar{R}_2^2 = -(f_\rho)^2 + (F_t)^2,$$

$$\bar{R}_3^3 = \frac{2}{\rho}(f+F)f_\rho - (f_\rho)^2 + (F_t)^2,$$

$$\bar{R}_4^4 = -2(f+F)F_{tt} - (f_\rho)^2 + (F_t)^2.$$

For non-null electromagnetic field, we have the following alternatives:

(a) $f_{\rho\rho} = 0, \quad \frac{1}{\rho} f_\rho = F_{tt}, \quad 2(f+F)(f_{\rho\rho} - F_{tt}) - 2(f_\rho)^2 + 2(F_t)^2 = 0,$

(b) $f_{\rho\rho} = \frac{1}{\rho} f_\rho, \quad F_{tt} = 0, \quad 2(f+F)(f_{\rho\rho} - F_{tt}) - 2(f_\rho)^2 + 2(F_t)^2 = 0,$

(c) $f_{\rho\rho} + F_{tt} = 0, \quad f_\rho = 0, \quad 2(f+F)f_{\rho\rho} - 2(f_\rho)^2 + 2(F_t)^2 = 0.$

In case (a), $f = 0$, $F =$ constant and hence it is trivial.
In case (b), $F_t =$ constant. Hence

$$(f+a)f_{\rho\rho} - (f_\rho)^2 + a^2 = 0,\ F_t = a.$$

But

$$f_{\rho\rho} = \frac{1}{\rho} f_\rho, \text{ i.e. } f = b\rho^2 + c.$$

Thus

$$2(b\rho^2 + c + a)b - 4b^2\rho^2 + a^2 = 0$$

which requires that

$$a = 0,\ b = 0, \text{ i.e. } f = \text{constant},\ F = \text{constant}.$$

In case (c), $f =$ constant, $F =$ constant. Hence when $\bar{R}_1^4 = 0$ the field is not non-null electromagnetic. However, the field turns out to be one of disordered radiation when we choose

$$e^\sigma = a + bt. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

If $\bar{R}_1^4 \neq 0$ we must have

$$\bar{R}_1^1 + \bar{R}_4^4 = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

for an electromagnetic field. Since $\bar{R} = 0$ this requires that

$$\bar{R}_2^2 + \bar{R}_3^3 = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

and

$$\left(\bar{R}_2^2\right)^2 = \left(\bar{R}_1^1\right)^2 - \left(\bar{R}_4^1\right)^2. \qquad \dots \quad \dots \quad \dots \quad (35)$$

Equations (33) and (34) lead to

$$\frac{\partial^2 \sigma}{\partial \rho^2} - \frac{\partial^2 \sigma}{\partial t^2} = 0, \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (36)$$

and

$$\left(\frac{\partial \sigma}{\partial \rho}\right)^2 + \frac{1}{\rho}\frac{\partial \sigma}{\partial \rho} - \left(\frac{\partial \sigma}{\partial t}\right)^2 = 0. \qquad \dots \quad \dots \quad \dots \quad (37)$$

Equation (36) has the solution

$$\sigma = f(\rho+t) + F(\rho-t). \qquad \dots \quad \dots \quad \dots \quad \dots \quad (38)$$

Using the new variables

$$\mu = \rho+t, \ \nu = \rho-t$$

as independent variables, equation (37) can be expressed in the form

$$\frac{\partial \sigma}{\partial \mu} + \frac{\partial \sigma}{\partial \nu} + 2(\mu+\nu)\frac{\partial \sigma}{\partial \mu}\frac{\partial \sigma}{\partial \nu} = 0. \qquad \dots \quad \dots \quad \dots \quad (39)$$

Substituting from (38) in (39) we have

$$f_\mu + F_\nu + 2(\mu+\nu) f_\mu F_\nu = 0$$

which requires that

$$\frac{1}{f_\mu} + 2\mu = -\frac{1}{F_\nu} - 2\nu = m,$$

where m is a constant. Hence

$$f = a - \tfrac{1}{2}\log(m-2\mu), \ F = b - \tfrac{1}{2}\log(m+2\nu). \qquad \dots \quad \dots \quad (40)$$

