THE RESOLUTION INTO MINIMAL IDEALS OF THE ENVELOPING ALGEBRA OF THE LIE-ALGEBRA OF THE ROTATION GROUP IN FOUR DIMENSIONS WITH SPIN 3/2

by K. N. SRINIVASA RAO[1] *and* D. SAROJA[2]

(Communicated by V. R. Thiruvenkatachar, F.N.I.)

(*Received September 25, 1965*)

It has been proved by Madhava Rao *et al.* that the enveloping algebra of the Lie-algebra of the Rotation Group R_n with half-odd-integral spin is the direct product of the Dirac algebra and another algebra called the ξ-algebra. In this paper, a complete resolution into minimal ideals of the ξ-algebra A associated with R_4 is carried out. The basis of A is shown to be of rank 14, and corresponding to the resolution $1^2+2^2+3^2 = 14$ three primitive orthogonal idempotent elements of the centre of A are determined, thus resolving A into a direct sum of two-sided ideals Ae_r $(r = 1, 2, 3)$. Each of these Ae_r is then resolved into minimal left ideals giving rise to the irreducible representations of A. Mutually orthogonal idempotents generating these minimal left ideals are also determined. Since such a minimal ideal resolution for the Dirac algebra is known, the resolution of the enveloping algebra of the Lie-algebra of R_4 is solved.

Guided by the fact that the relativistic wave equations describing elementary particles of spin 0, $\frac{1}{2}$, 1 can all be cast into the form

$$(\beta_\mu \partial_\mu + \chi)\psi = 0, \qquad \qquad (1)$$

Bhabha (1945) made the assumption that the relativistic wave equation describing an elementary particle of arbitrary spin must also be of the form (1) where the β_μ are matrices of appropriate dimensions. By taking $\beta_\mu = iI_{\mu 5}$ $(\mu = 1, 2, 3, 4)$ where $I_{rs} = -I_{sr}$ $(r, s = 1, 2, 3, 4, 5;\ r \neq s)$ are the ten infinitesimal generators of R_5, the rotation group in five dimensions, he showed that the β-algebra is the same as the enveloping algebra of the Lie-algebra of R_5. The algebra becomes finite-dimensional if one assumes that any one of the β_μ satisfies a polynomial relation. Jacobson (1949) showed that even in the case of the rotation group R_n of arbitrary dimension the polynomial relation satisfied is

$$\{x^2-j^2\}\{x^2-(j-1)^2\}\ldots. = 0,$$

where the last factor is either x or $\{x^2-\frac{1}{4}\}$ according as j is integral or half-odd-integral. In view of its application in Relativistic Quantum Mechanics, the polynomial relation will be called the 'spin equation' and j the 'spin'.

---

[1] Reader in Physics, Manasa Gangotri, University of Mysore, Mysore 6. Formerly N.I.S.I. Research Fellow.

[2] C.S.I.R. Junior Research Fellow, Department of Physics, Central College, Bangalore.

The choice of $j = \frac{1}{2}$ leads to the Clifford-Dirac algebra and $j = 1$ to the Duffin-Kemmer algebra.

The algebra of R_5 with $j = \frac{3}{2}$ was investigated by Madhava Rao *et al.* (1946). The β_μ satisfy in this algebra the polynomial relation

$$\beta_\mu^4 - \tfrac{5}{2}\beta_\mu^2 + \tfrac{9}{16} = 0,$$

whereas in the Kemmer algebra it is the comparatively simpler relation $\beta^3 = \beta$. Thus the investigation of this algebra would be considerably complex. However, Madhava Rao and others succeeded in showing that even in the general case of R_n the algebra with a half-odd-integral spin can be broken up as the direct product of the Dirac algebra and another algebra which they called the ξ-algebra generated by the symbols ξ_r $(r = 1, 2, \ldots n)$ satisfying the commutation rules

$$\{\xi_r, \{\xi_r, \xi_s\}\} = \xi_s \quad \ldots \quad (2)$$

$$[\xi_r, \{\xi_s, \xi_t\}] = 0, \quad (r \neq s \neq t) \quad \ldots \quad (3)$$

where $\{a, b\}$ is the anti-commutator $ab+ba$ and $[a, b]$ is the commutator $ab-ba$. They showed that for the case of spin $\frac{3}{2}$, the ξ_r satisfy the quadratic

$$\xi_r^2 = \tfrac{3}{4} - \xi_r \quad \ldots \quad (4)$$

and they also determined the idempotents generating the two-sided ideals and the irreducible representations of the ξ-algebra generated by four symbols with spin $\frac{3}{2}$. Venkatachaliengar and Srinivasa Rao (1954) determined the centre and all the irreducible representations of the ξ-algebra generated by n symbols with spin $\frac{3}{2}$.

In this paper we effect a complete resolution of the ξ-algebra generated by three symbols into minimal left ideals. Since such a resolution for the Dirac algebra is known (Srinivasa Rao and Venkatanarasimhiah 1950), the problem of minimal ideal resolution of the algebra of the rotation group in four dimensions is thus solved. One could expect a possible physical application of this algebra generated by three symbols, as the corresponding Kemmer algebra generated by three symbols has found an application in the theory of isobaric spin connected with elementary particles.

Let A be the ξ-algebra generated by the three symbols ξ_1, ξ_2, ξ_3 which satisfy the relations (2), (3), (4). We shall first enumerate the number of independent elements of the algebra. On simplifying (2) we obtain

$$\xi_{rsr} = -\tfrac{1}{4}\xi_s + \tfrac{1}{2}\{\xi_r, \xi_s\} \quad \ldots \quad (5)$$

where we write $\xi_r\xi_s\xi_t \equiv \xi_{rst}$. Equation (3) yields the following relations among the elements ξ_{rst}:

$$r = 1, s = 2, t = 3: \ \xi_{123} + \xi_{132} - \xi_{231} = \xi_{321}$$

$$r = 2, s = 1, t = 3: \ \xi_{213} + \xi_{231} - \xi_{132} = \xi_{312}.$$

It therefore follows that the algebra is of rank 14 and we can take as a basis the set of elements

$$1,\ \xi_1,\ \xi_2,\ \xi_3,\ \xi_{12},\ \xi_{21},\ \xi_{23},\ \xi_{32},\ \xi_{31},\ \xi_{13},\ \xi_{123},\ \xi_{132},\ \xi_{231},\ \xi_{213}.$$

