# STUDY OF TRANSITION ZONE OF LAMINAR FLOW AT THE ENTRANCE TO A PIPE BASED ON VARYING FRICTION

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(*Received November 5, 1965; after revision February 19, 1966*)

This paper presents the results of an analytical solution for the velocity profiles in the transition zone of laminar flow by assuming a proper varying frictional loss in the zone. It has been shown that the transition is much more gradual than reported by others. A suitable criterion for characterizing the fully-established laminar flow is suggested.

## SYNOPSIS

A theoretical solution has been developed for the velocity profiles in the transition zone of laminar flow at the entrance to a pipe by assuming a proper varying frictional loss, along the transitional length. This solution is found to be in good agreement with the experimental data of Nikuradse as given by Prandtl and Tietjans (1934) and recent data of Asthana (1951) and Shapiro *et al.* (1954).

It has been shown from the experimental data of Nikuradse and Asthana that the kinetic energy correction factor, $\alpha$, attains a value of two much more gradually than the ratio of core velocity to the average velocity does. The authors' theoretical solution for the kinetic energy correction factor shows a similar trend and is found to be in good agreement with the above data.

A suggestion has been made, in the light of the above, for a suitable criterion for characterizing the fully-established laminar flow.

## INTRODUCTION

When a fluid enters a pipe from a cistern through a smooth entry, the distribution of velocity in laminar flow changes from a rectangle (uniform) to a parabola with the gradual growth of the boundary layer (Fig. 1). In this process both the ratio of the core velocity to the average velocity and the kinetic energy correction factor, $\alpha$, change from a value of one at the entry to a value of two where the flow is fully established. Though the fully-established profile is attained only at an infinite distance from the entry, in a finite length from the entry, the difference from the fully-developed distribution becomes less than the experimental error. This length is defined as the *length of transition*.

The theoretical solutions of many of the previous investigators have some weakness or other and their solutions are reliable only in certain regions of the transition zone, as rightly pointed out by Shapiro *et al.* (1954). It is, therefore, felt desirable to solve this problem with better assumptions so that the solution will satisfy the experimental data available throughout the transition zone.

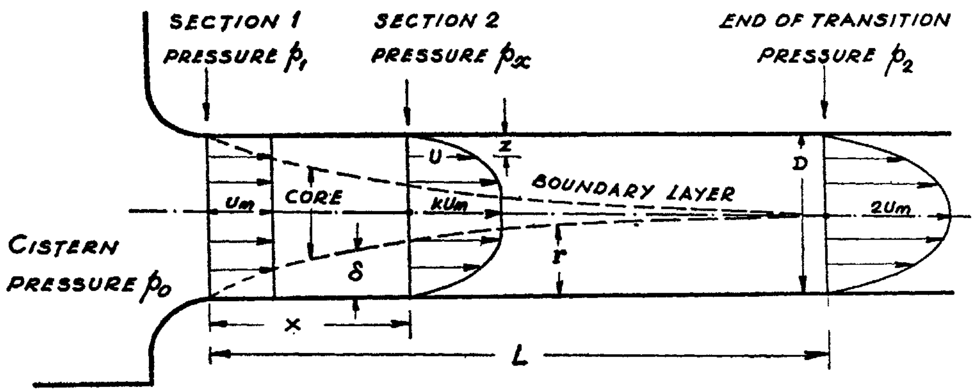


FIG. 1. Definition sketch.

In this paper the authors have assumed a law for the frictional loss in this zone which is proved to be logical. With this assumption a complete study of the zone has been made and compared with the analytical and experimental data of the previous investigators.

## Previous Work

Until recently the only measured velocity profiles of the laminar flow at the entrance to a smooth pipe were those of Nikuradse. All the theoretical investigations so far made have been compared with only Nikuradse's experimental data. Recently Asthana (1951) has measured the velocity profiles which have been discussed in detail later.

Many investigators have tackled this problem and Boussinesq (1960) can be said to be the first to have made a theoretical investigation of the length of transition. He arrived at the following equation for the length of transition:

$$L = 0{\cdot}065\,RD. \qquad (1)$$

His solution agrees with Nikuradse's experimental values beyond a length $x$ from the bellmouth entrance, approximately equal to 0·025 $RD$ (Fig. 2). For a lower value of $x$, less than 0·025 $RD$, the velocity profiles assumed are not correct as indicated by actual experimental observations of Nikuradse.