Equation (35) leads to

$$\left\{\left(\frac{\partial \sigma}{\partial \rho}\right)^2 - \left(\frac{\partial \sigma}{\partial t}\right)^2\right\}^2 = 2\left\{\frac{\partial^2 \sigma}{\partial \rho^2} - \left(\frac{\partial \sigma}{\partial \rho}\right)^2 - \left(\frac{\partial \sigma}{\partial t}\right)^2\right\}^2 - 4\left\{\frac{\partial^2 \sigma}{\partial \rho \partial t} - \frac{\partial \sigma}{\partial \rho}\frac{\partial \sigma}{\partial t}\right\}^2. \qquad (41)$$

The equation (41) is satisfied by virtue of (38) and (40). Thus

$$\sigma = C - \tfrac{1}{2}\log(m-2\mu)(m+2\nu) \qquad \dots \quad \dots \quad \dots \quad (42)$$

and the metric for the non-null electromagnetic field turns out to be of the form

$$ds^2 = \frac{A}{(t-B)^2 - \rho^2}\left(-d\rho^2 - dz^2 - \rho^2 d\phi^2 + dt^2\right). \qquad \dots \quad \dots \quad (43)$$

The electromagnetic field tensor F_{ij} for the metric (43) has the non-vanishing components given by

$$F_{12} = \frac{M\rho}{\{(t-B)^2 - \rho^2\}^{3/2}}, \quad F_{13} = \frac{N\rho(t-B)}{\{(t-B)^2 - \rho^2\}^{3/2}},$$

$$F_{24} = \frac{M(t-B)}{\{(t-B)^2 - \rho^2\}^{3/2}}, \quad F_{34} = \frac{N\rho^2}{\{(t-B)^2 - \rho^2\}^{3/2}},$$

where

$$M^2+N^2 = \frac{A}{4\pi}.$$

In this case also $F_{ij;k} = 0$, i.e. the electromagnetic field is uniform.

Case (iv): $\sigma = \sigma(\rho, z)$. Using cylindrical polar coordinates,

$$\bar{R}^1_1 = -e^{-2\sigma}\left[2\frac{\partial^2\sigma}{\partial\rho^2}-\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\left(\frac{\partial\sigma}{\partial z}\right)^2\right],$$

$$\bar{R}^2_2 = -e^{-2\sigma}\left[2\frac{\partial^2\sigma}{\partial z^2}+\left(\frac{\partial\sigma}{\partial\rho}\right)^2-\left(\frac{\partial\sigma}{\partial z}\right)^2\right],$$

$$\bar{R}^3_3 = -e^{-2\sigma}\left[\frac{2}{\rho}\frac{\partial\sigma}{\partial\rho}+\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\left(\frac{\partial\sigma}{\partial z}\right)^2\right],$$

$$\bar{R}^4_4 = -e^{-2\sigma}\left[\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\left(\frac{\partial\sigma}{\partial z}\right)^2\right],$$

$$\bar{R}^1_2 = -2e^{-2\sigma}\left[\frac{\partial^2\sigma}{\partial\rho\,\partial z}-\frac{\partial\sigma}{\partial\rho}\frac{\partial\sigma}{\partial z}\right].$$

As in case (iii) for an electromagnetic field, either

$$\bar{R}^1_2 = 0$$

or

$$\bar{R}^1_2 \neq 0, \quad \bar{R}^1_1+\bar{R}^2_2 = 0.$$

If $\bar{R}^1_2 = 0$ it is easily seen that the field is neither electromagnetic nor one of disordered radiation. If $\bar{R}^1_1+\bar{R}^2_2 = 0$ we must have $\bar{R}^3_3+\bar{R}^4_4 = 0$. Hence

$$\frac{\partial^2\sigma}{\partial\rho^2}+\frac{\partial^2\sigma}{\partial z^2} = 0, \quad \ldots \quad (44)$$

and

$$\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\frac{1}{\rho}\frac{\partial\sigma}{\partial\rho}+\left(\frac{\partial\sigma}{\partial z}\right)^2 = 0. \quad \ldots \quad (45)$$