It can be easily verified that

$$\{\xi_r, \xi_s\}^2 = \tfrac{3}{4} - \{\xi_r, \xi_s\}.$$

This suggests that the calculations can be simplified to some extent if we introduce a new symbol η defined by

$$\{\xi_r, \xi_s\} = \eta_t \ (r \neq s \neq t). \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

The η_t satisfy the following relations:

$$\eta_t^2 = \tfrac{3}{4} - \eta_t \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

$$(a)\ \{\xi_r, \eta_t\} = \xi_s \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$(b)\ \{\xi_r, \{\xi_r, \eta_t\}\} = \eta_t$$

$$\xi_t\eta_t = \eta_t\xi_t \ \text{(no summation)} \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

$$\{\eta_r, \eta_s\} = \eta_t \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

$$\xi_{rsr} = -\tfrac{1}{4}\xi_s + \tfrac{1}{2}\eta_t. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

We can now rewrite the basis of the algebra in the form

$$1,\ \xi_1,\ \xi_2,\ \xi_3,\ \eta_1,\ \eta_2,\ \eta_3,\ \xi_{12},\ \xi_{23},\ \xi_{31},\ \xi_1\eta_1,\ \xi_2\eta_2,\ \xi_3\eta_3,\ \xi_{123}.$$

Taking θ as a linear combination of the above 14 basis elements and using the relations

$$[\theta, \xi_r] = 0 \qquad r = 1, 2, 3,$$

we find that apart from the unit element 1 the only other linearly independent central elements are

$$\theta_1 = \xi_1 + \xi_2 + \xi_3 + \eta_1 + \eta_2 + \eta_3 \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\theta_2 = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

showing that the centre of the algebra A is of rank 3 with the basis

$$1,\ \theta_1,\ \theta_2.$$

Since $1^2 + 2^2 + 3^2 = 14$ is the only possible resolution of 14 into a sum of three squares, the algebra A must have three inequivalent irreducible representations of dimensions, 1, 2 and 3.

We obtain after some calculation the following relations:

$$\left.\begin{aligned} \theta_1^2 &= \tfrac{9}{2} + \theta_1 + 2\theta_2 \\ \theta_2^2 &= \tfrac{45}{16} - \tfrac{5}{4}\theta_1 + 2\theta_2 \\ \theta_1\theta_2 = \theta_2\theta_1 &= \tfrac{5}{4}\theta_1 - 2\theta_2 \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

Using these relations we determine the primitive idempotent elements e_1, e_2, e_3 of the centre of the algebra satisfying the condition

$$e_1+e_2+e_3 = 1$$

and the orthogonality relations

$$e_1e_2 = 0,\ e_2e_3 = 0,\ e_3e_1 = 0.$$

We get

$$\left.\begin{aligned} e_1 &= \tfrac{1}{24}(3-2\theta_1+4\theta_2) \\ e_2 &= \tfrac{1}{16}(9-2\theta_1-4\theta_2) \\ e_3 &= \tfrac{1}{48}(15+10\theta_1+4\theta_2) \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

We thus have the resolution

$$A = Ae_1\dot{+}Ae_2\dot{+}Ae_3$$

of the algebra A as a direct sum of two-sided ideals Ae_r which establishes that A is semi-simple.

We now proceed to determine explicitly the three two-sided ideals Ae_r $(r = 1, 2, 3)$. It is clear from (15) that it is necessary for this purpose to evaluate $A\theta_1$ and $A\theta_2$. We merely write down in the form of a table the set of elements $A\theta_1$ and $A\theta_2$ which can be obtained after a considerable amount of calculation.