Schillar (1938) found the length of transition as 0·02875 $RD$ which is considerably smaller than the experimental value found by Nikuradse. His core velocity does not approach twice the average velocity asymptotically and an abrupt break is noticed at $x = 0{\cdot}02875\ RD$ (Fig. 3). His values cannot be accepted for values of $x > 0{\cdot}02\ RD$.

Atkinson and Goldstein (1938) extended Boussinesq's work and arrived at a more accurate solution for this problem for values of $x > 0{\cdot}0075\ RD$. They

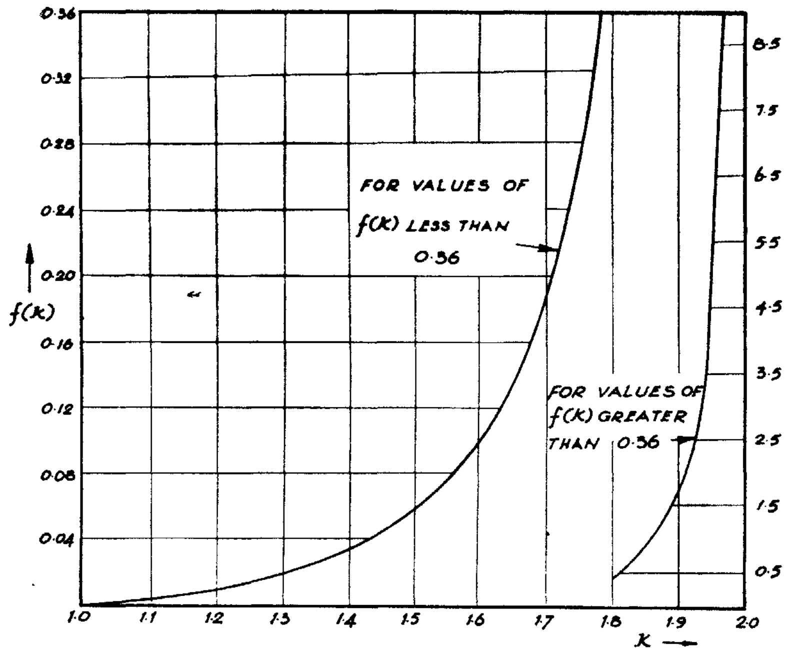


FIG. 2. A plot of $f(K)$ versus $K$.

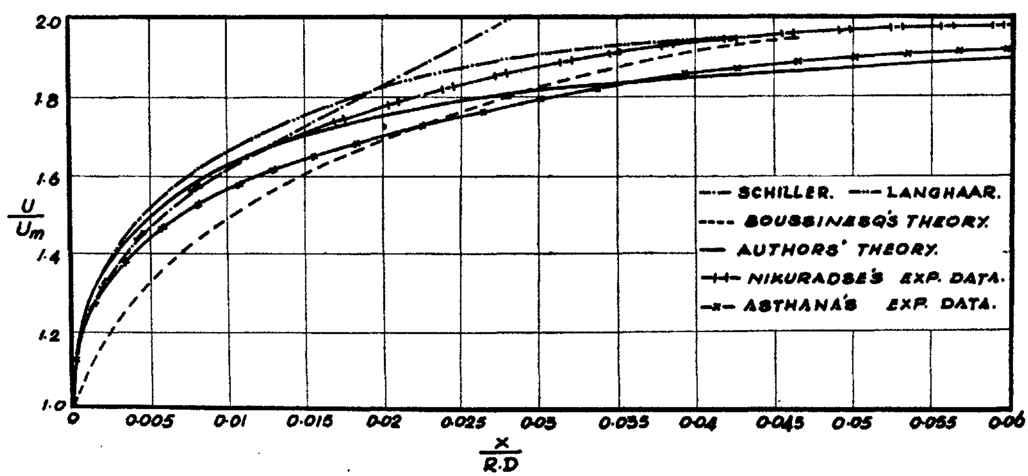


FIG. 3. Centre-line velocities compared.

solved the differential equation of motion by the use of series expansions and found an expression valid near the entrance of the pipe. This solution was joined to a second expression valid at larger distances from the entrance derived from an extension of the method used by Boussinesq.