Equation (44) shows that σ is a harmonic function.
Hence

$$\sigma = f(\rho+iz)+F(\rho-iz). \quad \ldots \quad (46)$$

From (45) and (46) we find that

$$\sigma = a-\tfrac{1}{2}\log\{\rho^2+(z+n)^2\}. \quad \ldots \quad (47)$$

For an electromagnetic field we also have

$$(\bar{R}^4_4)^2 = (\bar{R}^1_1)^2+(\bar{R}^1_2)^2$$

which leads to

$$\left[\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\left(\frac{\partial\sigma}{\partial z}\right)^2\right]^2 = \left[2\frac{\partial^2\sigma}{\partial\rho^2}-\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\left(\frac{\partial\sigma}{\partial z}\right)^2\right]^2+4\left[\frac{\partial^2\sigma}{\partial\rho\,\partial z}-\frac{\partial\sigma}{\partial\rho}\frac{\partial\sigma}{\partial z}\right]^2.$$

The last equation is easily seen to be satisfied for σ given by (47). The metric thus takes the form

$$ds^2 = -\frac{A}{\rho^2+(z+n)^2}(-dt^2+d\rho^2+dz^2+\rho^2\,d\phi^2) \quad \ldots \quad \ldots \quad (48)$$

which can be reduced to the spherically symmetric form

$$ds^2 = -\frac{A}{r^2}(-dt^2+dx^2+dy^2+dz^2)$$

by a translation in z and which thus is identical with (29). It is verified that the solutions (29) and (43) satisfy the conditions (10).

Space-time Conformal to that of Plane Gravitational Wave

The field of plane gravitational wave is described by the metric

$$ds^2 = dt^2-dx^2-e^{\alpha}dy^2-e^{\beta}dz^2, \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

where α and β are functions of $x-t$. The condition that (49) is an empty space leads to

$$\left.\begin{array}{r}\alpha_{\lambda\lambda}+\beta_{\lambda\lambda}+\frac{1}{2}\{(\alpha_\lambda)^2+(\beta_\lambda)^2\} = 0\\ \alpha_{\lambda\lambda}+\frac{1}{2}(\alpha_\lambda)^2 \neq 0\end{array}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

where $\lambda = x-t$.

The condition (8) leads to

$$4\frac{\partial^2(e^\sigma)}{\partial\lambda\partial\tau} = \frac{\partial(e^\sigma)}{\partial\tau}(\alpha_\lambda+\beta_\lambda). \quad \tau = x+t \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

which has the solution

$$e^\sigma = e^{1/4(\alpha+\beta)}[f(\lambda)+F(\tau)]. \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

The non-vanishing components of $\bar{R}^m_l$ are

$$\bar{R}^1_1 = e^{-2\sigma}\left[-\tfrac{1}{2}(\alpha_{\lambda\lambda}+\beta_{\lambda\lambda})+\tfrac{1}{8}(\alpha_\lambda+\beta_\lambda)^2-\frac{2(f_{\lambda\lambda}+F_{\tau\tau})}{f+F}\right.$$
$$\left.+\frac{4\{(f_\lambda)^2+f_\lambda F_\tau+(F_\tau)^2\}}{(f+F)^2}+\frac{(\alpha_\lambda+\beta_\lambda)f_\lambda}{f+F}\right],$$

$$\bar{R}^4_1 = e^{-2\sigma}\left[-\tfrac{1}{2}(\alpha_{\lambda\lambda}+\beta_{\lambda\lambda})+\tfrac{1}{8}(\alpha_\lambda+\beta_\lambda)^2-\frac{2(f_{\lambda\lambda}-F_{\tau\tau})}{f+F}\right.$$
$$\left.+\frac{4\{(f_\lambda)^2-(F_\tau)^2\}}{(f+F)^2}+\frac{(\alpha_\lambda+\beta_\lambda)f_\lambda}{f+F}\right],$$

$$\bar{R}^2_2 = e^{-2\sigma}\left[-\frac{2F_\tau\alpha_\lambda}{f+F}-\frac{4F_\tau f_\lambda}{(f+F)^2}\right],$$