A	$A\theta_1$	$A\theta_2$
1	$\xi_1+\xi_2+\xi_3+\eta_1+\eta_2+\eta_3$	$\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3$
ξ_r	$\frac{3}{4}-\xi_r+\frac{1}{2}\xi_s+\frac{1}{2}\xi_t+\frac{1}{2}\eta_s+\frac{1}{2}\eta_t+\xi_r\eta_r$	$\frac{5}{4}\eta_r-2\xi_r\eta_r+\frac{1}{2}\xi_s\eta_s+\frac{1}{2}\xi_t\eta_t$
η_r	$\frac{3}{4}+\frac{1}{2}\xi_s+\frac{1}{2}\xi_t-\eta_r+\frac{1}{2}\eta_s+\frac{1}{2}\eta_t+\xi_r\eta_r$	$\frac{5}{4}\xi_r-2\xi_r\eta_r+\frac{1}{2}\xi_s\eta_s+\frac{1}{2}\xi_t\eta_t$
$\xi_r\eta_r$	$\frac{3}{4}\xi_r+\frac{1}{4}\xi_s+\frac{1}{4}\xi_t+\frac{3}{4}\eta_r+\frac{1}{4}\eta_s+\frac{1}{4}\eta_t-2\xi_r\eta_r$	$\frac{15}{16}-\frac{5}{4}\xi_r-\frac{5}{4}\eta_r+\frac{3}{2}\xi_r\eta_r+\frac{1}{4}\xi_s\eta_s+\frac{1}{4}\xi_t\eta_t$
ξ_{12}	$\left\{\begin{array}{l}\frac{3}{8}+\frac{1}{4}\xi_1+\frac{1}{4}\xi_2+\frac{1}{2}\eta_1+\frac{1}{2}\eta_2+\frac{1}{4}\eta_3-\frac{3}{2}\xi_{12}\\ -\frac{1}{2}\xi_{23}-\frac{1}{2}\xi_{31}-\frac{1}{2}\xi_1\eta_1+\frac{1}{2}\xi_2\eta_2+\xi_{123}\end{array}\right\}$	$\left\{\begin{array}{l}\frac{5}{8}\xi_3-\frac{5}{8}\eta_1-\frac{5}{8}\eta_2+\frac{5}{4}\xi_{23}+\frac{5}{4}\xi_{31}\\ +\frac{3}{2}\xi_1\eta_1-\xi_2\eta_2+\frac{1}{4}\xi_3\eta_3-\frac{5}{2}\xi_{123}\end{array}\right\}$
ξ_{23}	$\left\{\begin{array}{l}\frac{3}{8}+\frac{1}{4}\xi_2+\frac{1}{4}\xi_3+\frac{1}{4}\eta_1+\frac{1}{2}\eta_2+\frac{1}{2}\eta_3-\frac{1}{2}\xi_{12}\\ -\frac{3}{2}\xi_{23}-\frac{1}{2}\xi_{31}+\frac{1}{2}\xi_2\eta_2-\frac{1}{2}\xi_3\eta_3+\xi_{123}\end{array}\right\}$	$\left\{\begin{array}{l}\frac{5}{8}\xi_1-\frac{5}{8}\eta_2-\frac{5}{8}\eta_3+\frac{5}{4}\xi_{12}+\frac{5}{4}\xi_{31}\\ +\frac{1}{4}\xi_1\eta_1-\xi_2\eta_2+\frac{3}{2}\xi_3\eta_3-\frac{5}{2}\xi_{123}\end{array}\right\}$
ξ_{31}	$\left\{\begin{array}{l}\frac{3}{8}+\frac{1}{4}\xi_1+\frac{1}{4}\xi_3+\frac{1}{2}\eta_1+\frac{1}{4}\eta_2+\frac{1}{2}\eta_3-\frac{1}{2}\xi_{12}\\ -\frac{1}{2}\xi_{23}-\frac{3}{2}\xi_{31}-\frac{1}{2}\xi_1\eta_1+\xi_2\eta_2-\frac{1}{2}\xi_3\eta_3+\xi_{123}\end{array}\right\}$	$\left\{\begin{array}{l}\frac{5}{8}\xi_2-\frac{5}{8}\eta_1-\frac{5}{8}\eta_3+\frac{5}{4}\xi_{12}+\frac{5}{4}\xi_{23}\\ +\frac{3}{2}\xi_1\eta_1-\frac{9}{4}\xi_2\eta_2+\frac{3}{2}\xi_3\eta_3-\frac{5}{2}\xi_{123}\end{array}\right\}$
ξ_{123}	$\left\{\begin{array}{l}\frac{3}{8}\xi_1-\frac{1}{8}\xi_2+\frac{3}{8}\xi_3+\frac{1}{4}\eta_1-\frac{1}{4}\eta_2+\frac{1}{4}\eta_3+\frac{1}{4}\xi_{12}\\ +\frac{1}{4}\xi_{23}+\frac{1}{4}\xi_{31}-\frac{1}{4}\xi_1\eta_1+\frac{1}{4}\xi_2\eta_2-\frac{1}{4}\xi_3\eta_3-\frac{3}{2}\xi_{123}\end{array}\right\}$	$\left\{\begin{array}{l}\frac{15}{32}-\frac{5}{8}\xi_1+\frac{5}{8}\xi_2-\frac{5}{8}\xi_3-\frac{5}{16}\eta_1+\frac{15}{16}\eta_2\\ -\frac{5}{16}\eta_3-\frac{5}{8}\xi_{12}-\frac{5}{8}\xi_{23}-\frac{5}{8}\xi_{31}+\frac{3}{4}\xi_1\eta_1\\ -\frac{1}{2}\xi_2\eta_2+\frac{3}{4}\xi_3\eta_3\end{array}\right\}$

[r, s, t take the values 1, 2, 3 in cyclic order in the above table.]

One verifies that

(i) $$\xi_r e_3 = \tfrac{1}{48}(15\xi_r + 10\xi_r\theta_1 + 4\xi_r\theta_2) = \tfrac{1}{2}e_3\,;\ r = 1,\ 2,\ 3.$$

Thus the idempotent e_3 generates a one-dimensional two-sided ideal $\Omega_3 = Ae_3$ and the corresponding first order representation is $\xi_r = \frac{1}{2}$ $(r = 1,\ 2,\ 3)$.

(ii) The two-sided ideal $\Omega_1 = Ae_1$ generated by the idempotent e_1 is of rank 4 and a basis can be chosen to be

$$\omega_1^i\ (i = 1,\ 2,\ 3,\ 4),$$

where

$$\left.\begin{aligned} \omega_1^r &\equiv \xi_r e_1 = \tfrac{1}{24}[-\tfrac{3}{2}+5\xi_r-\xi_s-\xi_t+5\eta_r-\eta_s-\eta_t-10\xi_r\eta_r+2\xi_s\eta_s+2\xi_t\eta_t] \\ \omega_1^4 &\equiv \xi_{31}e_1 = \tfrac{1}{24}[-\tfrac{3}{4}-\tfrac{1}{2}\xi_1+\tfrac{5}{2}\xi_2-\tfrac{1}{2}\xi_3-\tfrac{7}{2}\eta_1-\tfrac{1}{2}\eta_2-\tfrac{7}{2}\eta_3+6\xi_{12}+6\xi_{23} \\ &\qquad +6\xi_{31}+7\xi_1\eta_1-11\xi_2\eta_2+7\xi_3\eta_3-12\xi_{123}] \end{aligned}\right\} \quad (16)$$

with $r,\ s,\ t = 1,\ 2,\ 3$ in cyclic order.

By virtue of the following relations

$$\left.\begin{aligned} &e_1 = -\tfrac{2}{3}[\omega_1^1+\omega_1^2+\omega_1^3] && \xi_{12}e_1 = \tfrac{1}{2}[-\omega_1^2+\omega_1^3+2\omega_1^4] \\ &\eta_r e_1 = \omega_1^r && \xi_{23}e_1 = \tfrac{1}{2}[\omega_1^1-\omega_1^2+2\omega_1^4] \\ &\xi_r\eta_r e_1 = -\tfrac{1}{2}[3\omega_1^r+\omega_1^s+\omega_1^t] && \xi_{123}e_1 = -\tfrac{1}{4}[3\omega_1^1-2\omega_1^2+3\omega_1^3+2\omega_1^4] \\ &(r,\ s,\ t = 1,\ 2,\ 3 \text{ in cyclic order}) && \end{aligned}\right\} \quad (17)$$

all elements of the form Ae_1 can be expressed as linear combinations of the ω_1^i.

