RADIAL PULSATIONS OF AN INFINITE CYLINDER IN THE PRESENCE OF HELICAL CURRENTS

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The radial pulsations of an infinite cylinder of uniform density and infinite electric conductivity in the presence of helical currents are studied for the case when $\xi = \frac{\delta r}{r_0}$ is a function of the displacement from the axis of the cylinder. It is found that the frequencies of pulsations are of the same order both when ξ is a function of displacement and when ξ is constant in space.

INTRODUCTION

Radial pulsations of infinitely long cylindrical masses are of great importance in the study of cosmic bodies like Spiral Arms, Solar Ion Streams, etc. This problem has been the subject of detailed investigations during recent years (Chandrasekhar and Fermi (1953), Lyttkens (1954), Chopra and Talwar (1955), Bhatnagar and Nagpaul (1957) and Talwar and Tandon (1958)). Here we explicitly solve the modified Eddington's equation given in the paper of Talwar and Tandon referred to above. It is shown that the frequency of pulsations does not differ considerably from the value it takes when ξ is constant in space.

FIRST CASE

The modified Eddington's equation for the radial pulsations of a cylinder of uniform density and infinite conductivity in the presence of a magnetic field is as follows (Talwar and Tandon 1958):

$$\frac{\partial^2 \xi}{\partial r_0^2} + \frac{\partial \xi}{\partial r_0}\left[\frac{3}{r_0} - \frac{2Gm(r_0)\rho_0}{p_0 r_0} + \frac{\Gamma+1}{\Gamma p_0}\left(\vec{J_0}\times\vec{H_0}\right)_{\text{rad}}\right]$$

$$+\frac{\xi}{p_0}\left[\frac{4(1-\Gamma)Gm(r_0)\rho_0}{\Gamma r_0^2} + \frac{(1+2\Gamma)\left(\vec{J_0}\times\vec{H_0}\right)_{\text{rad}}}{\Gamma r_0} + \frac{\sigma^2\rho_0}{\Gamma}\right]$$

$$+\left[\frac{\left\{\left(\delta\vec{J_0}\times\vec{H_0}\right)+\left(\vec{J_0}\times\delta\vec{H_0}\right)\right\}_{\text{rad}}}{\Gamma p_0 r_0}\right] = 0. \qquad (1)$$

We have also

$$\text{curl } \vec{H} = 4\pi\vec{J}, \quad \ldots \quad (2)$$

$$\text{div } H = 0 \quad \ldots \quad (3)$$

and

$$\frac{\delta r}{r_0} = \xi = \xi_0(r)e^{i\sigma t}. \quad \ldots \quad (4)$$

Further, Γ is the ratio of the specific heats of the material of the cylinder, $m(r_0)$ the mass per unit length of the cylinder within a radius r_0 and p_0, ρ_0 and $\vec{H}_0$ are the gas pressure, the density and the magnetic field respectively in the equilibrium state.

The change $\delta\vec{H}$ in the magnetic field following the motion is given by

$$\delta\vec{H} = \text{curl}\left(\delta\vec{r}\times\vec{H}\right)+\left(\delta\vec{r}\,.\,\text{grad}\right)\vec{H}. \quad \ldots \quad (5)$$

The change $\delta\vec{J}$ in the current density, following the motion is evaluated by using eqn. (5) in conjunction with eqn. (2). For the equilibrium configuration

$$\frac{\partial p_0}{\partial r_0} = -\frac{2Gm(r_0)\rho_0}{r_0}+\left(\vec{J}_0\times\vec{H}_0\right)_{\text{rad}}. \quad \ldots \quad (6)$$

We assume the helical magnetic field inside the cylinder to be of the form

with

$$\left.\begin{aligned}\vec{H}_0^{(\text{int})} &= (0,\ K_1r_0,\ K(r_0^2-R^2)/2)\\ \vec{J}_0 &= (0,\ -Kr_0/4\pi,\ K_1/2\pi)\end{aligned}\right\}, \quad \ldots \quad (7)$$

where R is the radius of the cylinder and r_0 is the distance of any particle from the axis in the equilibrium state; while the external field is governed by the relation

$$\vec{H}^{(\text{ext})} = (0,\ K_1R^2/r_0,\ 0). \quad \ldots \quad (8)$$

Equation (1) is solved under the boundary conditions :

$$\left.\begin{aligned}&\text{(i)}\ \ \delta r = 0 \ \text{ at } \ r_0 = 0\\ &\text{(ii)}\ \ \delta P = 0 \ \text{ at } \ r_0 = R\end{aligned}\right\}, \quad \ldots \quad (9)$$

where P stands for the total pressure. Now across the boundary of the cylinder P should be continuous, *i.e.*

$$p^{(\text{int})}+H^{(\text{int})^2} = H^{(\text{ext})^2}/8\pi. \quad \ldots \quad (10)$$

We assume a vacuum outside the cylinder. For no surface current $\vec{H}^{(\text{int})} = \vec{H}^{(\text{ext})}$ and hence on the boundary of the cylinder $\vec{p}^{(\text{int})}=0$. Throughout the space outside the cylinder, $\vec{H}^{(\text{ext})}$ is uniform.