Langhaar (1951) solved the Navier-Stokes equation for the case of steady flow in the length of transition of a straight pipe, by assuming the pressure gradient to be a function of $x$ alone and the acceleration to be a function of velocity and $x$. Even though his velocity profiles agree with those of Nikuradse, they radically depart from Asthana's experimental data as seen from Fig. 3.

Asthana (1951) measured velocity profiles in the length of transition and also gave an analytical solution. He has reported that 98 per cent transition is achieved at $x = 0{\cdot}1925\ RD$. It can be seen that none of the previous theoretical solutions is in agreement with this recent experimental value (Fig. 3).

With Asthana's theoretical solution, it has been computed by the authors that 99·5 per cent transition is achieved at a value of $x = 0{\cdot}072\ RD$, which is far less than his own experimental value. This is because his assumed value for the pressure drop is considerably smaller than those given by other investigators. He should have taken more of frictional loss as rightly pointed out by himself.

In 1954, Shapiro reported about integral methods of Thwaite and Pohlhausen. Thwaite's method is reliable for $\frac{x}{RD} < 4 \times 10^{-4}$ and modified Pohlhausen's profiles suffer from more errors in the region $10^{-2} < \frac{x}{RD} < 10^{-1}$. Shapiro and others presented experimental data for the mean apparent friction factor and their theoretical equation is:

$$\bar{f}\left(\frac{x}{D}\right) = 13.74 \sqrt{\frac{x}{RD}} \text{ for } \frac{x}{RD} < 10^{-3}$$

where

$$\bar{f} = (p_1 - p_x)/\tfrac{1}{2}\rho U_m^2 .$$

Even though the apparent friction factor computed by these methods fit the experimental data, the lengths of transitions obtained by these methods are of the order $\frac{x}{RD} = 0{\cdot}03$ which are far less than the experimental findings. Hence, as mentioned earlier, it has been sought to solve this problem theoretically so that the solution is unique for the entire range of transition and checks well with the experimental data available.

## Solution

In order to solve the problem, the following assumptions have been made:

1. The central core of flow (Fig. 1) has constant velocity at any section.
2. Within the boundary layer the velocity distribution is parabolic with zero velocity at the wall of the pipe and reaching the core velocity tangentially at a distance $\delta$, the boundary layer thickness, from the wall.
3. The core flow obeys Bernoulli's equation for irrotational flow and the same pressure exists in the boundary layer also.

These assumptions are the same as those made by Schillar. An assumption for frictional loss has been made at a later stage of the development of the equation.

*Energy Equation*

Consider section 1 at the entry and section 2 at a distance $x$ from the entry (Fig. 1). Comparing the total energies between the two sections we can write in the core flow

$$\frac{p_1}{\gamma}+\frac{U_m^2}{2g}=\frac{p_x}{\gamma}+\frac{\alpha U_m^2}{2g}+h_f \quad \ldots \quad (2)$$

where $p_1$ and $p_x$ are the pressures at sections 1 and 2 respectively, $h_f$ the frictional loss and $\gamma$ the unit weight of the fluid.

$$\therefore \quad p_1-p_x=\frac{\rho U_m^2}{2}(\alpha-1)+h_f.\gamma. \quad \ldots \quad (2a)$$

*Continuity Equation*

At section 2, let $U$ be the velocity at a point in the boundary layer at a distance $Z$ from the wall of the pipe and $KU_m$ the core velocity.

Let the boundary layer thickness be $\delta$ at section 2.

By assumption 2,

$$U=KU_m\left[\frac{2Z}{\delta}-\left(\frac{Z}{\delta}\right)^2\right]. \quad \ldots \quad (3)$$

Applying the equation of continuity between sections 1 and 2, we get

$$U_m.\pi r^2=\pi(r-\delta)^2KU_m+\int_0^\delta 2\pi(r-Z)dZ\,.\,U. \quad \ldots \quad (4)$$

Solving for $\frac{\delta}{r}$, we get

$$\frac{\delta}{r}=2-\sqrt{\frac{6}{K}-2}. \quad \ldots \quad (5)$$

*Momentum Equation*

$$F=\frac{d}{dt}(\text{momentum at section 2}-\text{momentum at section 1}) \quad \ldots \quad (6)$$

where $F$ = the resultant of the external forces acting in the direction of motion
$t$ = time.