(iii) The two-sided ideal $\Omega_2 = Ae_2$ generated by the idempotent e_2 is of rank 9 and a basis for Ω_2 is chosen to be

$$\omega_2^i\ (i = 1,\ 2,\ 3, \ldots .\ 9),$$

where

$$\left.\begin{aligned} \omega_2^r &\equiv \xi_r e_2 = \tfrac{1}{16}[-\tfrac{3}{2}+11\xi_r-\xi_s-\xi_t-5\eta_r-\eta_s-\eta_t+6\xi_r\eta_r-2\xi_s\eta_s-2\xi_t\eta_t] \\ \omega_2^{r+3} &\equiv \eta_r e_2 = \tfrac{1}{16}[-\tfrac{3}{2}-5\xi_r-\xi_s-\xi_t+11\eta_r-\eta_s-\eta_t+6\xi_r\eta_r-2\xi_s\eta_s-2\xi_t\eta_t] \\ \omega_2^7 &\equiv \xi_{12}e_2 = \tfrac{1}{16}[-\tfrac{3}{4}-\tfrac{1}{2}\xi_1-\tfrac{1}{2}\xi_2-\tfrac{5}{2}\xi_3+\tfrac{3}{2}\eta_1+\tfrac{3}{2}\eta_2-\tfrac{1}{2}\eta_3+12\xi_{12} \\ &\qquad -4\xi_{23}-4\xi_{31}-5\xi_1\eta_1+3\xi_2\eta_2-\xi_3\eta_3+8\xi_{123}] \\ \omega_2^8 &\equiv \xi_{23}e_2 = \tfrac{1}{16}[-\tfrac{3}{4}-\tfrac{5}{2}\xi_1-\tfrac{1}{2}\xi_2-\tfrac{1}{2}\xi_3-\tfrac{1}{2}\eta_1+\tfrac{3}{2}\eta_2+\tfrac{3}{2}\eta_3-4\xi_{12}+12\xi_{23} \\ &\qquad -4\xi_{31}-\xi_1\eta_1+3\xi_2\eta_2-5\xi_3\eta_3+8\xi_{123}] \\ \omega_2^9 &\equiv \xi_{31}e_2 = \tfrac{1}{16}[-\tfrac{3}{4}-\tfrac{1}{2}\xi_1-\tfrac{5}{2}\xi_2-\tfrac{1}{2}\xi_3+\tfrac{3}{2}\eta_1-\tfrac{1}{2}\eta_2+\tfrac{3}{2}\eta_3-4\xi_{12}-4\xi_{23} \\ &\qquad +12\xi_{31}-5\xi_1\eta_1+7\xi_2\eta_2-5\xi_3\eta_3+8\xi_{123}] \end{aligned}\right\} \quad (18)$$

with $r,\ s,\ t = 1,\ 2,\ 3$ in cyclic order.

By virtue of the following relations

$$\left.\begin{aligned} e_2 &= -[\omega_2^1+\omega_2^2+\omega_2^3+\omega_2^4+\omega_2^5+\omega_2^6] \\ \xi_r\eta_r e_2 &= \tfrac{1}{4}[3\omega_2^r+\omega_2^s+\omega_2^t+3\omega_2^{r+3}+\omega_2^{s+3}+\omega_2^{t+3}]\ (r, s, t = 1, 2, 3 \text{ in cyclic order}) \\ \xi_{123}e_2 &= \tfrac{1}{8}[3\omega_2^1-\omega_2^2+3\omega_2^3+\omega_2^4-3\omega_2^5+\omega_2^6+4\omega_2^7+4\omega_2^8+4\omega_2^9] \end{aligned}\right\} \quad (19)$$

we see, as before, that all elements of the form Ae_2 are expressible in terms of the ω_2^i.

This completes the resolution of the semi-simple algebra A into a direct sum of simple algebras.

We now proceed to resolve each of the two-sided ideals Ω_1 and Ω_2 into minimal left ideals. We first of all consider the resolution of Ω_1. One would normally expect a reduction in the set of elements $\Omega_1\xi_r$, $\Omega_1\xi_r\xi_s$, etc., when elements of Ω_1 are multiplied by the generators ξ_r successively on the right. Such a reduction does not result in this case and we therefore adopt the following procedure for finding the minimal left ideals. For this purpose we make use of the fact that the irreducible representations of A are known (Venkatachaliengar and Srinivasa Rao 1954). Let $\xi \rightarrow M$ be the 4-dimensional regular representation corresponding to the simple algebra Ω_1, and $\xi \rightarrow N$ be the irreducible representation of dimension 2. Then we determine a non-singular transformation matrix $T = (t_{ij})$ such that

$$T^{-1}MT = M^* = N\dot{+}N \quad \text{or} \quad MT = TM^*. \qquad \ldots \quad \ldots \quad (20)$$

If $\omega_i \equiv \omega_1^i$ is the basis giving the representation $\xi \rightarrow M$ and ω_i^* the basis giving $\xi \rightarrow M^*$, then

$$\omega_j^* = t_{ij}\omega_i\,;\ i, j = 1, 2, 3, 4\ (i \text{ a dummi}) \qquad \ldots \quad \ldots \quad (21)$$

and therefore ω_j^*, the basis elements of the minimal left ideals in Ω_1, can be obtained when once the matrix T is found out.