$$\bar{R}^3_3 = e^{-2\sigma}\left[-\frac{2F_\tau\beta_\lambda}{f+F}-\frac{4F_\tau f_\lambda}{(f+F)^2}\right].$$

$$\bar{R}^4_4 = e^{-2\sigma}\left[\tfrac{1}{2}(\alpha_{\lambda\lambda}+\beta_{\lambda\lambda})-\tfrac{1}{8}(\alpha_\lambda+\beta_\lambda)^2+\frac{2(f_{\lambda\lambda}+F_{\tau\tau})}{f+F}\right.$$
$$\left.-\frac{4\{(f_\lambda)^2-f_\lambda F_\tau+(F_\tau)^2\}}{(f+F)^2}-\frac{(\alpha_\lambda+\beta_\lambda)f_\lambda}{f+F}\right]. \quad (53)$$

For an electromagnetic field we have either $\bar{R}_1^4 \neq 0$ or $\bar{R}_1^4 = 0$.
In the former case we have

$$\bar{R}_1^1 + \bar{R}_4^4 = 0 \quad \text{and} \quad \bar{R}_2^2 + \bar{R}_3^3 = 0$$

which lead to

$$f_\lambda F_\tau = 0$$

and

$$F_\tau\left(\alpha_\lambda + \beta_\lambda + \frac{2f_\lambda}{f+F}\right) = 0.$$

Hence $F_\tau = 0$. The field in this case turns out to be null electromagnetic.

If $\bar{R}_1^4 = 0$ we must have $\bar{R}_1^1 - \bar{R}_4^4 = 0$ and $\bar{R}_2^2 - \bar{R}_3^3 = 0$. The latter condition leads to

$$\alpha_\lambda = \beta_\lambda$$

which violates the conditions (50). Hence $\bar{R}_1^4$ cannot vanish if the field is to be electromagnetic. The only possible electromagnetic field is obtained when σ is a function of $x-t$ alone and in that case the resulting field is null electromagnetic.

Space-times Conformal to Some Well-known Empty Space-times

Case (i): The Schwarzschild metric—

$$ds^2 = -e^{-\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^\lambda dt^2,\; e^\lambda = 1 - \frac{2m^2}{r}. \qquad \text{(54)}$$

Let $\sigma = \sigma(r, t)$ so that the conformal metric is also spherically symmetric. The eigenvalues of $\bar{R}_{ij}$ are $\bar{R}_2^2$, $\bar{R}_3^3$ and the roots of the equation

$$\lambda^2 - \lambda(\bar{R}_1^1 + \bar{R}_4^4) + \bar{R}_1^1\bar{R}_4^4 - \bar{R}_4^1\bar{R} = 0.$$

For a null electromagnetic field

$$\bar{R}_1^1 + \bar{R}_4^4 = 0,$$

$$\bar{R}_1^1\bar{R}_4^4 - \bar{R}_4^1\bar{R}_1^4 = 0,$$

$$\bar{R}_2^2 = \bar{R}_3^3 = 0.$$

These lead to the equations

$$2e^\lambda \frac{\partial\sigma}{\partial r} + re^\lambda\left(\frac{\partial\sigma}{\partial r}\right)^2 - re^{-\lambda}\left(\frac{\partial\sigma}{\partial t}\right)^2 = 0, \qquad \text{(55)}$$

$$\frac{\partial^2\sigma}{\partial r^2} + \left(\frac{\partial\sigma}{\partial r}\right)^2 + \left(\frac{d\lambda}{dr} + \frac{2}{r}\right)\frac{\partial\sigma}{\partial r} - e^{-2\lambda}\left\{\frac{\partial^2\sigma}{\partial t^2} + \left(\frac{\partial\sigma}{\partial t}\right)^2\right\} = 0, \qquad \text{(56)}$$

$$\left\{\frac{\partial^2 e^{-\sigma}}{\partial r^2} + e^{-2\lambda}\frac{\partial^2 e^{-\sigma}}{\partial t^2}\right\}^2 = e^{-2\lambda}\left\{2\frac{\partial^2 e^{-\sigma}}{\partial\rho\partial t} - \frac{d\lambda}{dr}\frac{\partial e^{-\sigma}}{\partial t}\right\}^2. \qquad \text{(57)}$$