In view of the above relations we have

$$p_0 = \pi G\rho_0^2(R^2-r_0^2)+\frac{K_1^2}{4\pi}(R^2-r_0^2)-\frac{K^2}{32\pi}(R^2-r_0^2)^2. \quad \ldots \quad (11)$$

Writing $x = r_0/R$, the eqn. (1) can be written as

$$\frac{\partial^2\xi}{\partial x^2}\left(\frac{b}{d}x^5+\frac{c}{d}x^3+x\right)+\frac{\partial\xi}{\partial x}\left(\frac{e}{d}x^4+\frac{f}{d}x^2+3\right)+\xi\left(\frac{h}{d}x-\frac{k}{d}x^3\right)=0, \quad \ldots \quad (12)$$

where

$$b = K^2R^3(1-\Gamma/2)/16\pi,$$

$$c = -2b+\frac{K_1^2R}{4\pi}(1-\Gamma)-\pi G\Gamma R\rho_0^2,$$

$$d = b+\pi\rho_0^2GR\Gamma+\frac{K_1^2\Gamma R}{4\pi},$$

$$e = 7b,\ f = 5c,\ g = 3d,\ k = -8b$$

and

$$h = \sigma^2\rho_0R+4\pi\rho_0^2GR+4c.$$

Equation (12) has singularities (given as $(1/\lambda)^{1/2}$ where λ is given by eqn. (16) and $x = 0$). We have to find out regular integrals which are finite in the range $0 \leqslant x \leqslant 1$ in order that the boundary conditions may be satisfied.

We take

$$\xi = \sum_{n=0}^{\infty} a_n x^{n+p} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where n is a positive integer. The roots of the indicial equation are -2 and 0. To avoid the singularity at the origin we neglect the former root. The coefficients are thus given by the relations

$$8a_2+(h/d)a_0 = 0$$

$$\frac{a_{2n+2}}{a_{2n}} = -\frac{h+2n(2n+4)c}{d(2n+2)(2n+4)}-\frac{a_{2n-2}}{a_{2n}}\cdot\frac{(2n-2)(2n+4)b-k}{d(2n+2)(2n+4)}. \quad \ldots \quad (14)$$

Writing N_{n+1}, $L(n)$ and $M(n)$ respectively for

$$\frac{a_{2n+2}}{a_{2n}},\ -\frac{h+2n(2n+4)c}{d(2n+2)(2n+4)} \text{ and } -\frac{b(2n-2)(2n+4)-k}{d(2n+2)(2n+4)}$$

we have

$$N_{n+1} = L(n)+\frac{M(n)}{N_n}. \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Let λ be the limit of N_n (and thus of N_{n+1} also) as $n \to \infty$. Taking the limit we find

$$\lambda^2+\lambda\frac{c}{d}+\frac{b}{d} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Let the root of eqn. (16) that is less than 1 be denoted by A.
For ξ to be regular in the interval $0 \leqslant x \leqslant 1$, N_n must tend to the limit A.
Equation (15) can be written in the form

$$N_n = \frac{-M(n)}{L(n)-N_{n+1}} \quad \ldots \quad (17)$$

(negative sign of c/d is taken into account and $F = h/d$ contains frequency). By a successive application of eqn. (17) we get a convergent continued fraction for the representation of N_n. The continued fraction for N_1 is

$$N_1 = \frac{-k/24d}{\frac{12c/d-F}{24} -} \; \frac{\left(16\frac{b}{d}-\frac{K}{d}\right)\Big/48}{\frac{32c/d-F}{48} -} \; \frac{\left(40\frac{b}{d}-\frac{K}{d}\right)\Big/80}{\frac{60c/d-F}{80} -} \cdots \quad \ldots \quad (18)$$

But $N_1 = -F/8$, from eqn. (14). Therefore, we have

$$\frac{F}{8} + \frac{-k/24d}{\frac{12c/d-F}{24} -} \; \frac{\left(16\frac{b}{d}-\frac{k}{d}\right)\Big/48}{\frac{32c/d-F}{48} -} \; \frac{\left(40\frac{b}{d}-\frac{k}{d}\right)\Big/80}{\frac{60c/d-F}{80} -} \cdots = 0. \quad \ldots \quad (19)$$

This equation gives the admissible values of F, which finally gives the value of σ, the frequency of oscillations. Taking different values of F, we plotted a graph for eqn. (14). Also for different values of F, we calculated N_5 by assuming it to have the limiting value A and then obtained N_1 using eqn. (17). We plotted a second graph for these values of N_1 and F. The point of intersection of the two graphs gives the approximate solution of eqn. (19). Hence the frequency can be determined.

For the Spiral Arm

$$R \simeq 75{\cdot}7 \times 10^{19} \text{ cm}, \quad \rho = 2 \times 10^{-24} \text{ gm/c.c.}, \quad \Gamma = 5/3,$$

$H_\theta = 10^{-6}$ gauss at the surface and $H_z = 10^{-6}$ gauss at the axis.

$$\therefore \quad \sigma^2 = 13{\cdot}34 \times 10^{-31} \text{ sec}^{-2}$$

and

$$\xi = a_0(1+0{\cdot}0094x^2-0{\cdot}0003x^4+0{\cdot}0001x^6\ldots).$$

Second Case

Here $$\xi = \text{constant}$$

$$\sigma^2 \int^M r^2 dm = 13{,}788{\cdot}56 \times 10^{26}.$$

Integrating, we get

$$\sigma^2 = 13{\cdot}38 \times 10^{-31} \text{ sec}^{-2}.$$

Similarly if H_θ were zero, we would get

$$\sigma^2 = 11{\cdot}48 \times 10^{-31} \text{ sec}^{-2}$$

with

$$\xi = a_0(1+0{\cdot}011x^2+0{\cdot}0002x^4+0{\cdot}0001x^6\ldots)$$

for the case $\xi \neq$ constant in space,

and

$$\sigma^2 = 11{\cdot}52\times10^{-31}\ \mathrm{sec}^{-2}$$

for the case $\xi =$ constant.

Hence we conclude that the frequency remains almost the same whether we take ξ to be a constant or as a function of the displacement.

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