We have from (16) and (17) the following regular representation M for Ω_1:

$$\xi_1 \rightarrow \begin{pmatrix} -3/2 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1 & 1/2 \\ -1/2 & 1/2 & 0 & -1/4 \\ 0 & 1 & -1 & 0 \end{pmatrix},\ \xi_2 \rightarrow \begin{pmatrix} 0 & -1/2 & 1/2 & -3/4 \\ 1/2 & -3/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 0 & 1/4 \\ -1 & 0 & 1 & -1/2 \end{pmatrix},\ \xi_3 \rightarrow \begin{pmatrix} 0 & 1/2 & -1/2 & 3/4 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & -3/2 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}. \quad (22)$$

The reduced representation M^* for Ω_1 can be taken to be

$$\xi_1 \rightarrow \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -3/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix},\ \xi_2 \rightarrow \begin{pmatrix} -1 & -1 & 0 & 0 \\ -3/4 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3/4 & 0 \end{pmatrix},\ \xi_3 \rightarrow \begin{pmatrix} -1 & 1 & 0 & 0 \\ 3/4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3/4 & 0 \end{pmatrix}. \quad (23)$$

Using (22) and (23) in (20), we obtain after some simplification the following 12 relations among the 16 matrix elements t_{ij}:

$$\left.\begin{array}{ll} t_{1,\,2p+1} = 0 & t_{1,\,2p+2} = \frac{4}{3}t_{4,\,2p+1} \\ t_{2,\,2p+1} = t_{3,\,2p+1}+\frac{1}{2}t_{4,\,2p+1} & t_{2,\,2p+2} = \frac{4}{3}t_{3,\,2p+1}+t_{4,\,2p+1} \\ & t_{3,\,2p+2} = -\frac{2}{3}t_{3,\,2p+1} \\ & t_{4,\,2p+2} = -\frac{4}{3}t_{3,\,2p+1}-\frac{2}{3}t_{4,\,2p+1} \end{array}\right\} \quad \ldots \quad (24)$$

with $p = 0, 1.$

The choice $t_{33} = 0$, $t_{41} = 0$, $t_{31} = 1$, $t_{43} = 1$ of the 4 arbitrary parameters in (24) leads to the following non-singular transformation matrix:

$$T = \begin{pmatrix} 0 & 0 & 0 & 4/3 \\ 1 & 4/3 & 1/2 & 1 \\ 1 & -2/3 & 0 & 0 \\ 0 & -4/3 & 1 & -2/3 \end{pmatrix}. \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Using (25) in (21) we get

$$\left.\begin{array}{l} \omega_1^{*1} = \omega_1^2+\omega_1^3 \\ \omega_1^{*2} = \frac{4}{3}\omega_1^2-\frac{2}{3}\omega_1^3-\frac{4}{3}\omega_1^4 \\ \omega_1^{*3} = \frac{1}{2}\omega_1^2+\omega_1^4 \\ \omega_1^{*4} = \frac{4}{3}\omega_1^1+\omega_1^2-\frac{2}{3}\omega_1^4 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

where we have restored the original notation for ω_j^*, ω_i as ω_1^{*j}, ω_1^i respectively. One can check that ω_1^{*1}, ω_1^{*2} form one minimal left ideal, L_{11}, and ω_1^{*3}, ω_1^{*4} another, L_{12}, in Ω_1, each of which gives rise to the two-dimensional irreducible representation used in (23). A convenient choice of basis (l_{11}^1, l_{11}^2) of L_{11} and (l_{12}^1, l_{12}^2) of L_{12} is the following:—

$$\left.\begin{array}{ll} l_{11}^1 \equiv \omega_1^{*1} = \omega_1^2+\omega_1^3 & l_{12}^1 \equiv \omega_1^{*3} = \frac{1}{2}\omega_1^2+\omega_1^4 \\ l_{11}^2 \equiv \frac{4}{3}\omega_1^{*1}-\omega_1^{*2} = 2\omega_1^3+\frac{4}{3}\omega_1^4 & l_{12}^2 \equiv \frac{2}{3}\omega_1^{*3}+\omega_1^{*4} = \frac{4}{3}\left(\omega_1^1+\omega_1^2\right) \end{array}\right\}. \quad \ldots \quad (27)$$

We now proceed to determine the idempotents e_{11}, e_{12} generating the minimal left ideals L_{11}, L_{12} respectively. For this purpose, we take e_{11} and e_{12} to be linear combinations respectively of (l_{11}^1, l_{11}^2) and (l_{12}^1, l_{12}^2), and make use of the condition

$$e_1 = e_{11}+e_{12}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

One finds, on using (17) and (27) and comparing coefficients of the ω_1^i, that e_{11} and e_{12} are uniquely determined. We obtain

$$\left.\begin{array}{l} e_{11} = -\frac{1}{6}l_{11}^1-\frac{1}{4}l_{11}^2 = -\frac{1}{6}\left[\omega_1^2+4\omega_1^3+2\omega_1^4\right] \\ \quad = -\frac{1}{24}\left[-\frac{3}{2}-\xi_1+\xi_2+3\xi_3-2\eta_1+2\eta_3+2\xi_{12}+2\xi_{23}+2\xi_{31}+4\xi_1\eta_1-4\xi_2\eta_2\right. \\ \quad \left.-4\xi_3\eta_3-4\xi_{123}\right] \\ e_{12} = \frac{1}{3}l_{12}^1-\frac{1}{2}l_{12}^2 = -\frac{1}{6}\left[4\omega_1^1+3\omega_1^2-2\omega_1^4\right] \\ \quad = -\frac{1}{24}\left[-\frac{3}{2}+3\xi_1+\xi_2-\xi_3+4\eta_1+2\eta_2-2\xi_{12}-2\xi_{23}-2\xi_{31}-8\xi_1\eta_1+4\xi_{123}\right] \end{array}\right\}. \quad (29)$$