Equation (55) gives

$$\sigma = at - \log r \pm \int \frac{1}{r} \sqrt{1+a^2r^2e^{-2\lambda}}\, dr. \qquad \ldots \quad \ldots \quad (58)$$

This value of σ does not satisfy (56) and (57). For non-null electromagnetic field we have either

$$\bar{R}_1^4 = 0$$

or

$$\bar{R}_1^4 \neq 0,\ \bar{R}_1^1 + \bar{R}_4^4 = 0.$$

In the first case

$$\bar{R}_1^1 = \bar{R}_4^4 = -\bar{R}_2^2 = -\bar{R}_3^3$$

which lead to

$$\frac{\partial^2\sigma}{\partial r^2} - \left(\frac{\partial\sigma}{\partial r}\right)^2 + \frac{1}{2}\frac{d\lambda}{dr}(1-e^{-2\lambda})\frac{\partial\sigma}{\partial r} + e^{-2\lambda}\left\{\frac{\partial^2\sigma}{\partial t^2} - \left(\frac{\partial\sigma}{\partial t}\right)^2\right\} = 0, \qquad \ldots \quad (59)$$

and

$$\frac{\partial^2\sigma}{\partial r^2} + \frac{1}{2}\left(\frac{d\lambda}{dr} + \frac{2}{r}\right)\frac{\partial\sigma}{\partial r} - e^{-2\lambda}\left(\frac{\partial\sigma}{\partial t}\right)^2 = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

Also $\bar{R} = 0$ leads to (56). Equations (56) and (59) give

$$2\frac{\partial^2\sigma}{\partial r^2} + \left(\frac{3}{2}\frac{d\lambda}{dr} + \frac{2}{r} - \tfrac{1}{2}e^{-2\lambda}\frac{d\lambda}{dr}\right)\frac{\partial\sigma}{\partial r} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (61)$$

which on integration gives

$$\frac{\partial\sigma}{\partial r} = \frac{L(t)}{r}\left(1 - \frac{2m}{r}\right)^{-\frac{3}{4}} e^{-\frac{1}{4}\left(1-\frac{2m}{r}\right)^{-2}}. \qquad \ldots \quad \ldots \quad (62)$$

From (60) and (62) we get

$$\frac{\partial\sigma}{\partial t} = \frac{1}{2}\left\{\frac{L(t)}{r}\right\}^{\frac{1}{2}}\left(1 - \frac{2m}{r}\right)^{-\frac{3}{8}} e^{-\frac{1}{16}\left(1-\frac{2m}{r}\right)^{-2}} \sqrt{(1-e^2\lambda)\frac{d\lambda}{dr}}. \qquad \ldots \quad (63)$$

It is easily verified that the integrability condition is not satisfied.

If $\bar{R}_1^4 \neq 0$ we have $\bar{R}_1^1 + \bar{R}_4^4 = 0$. This requires that $\bar{R}_2^2 = \bar{R}_3^3 = 0$ so that the field ceases to be non-null. We have already seen that null electromagnetic field is not possible in this case.

Case (ii): Einstein-Rosen metric—

$$ds^2 = -e^{2\gamma-2\psi}(d\rho^2 - dt^2) - e^{2\psi}dz^2 - \rho^2e^{-2\psi}d\phi^2. \qquad \ldots \quad \ldots \quad (64)$$

$$\gamma = \gamma(\rho, t), \quad \psi = \psi(\rho, t).$$

The functions ψ and γ satisfy the equations

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2\psi}{\partial t^2} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (65)$$

$$\frac{\partial\gamma}{\partial\rho} = \rho\left[\left(\frac{\partial\psi}{\partial\rho}\right)^2 + \left(\frac{\partial\psi}{\partial t}\right)^2\right], \qquad \ldots \quad \ldots \quad \ldots \quad (66)$$

$$\frac{\partial\gamma}{\partial t} = 2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\gamma}{\partial t}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