One verifies that the e_{11}, e_{12} thus determined are a pair of mutually orthogonal idempotents satisfying the conditions

$$e_{11}^2 = e_{11},\ e_{12}^2 = e_{12} \text{ and } e_{11}e_{12} = 0.$$

We now go on to resolve the two-sided ideal Ω_2 into minimal left ideals following the same procedure as was adopted for Ω_1. We have from (18) and (19) the following 9-dimensional regular representation M for Ω_2:

$$\xi_1 \to \begin{pmatrix} -7/4 & 0 & 0 & 3/4 & 0 & 0 & 0 & 3/8 & 0 \\ -3/4 & 0 & 0 & 1/4 & 0 & 1/2 & 3/4 & -1/8 & 0 \\ -3/4 & 0 & 0 & 1/4 & 1/2 & 0 & 0 & 3/8 & -1/4 \\ -3/4 & 0 & 0 & 3/4 & 0 & 0 & 0 & 1/8 & 0 \\ -3/4 & 0 & 1 & 1/4 & -1/2 & 0 & 0 & -3/8 & 1/2 \\ -3/4 & 0 & 0 & 1/4 & 0 & 1/2 & 0 & 1/8 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1/2 & 0 \end{pmatrix},$$

$$\xi_2 \to \begin{pmatrix} 0 & -3/4 & 0 & 0 & 1/4 & 1/2 & -1/4 & 0 & 3/8 \\ 0 & -7/4 & 0 & 0 & 3/4 & 0 & 0 & 0 & 3/8 \\ 0 & -3/4 & 0 & 1/2 & 1/4 & 0 & 0 & 3/4 & -1/8 \\ 0 & -3/4 & 0 & 1/2 & 1/4 & 0 & 0 & 0 & 1/8 \\ 0 & -3/4 & 0 & 0 & 3/4 & 0 & 0 & 0 & 1/8 \\ 1 & -3/4 & 0 & 0 & 1/4 & -1/2 & 1/2 & 0 & -3/8 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix},$$

$$\xi_3 \to \begin{pmatrix} 0 & 0 & -3/4 & 0 & 1/2 & 1/4 & -1/8 & 0 & 3/4 \\ 0 & 0 & -3/4 & 1/2 & 0 & 1/4 & 3/8 & -1/4 & 0 \\ 0 & 0 & -7/4 & 0 & 0 & 3/4 & 3/8 & 0 & 0 \\ 0 & 1 & -3/4 & -1/2 & 0 & 1/4 & -3/8 & 1/2 & 0 \\ 0 & 0 & -3/4 & 0 & 1/2 & 1/4 & 1/8 & 0 & 0 \\ 0 & 0 & -3/4 & 0 & 0 & 3/4 & 1/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 1/2 & 0 & -1 \end{pmatrix}. \quad \ldots (30)$$

The reduced representation M^* for Ω_2 can be taken to be:

$$\xi_1 \to \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/2 \end{pmatrix}, \quad \xi_2 \to \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/4 & 0 \end{pmatrix},$$

$$\xi_3 \to \begin{pmatrix} -5/6 & -1/2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4/9 & 1/3 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2/3 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5/6 & -1/2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4/9 & 1/3 & -1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2/3 & -1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5/6 & -1/2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4/9 & 1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2/3 & -1/4 & 0 \end{pmatrix}. \quad \ldots (31)$$

Using (30) and (31) in (20), we obtain after a considerable amount of routine work the following 72 relations among the 81 matrix elements t_{ij} $(i, j = 1, 2, 3, \ldots 9)$:

$$\left.\begin{aligned}
t_{1,\,3p+1} &= -\tfrac{1}{6}t_{8,\,3p+1} + \tfrac{2}{9}t_{9,\,3p+3} \\
t_{2,\,3p+1} &= \tfrac{1}{6}t_{9,\,3p+1} + \tfrac{2}{9}t_{9,\,3p+3} \\
t_{3,\,3p+1} &= \tfrac{1}{2}t_{8,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+1} + \tfrac{2}{3}t_{9,\,3p+3} \\
t_{4,\,3p+1} &= -t_{8,\,3p+1} + \tfrac{2}{3}t_{9,\,3p+3} \\
t_{5,\,3p+1} &= \tfrac{2}{3}t_{9,\,3p+3} \\
t_{6,\,3p+1} &= -\tfrac{1}{3}t_{9,\,3p+1} + \tfrac{2}{9}t_{9,\,3p+3} \\
t_{7,\,3p+1} &= \tfrac{1}{3}t_{8,\,3p+1} + \tfrac{1}{3}t_{9,\,3p+1} \\
t_{1,\,3p+2} &= \tfrac{1}{8}t_{8,\,3p+1} - \tfrac{1}{6}t_{9,\,3p+3} \\
t_{2,\,3p+2} &= \tfrac{9}{8}t_{8,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+1} + \tfrac{1}{12}t_{9,\,3p+3} \\
t_{3,\,3p+2} &= \tfrac{3}{8}t_{8,\,3p+1} + \tfrac{1}{4}t_{9,\,3p+3} \\
t_{4,\,3p+2} &= \tfrac{3}{8}t_{8,\,3p+1} \\
t_{5,\,3p+2} &= \tfrac{3}{8}t_{8,\,3p+1} + \tfrac{1}{2}t_{9,\,3p+3} \\
t_{6,\,3p+2} &= \tfrac{3}{8}t_{8,\,3p+1} + t_{9,\,3p+1} + \tfrac{1}{3}t_{9,\,3p+3} \\
t_{7,\,3p+2} &= \tfrac{1}{2}t_{8,\,3p+1} - t_{9,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+3} \\
t_{8,\,3p+2} &= -t_{9,\,3p+3} \\
t_{9,\,3p+2} &= -\tfrac{1}{2}t_{9,\,3p+3} \\
t_{1,\,3p+3} &= \tfrac{3}{4}t_{8,\,3p+1} - \tfrac{3}{2}t_{9,\,3p+1} \\
t_{2,\,3p+3} &= \tfrac{3}{4}t_{8,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+1} - \tfrac{1}{6}t_{9,\,3p+3} \\
t_{3,\,3p+3} &= \tfrac{1}{4}t_{8,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+1} + \tfrac{1}{2}t_{9,\,3p+3} \\
t_{4,\,3p+3} &= \tfrac{1}{4}t_{8,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+1} \\
t_{5,\,3p+3} &= \tfrac{1}{4}t_{8,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+1} - t_{9,\,3p+3} \\
t_{6,\,3p+3} &= \tfrac{1}{4}t_{8,\,3p+1} - \tfrac{1}{2}t_{9,\,3p+1} \\
t_{7,\,3p+3} &= -t_{8,\,3p+1} + \tfrac{1}{3}t_{9,\,3p+3} \\
t_{8,\,3p+3} &= 0
\end{aligned}\right\} \quad (32)$$

with $p = 0, 1, 2.$

The choice $t_{81} = 0$, $t_{91} = 0$, $t_{93} = 1$, $t_{84} = 1$, $t_{94} = 0$, $t_{96} = 0$, $t_{87} = 0$, $t_{97} = 1$, $t_{99} = 0$ of the 9 arbitrary parameters in (32) leads to the following non-singular transformation matrix:

$$T=\begin{pmatrix} 2/9 & -1/6 & 0 & -1/6 & 1/8 & 3/4 & 0 & 0 & -3/2 \\ 2/9 & 1/12 & -1/6 & 0 & 9/8 & 3/4 & 1/6 & -1/2 & -1/2 \\ 2/3 & 1/4 & 1/2 & 1/2 & 3/8 & 1/4 & -1/2 & 0 & -1/2 \\ 2/3 & 0 & 0 & -1 & 3/8 & 1/4 & 0 & 0 & -1/2 \\ 2/3 & 1/2 & -1 & 0 & 3/8 & 1/4 & 0 & 0 & -1/2 \\ 2/9 & 1/3 & 0 & 0 & 3/8 & 1/4 & -1/3 & 1 & -1/2 \\ 0 & -1/2 & 1/3 & 1/3 & 1/2 & -1 & 1/3 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \ldots \quad (33)$$

Using (33) in (21) we get, as before,

$$\left.\begin{aligned}
\omega_2^{*1} &= \tfrac{2}{9}\omega_2^1+\tfrac{2}{9}\omega_2^2+\tfrac{2}{3}\omega_2^3+\tfrac{2}{3}\omega_2^4+\tfrac{2}{3}\omega_2^5+\tfrac{2}{9}\omega_2^6 \\
\omega_2^{*2} &= -\tfrac{1}{6}\omega_2^1+\tfrac{1}{12}\omega_2^2+\tfrac{1}{4}\omega_2^3+\tfrac{1}{2}\omega_2^5+\tfrac{1}{3}\omega_2^6-\tfrac{1}{2}\omega_2^7-\omega_2^8-\tfrac{1}{2}\omega_2^9 \\
\omega_2^{*3} &= -\tfrac{1}{6}\omega_2^2+\tfrac{1}{2}\omega_2^3-\omega_2^5+\tfrac{1}{3}\omega_2^7+\omega_2^9 \\
\omega_2^{*4} &= -\tfrac{1}{6}\omega_2^1+\tfrac{1}{2}\omega_2^3-\omega_2^4+\tfrac{1}{3}\omega_2^7+\omega_2^8 \\
\omega_2^{*5} &= \tfrac{1}{8}\omega_2^1+\tfrac{9}{8}\omega_2^2+\tfrac{3}{8}\omega_2^3+\tfrac{3}{8}\omega_2^4+\tfrac{3}{8}\omega_2^5+\tfrac{3}{8}\omega_2^6+\tfrac{1}{2}\omega_2^7 \\
\omega_2^{*6} &= \tfrac{3}{4}\omega_2^1+\tfrac{3}{4}\omega_2^2+\tfrac{1}{4}\omega_2^3+\tfrac{1}{4}\omega_2^4+\tfrac{1}{4}\omega_2^5+\tfrac{1}{4}\omega_2^6-\omega_2^7 \\
\omega_2^{*7} &= \tfrac{1}{6}\omega_2^2-\tfrac{1}{2}\omega_2^3-\tfrac{1}{3}\omega_2^6+\tfrac{1}{3}\omega_2^7+\omega_2^9 \\
\omega_2^{*8} &= -\tfrac{1}{2}\omega_2^2+\omega_2^6-\omega_2^7 \\
\omega_2^{*9} &= -\tfrac{3}{2}\omega_2^1-\tfrac{1}{2}\omega_2^2-\tfrac{1}{2}\omega_2^3-\tfrac{1}{2}\omega_2^4-\tfrac{1}{2}\omega_2^5-\tfrac{1}{2}\omega_2^6
\end{aligned}\right\} \quad \ldots \quad (34)$$

One can check that $(\omega_2^{*1}, \omega_2^{*2}, \omega_2^{*3})$, $(\omega_2^{*4}, \omega_2^{*5}, \omega_2^{*6})$, $(\omega_2^{*7}, \omega_2^{*8}, \omega_2^{*9})$ form three minimal left ideals L_{21}, L_{22}, L_{23} respectively in Ω_2, each of which gives rise to the three-dimensional irreducible representation used in (31). A convenient choice of basis $(l_{21}^1, l_{21}^2, l_{21}^3)$ of L_{21}, $(l_{22}^1, l_{22}^2, l_{22}^3)$ of L_{22}, $(l_{23}^1, l_{23}^2, l_{23}^3)$ of L_{23} is the following:

$$\left.\begin{aligned}
l_{21}^1 &\equiv \omega_2^{*1} &&= \tfrac{2}{9}\left[\omega_2^1+\omega_2^2+3\omega_2^3+3\omega_2^4+3\omega_2^5+\omega_2^6\right] \\
l_{21}^2 &\equiv \omega_2^{*2}+\tfrac{1}{2}\omega_2^{*3} &&= \tfrac{1}{6}\left[-\omega_2^1+3\omega_2^3+2\omega_2^6-2\omega_2^7-6\omega_2^8\right] \\
l_{21}^3 &\equiv \omega_2^{*3} &&= \tfrac{1}{6}\left[-\omega_2^2+3\omega_2^3-6\omega_2^5+2\omega_2^7+6\omega_2^9\right] \\
l_{22}^1 &\equiv \omega_2^{*4}-\tfrac{1}{6}\left(\omega_2^{*5}-\tfrac{3}{2}\omega_2^{*6}\right) &&= \tfrac{1}{2}\left[\omega_2^3-2\omega_2^4+2\omega_2^8\right] \\
l_{22}^2 &\equiv \omega_2^{*5}-\tfrac{3}{2}\omega_2^{*6} &&= \left[-\omega_2^1+2\omega_2^7\right] \\
l_{22}^3 &\equiv \omega_2^{*5}+\tfrac{1}{2}\omega_2^{*6} &&= \tfrac{1}{2}\left[\omega_2^1+3\omega_2^2+\omega_2^3+\omega_2^4+\omega_2^5+\omega_2^6\right] \\
l_{23}^1 &\equiv \omega_2^{*7}+\tfrac{1}{3}\omega_2^{*8} &&= \tfrac{1}{2}\left[-\omega_2^3+2\omega_2^9\right] \\
l_{23}^2 &\equiv \omega_2^{*8} &&= \tfrac{1}{2}\left[-\omega_2^2+2\omega_2^6-2\omega_2^7\right] \\
l_{23}^3 &\equiv \omega_2^{*9} &&= -\tfrac{1}{2}\left[3\omega_2^1+\omega_2^2+\omega_2^3+\omega_2^4+\omega_2^5+\omega_2^6\right]
\end{aligned}\right\}. \quad \ldots \quad (35)$$