Let $\sigma = \sigma(\rho, t)$. For a null electromagnetic field, since the eigenvalues of $\bar{R}_i^j$ are all zero, we get

$$2\frac{\partial\sigma}{\partial\rho}\frac{\partial\psi}{\partial\rho} - 2\frac{\partial\sigma}{\partial t}\frac{\partial\psi}{\partial t} + \left(\frac{\partial\sigma}{\partial\rho}\right)^2 - \left(\frac{\partial\sigma}{\partial t}\right)^2 = 0, \quad \ldots \quad (68)$$

$$2\frac{\partial\sigma}{\partial\rho}\frac{\partial\psi}{\partial\rho} - 2\frac{\partial\sigma}{\partial t}\frac{\partial\psi}{\partial t} - \left(\frac{\partial\sigma}{\partial\rho}\right)^2 + \left(\frac{\partial\sigma}{\partial t}\right)^2 = \frac{2}{\rho}\frac{\partial\sigma}{\partial\rho}, \quad \ldots \quad (69)$$

$$\frac{\partial^2\sigma}{\partial\rho^2} + \left(\frac{\partial\sigma}{\partial\rho}\right)^2 + \frac{1}{\rho}\frac{\partial\sigma}{\partial\rho} - \frac{\partial^2\sigma}{\partial t^2} - \left(\frac{\partial\sigma}{\partial t}\right)^2 = 0. \quad \ldots \quad (70)$$

Hence

$$\frac{\partial^2\sigma}{\partial\rho^2} - \frac{\partial^2\sigma}{\partial t^2} = 0 \quad \ldots \quad (71)$$

and

$$\left(\frac{\partial\sigma}{\partial\rho}\right)^2 + \frac{1}{\rho}\frac{\partial\sigma}{\partial\rho} - \left(\frac{\partial\sigma}{\partial t}\right)^2 = 0 \quad \ldots \quad (72)$$

which have the solution (42). (68) gives

$$2\rho\frac{\partial\psi}{\partial\rho} - (m-2t)\frac{\partial\psi}{\partial t} = 1 \quad \ldots \quad (73)$$

which has the solution

$$\psi = \tfrac{1}{4}\log\rho\left(t - \frac{m}{2}\right) + f\left(\frac{\rho}{t - \frac{m}{2}}\right), \quad \ldots \quad (74)$$

where f satisfies the equation

$$(1-u^2)\frac{d^2f}{du^2} + \frac{1-2u^2}{u}\frac{df}{du} + \tfrac{1}{4} = 0 \quad \ldots \quad (75)$$

by virtue of (65), u being the variable $\dfrac{\rho}{t - \frac{m}{2}}$.

Substituting the values of σ and ψ in the equation

$$(\bar{R}_{11})^2 = (\bar{R}_{14})^2$$

we find that it is not satisfied.

For a non-null field either $\bar{R}_1^4 = 0$ or $\bar{R}_1^4 \neq 0$, $\bar{R}_1^1 + \bar{R}_4^4 = 0$. If $\bar{R}_1^4 = 0$ we must have

(a) $\bar{R}_1^1 = \bar{R}_4^4$, $\bar{R}_2^2 = \bar{R}_3^3$, and $\bar{R}_1^1 = -\bar{R}_2^2$

or

(b) $\bar{R}_1^1 = -\bar{R}_4^4$, $\bar{R}_2^2 = -\bar{R}_3^3$, and $\bar{R}_1^1 = \bar{R}_2^2$.

Case (a) leads to

$$\frac{\partial^2\sigma}{\partial\rho^2}+\frac{\partial^2\sigma}{\partial t^2}-2\frac{\partial\sigma}{\partial\rho}\left(\frac{\partial\gamma}{\partial\rho}-\frac{\partial\psi}{\partial\rho}\right)-2\frac{\partial\sigma}{\partial t}\left(\frac{\partial\gamma}{\partial t}-\frac{\partial\psi}{\partial t}\right)-\left(\frac{\partial\sigma}{\partial\rho}\right)^2-\left(\frac{\partial\sigma}{\partial t}\right)^2=0,$$

$$2\frac{\partial\sigma}{\partial\rho}\frac{\partial\psi}{\partial\rho}-2\frac{\partial\sigma}{\partial t}\frac{\partial\psi}{\partial t}=\frac{1}{\rho}\frac{\partial\sigma}{\partial\rho},$$

$$\frac{\partial^2\sigma}{\partial\rho^2}+\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\frac{1}{\rho}\frac{\partial\sigma}{\partial\rho}-\frac{\partial^2\sigma}{\partial t^2}-\left(\frac{\partial\sigma}{\partial t}\right)^2=0,$$

$$\frac{\partial^2\sigma}{\partial\rho\partial t}-\frac{\partial\sigma}{\partial\rho}\left(\frac{\partial\gamma}{\partial t}-\frac{\partial\psi}{\partial t}\right)-\frac{\partial\sigma}{\partial t}\left(\frac{\partial\gamma}{\partial\rho}-\frac{\partial\gamma}{\partial\rho}\right)-\frac{\partial\sigma}{\partial\rho}\frac{\partial\sigma}{\partial t}=0. \quad \ldots \quad (76)$$

From the second equation,

$$e^{\sigma}=F\{f(\rho,t)\}, \quad \ldots \quad (77)$$

where

$$\frac{\partial f}{\partial\rho}=2\rho\frac{\partial\psi}{\partial t},\ \frac{\partial f}{\partial t}=2\rho\frac{\partial\psi}{\partial\rho}-1.$$

It is verified that (77) does not satisfy the other equations.

Case (b) leads to

$$\frac{\partial^2\sigma}{\partial\rho^2}-\frac{\partial^2\sigma}{\partial t^2}=0,$$

$$\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\frac{1}{\rho}\frac{\partial\sigma}{\partial\rho}-\left(\frac{\partial\sigma}{\partial t}\right)^2=0,$$

$$\frac{\partial^2\sigma}{\partial\rho\partial t}-\frac{\partial\sigma}{\partial\rho}\left(\frac{\partial\gamma}{\partial t}-\frac{\partial\psi}{\partial t}\right)-\frac{\partial\sigma}{\partial t}\left(\frac{\partial\gamma}{\partial\rho}-\frac{\partial\psi}{\partial\rho}\right)-\frac{\partial\sigma}{\partial\rho}\frac{\partial\sigma}{\partial t}=0.$$

The first two equations have the solution (42). The last equation is not satisfied for this value of σ.

If $\bar{R}_1^4\neq 0$, we must have

$$\bar{R}_1^1+\bar{R}_4^4=0 \text{ and } \bar{R}_2^2+\bar{R}_3^3=0.$$

Hence

$$\frac{\partial^2\sigma}{\partial\rho^2}-\frac{\partial^2\sigma}{\partial t^2}=0,$$

and

$$\left(\frac{\partial\sigma}{\partial\rho}\right)^2+\frac{1}{\rho}\frac{\partial\sigma}{\partial\rho}-\left(\frac{\partial\sigma}{\partial t}\right)^2=0$$

which have the solution (42).
We have also the equation

$$(\bar{R}_2^2)^2=(\bar{R}_1^1)^2-(\bar{R}_1^4)^2$$

which gives

$$\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}=\left(\frac{\partial\psi}{\partial\rho}\right)^2-\left(\frac{\partial\psi}{\partial t}\right)^2. \quad \ldots \quad (78)$$

Equation (78) together with the field equations for empty space, viz. (65) to (67), imply that V_4 is flat. The corresponding conformal space-time which represents an electromagnetic field has already been discussed in Section III. Hence there is no cylindrically symmetric non-flat empty space-time corresponding to which a conformal cylindrically symmetric electromagnetic field can exist.

Case (iii):

$$ds^2 = -e^{-2\gamma-2\psi}(d\rho^2+dz^2)-\rho^2 e^{-2\psi}d\phi^2+e^{2\psi}dt^2,$$

$$\psi = \psi(\rho, z), \gamma = \gamma(\rho, z).$$

Since we can go over from cylindrical symmetry to static axial symmetry by the substitution

$$T = iz \text{ and } Z = it$$

we can conclude that static axially symmetric spaces conformal to static axially symmetric empty fields cannot represent an electromagnetic field.

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