We determine the idempotents e_{21}, e_{22}, e_{23} generating the minimal left ideals L_{21}, L_{22}, L_{23}, respectively, in the same way as we did for Ω_1, taking them to be linear combinations of $(l^1_{21}, l^2_{21}, l^3_{21})$, $(l^1_{22}, l^2_{22}, l^3_{22})$, $(l^1_{23}, l^2_{23}, l^3_{23})$ and making use of the condition

$$e_2 = e_{21}+e_{22}+e_{23}. \qquad \ldots \quad \ldots \quad \ldots \quad (36)$$

We obtain, using (19) and (35):

$$\left.\begin{aligned} e_{21} &= -\tfrac{3}{4}l^1_{21} = -\tfrac{1}{6}(\omega^1_2+\omega^2_2+3\omega^3_2+3\omega^4_2+3\omega^5_2+\omega^6_2) \\ &= -\tfrac{1}{24}[-\tfrac{9}{2}-3\xi_1-3\xi_2+5\xi_3+5\eta_1+5\eta_2-3\eta_3+2\xi_1\eta_1+2\xi_2\eta_2+2\xi_3\eta_3] \\ e_{22} &= -\tfrac{1}{6}l^2_{22}-\tfrac{1}{2}l^3_{22} = -\tfrac{1}{12}(\omega^1_2+9\omega^2_2+3\omega^3_2+3\omega^4_2+3\omega^5_2+3\omega^6_2+4\omega^7_2) \\ &= -\tfrac{1}{24}[-\tfrac{9}{2}-3\xi_1+9\xi_2-\xi_3+2\eta_1-2\eta_2+6\xi_{12}-2\xi_{23}-2\xi_{31}-4\xi_1\eta_1 \\ &\quad +8\xi_2\eta_2+4\xi_{123}] \\ e_{23} &= -\tfrac{1}{3}l^2_{23}+\tfrac{1}{2}l^3_{23} = -\tfrac{1}{12}(9\omega^1_2+\omega^2_2+3\omega^3_2+3\omega^4_2+3\omega^5_2+7\omega^6_2-4\omega^7_2) \\ &= -\tfrac{1}{24}[-\tfrac{9}{2}+9\xi_1-3\xi_2-\xi_3-4\eta_1+6\eta_3-6\xi_{12}+2\xi_{23}+2\xi_{31}+8\xi_1\eta_1-4\xi_2\eta_2 \\ &\quad +4\xi_3\eta_3-4\xi_{123}]. \end{aligned}\right\} \quad \ldots \quad (37)$$

One can check as before that the e_{21}, e_{22}, e_{23} given by (37) are a set of mutually orthogonal idempotents satisfying the conditions:

$$e^2_{21} = e_{21},\ e^2_{22} = e_{22},\ e^2_{23} = e_{23}$$

and

$$e_{21}e_{22} = 0,\ e_{22}e_{23} = 0,\ e_{23}e_{21} = 0.$$

Conclusion

The linear associative algebra A generated by the three symbols ξ_r $(r = 1, 2, 3)$ satisfying the relations

$$\{\xi_r, \{\xi_r, \xi_s\}\} = \xi_s$$

$$[\xi_r, \{\xi_s, \xi_t\}] = 0 \quad (r \neq s \neq t)$$

$$\xi^2_r = \tfrac{3}{4}-\xi_r$$

is resolved into a direct sum of minimal left ideals. Such a resolution for the Dirac algebra D generated by three symbols is already known. Since the enveloping algebra $\mathcal{L}$ of the Lie-algebra of the rotation group in four dimensions, with the infinitesimal transformations I_{rs} $(r \neq s)$ satisfying the quartic

$$(I^2_{rs}+\tfrac{9}{4})(I^2_{rs}+\tfrac{1}{4}) = 0,$$

is the direct product $D \times A$, the problem of resolving $\mathcal{L}$ into minimal left ideals is thus solved.

ACKNOWLEDGEMENT

It is a pleasure to express our deep sense of gratitude to Dr. K. Venkatachaliengar, Professor of Mathematics, Manasa Gangotri, Mysore, for having originally suggested the problem and for helpful discussions, and to the Council of Scientific and Industrial Research for the award of a Fellowship to one of us (D.S.) for carrying out the investigation.

REFERENCES

Bhabha, H. J. (1945). Relativistic Wave Equations for the Elementary Particles. *Rev. mod. Phys.*, **17**, 200–216.

Jacobson, N. (1949). Lie and Jordan Triple Systems. *Am. J. Math.*, **71**, 149–170.

Madhava Rao, B. S., Thiruvenkatachar, V. R., and Venkatachaliengar, K. (1946). Algebra related to elementary particles of spin 3/2. *Proc. R. Soc.*, A **187**, 385–397.

Srinivasa Rao, K. N., and Venkatanarasimhiah, C. K. (1950). The resolution of the Clifford Algebra (Dirac Algebra) with any number of symbols as the direct sum of minimal left ideals. *Proc. Indian Acad. Sci.*, A **32**, 46–51.

Venkatachaliengar, K., and Srinivasa Rao, K. N. (1954). The irreducible representations of the Lie-algebra of the orthogonal group with spin 3/2. *Proc. natn. Inst. Sci. India*, **20**, 509–523.