The external forces acting are the thrust due to the difference of pressures between the two sections acting in the direction of motion and the viscous drag along the boundary acting against the motion.

$$\therefore \quad F = \pi r^2 (p_1 - p_x) - \int_0^x \tau_0 \,.\, 2\pi \,.\, r \,.\, dx \qquad \ldots \quad \ldots \quad (7)$$

where $\tau_0$ is the shear stress at the wall at the section which is at a distance $x$ from the entry.

Momentum at section 2

$$= \pi(r-\delta)^2 K U_m \rho K \,.\, U_m\,(dt) + \int_0^\delta 2\pi(r-Z)\,dZ \,.\, U \,.\, \rho \,.\, U \,.\, dt \qquad \ldots \quad (8)$$

Momentum at section 1

$$= \pi \rho r^2 U_m^2\, dt$$

$$\therefore \quad \frac{d}{dt}\,(\text{momentum at section } 2 - \text{momentum at section } 1)$$

$$= \pi \rho U_m^2 \left[K^2 (r-\delta)^2 + \tfrac{2}{15} K^2 \left(8r\delta - \tfrac{11}{2}\delta^2\right) - r^2\right]. \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

Now eqn. 6 can be written as

$$\pi r^2 (p_1 - p_x) - \int_0^x \tau_0 \,.\, 2\pi r \,.\, dx = \pi \rho U_m^2 \left[K^2 (r-\delta)^2 + \tfrac{2}{15} K^2 \left(8r\delta - \tfrac{11}{2}\delta^2\right) - r^2\right]. \qquad \ldots \quad (10)$$

Substituting for $(p_1 - p_x)$ from eqn. (2*a*), differentiating the above equation with respect to $x$ and rearranging, we get

$$\tau_0 \,.\, 2\pi r = \pi r^2 \left[\frac{\rho U_m^2}{2} \cdot \frac{d\alpha}{dx} + \gamma \frac{dh_f}{dx}\right] - \pi \rho U_m^2 \frac{d}{dx} \left[K^2 (r-\delta)^2 + \tfrac{2}{15} K^2 \left(8r\delta - \tfrac{11}{2}\delta^2\right) - r^2\right]. \qquad \ldots \quad \ldots \quad (11)$$

Now we have to get expressions for $\tau_0$, $\alpha$ and $h_f$ to solve the equation (11).

(*a*) Expression for $\tau_0$:

$$\tau_0 = \mu \left(\frac{du}{dz}\right)_{z=0}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

From eqn. (3)

$$\left(\frac{du}{dz}\right)_{z=0} = K U_m \left(\frac{2}{\delta}\right) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$\therefore \quad \tau_0 = \frac{2K\mu U_m}{2r - r\sqrt{\frac{6}{K} - 2}}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

(*b*) EXPRESSION FOR $\alpha$:

The kinetic energy per one unit of weight of fluid, passing through section 2, can be written as

$$\alpha \frac{U_m^2}{2_g} = \frac{K \cdot E}{\text{unit of wt.}} = \frac{\pi(r-\delta)^2 K U_m \gamma \frac{K^2 U_m^2}{2_g} + \int_0^\delta 2\pi(r-z)\, dz \frac{U \cdot U^2}{2_g} \cdot \gamma}{\pi r^2 U_m \cdot \gamma}. \quad (15)$$

Substituting for values of $U$ and $\delta$ from eqns. (3) and (5) respectively in the above equation and solving for $\alpha$ we get

$$\alpha = -\frac{K^3}{2} - \tfrac{9}{35} K^2 \sqrt{6K - 2K^2} + \tfrac{141}{70} K^2. \quad \ldots \quad \ldots \quad (16)$$

(*c*) EXPRESSION FOR $h_f$:

In fully-established laminar flow the whole of the flow is influenced by viscosity and there is no further change in the velocity profile. Hence, all the pressure drop is used to overcome the viscous friction alone. But in the transition zone, the fluid in the core is being accelerated while the fluid within the boundary layer experiences a retardation.

If the entire flow is affected by viscosity for the same value of $\tau_0$ at the boundary as at section 2, the corresponding pressure drop would be

$$\Delta p = \frac{4\mu K U_m \Delta x}{r\delta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$\left[\because \quad \Delta p \pi r^2 = \tau_0 \cdot 2\pi r \cdot \Delta x = \frac{4\mu K U_m \Delta x \cdot \pi r}{\delta}\right].$$

But actually the area that is affected by viscosity here in this case is only $\pi[r^2 - (r-\delta)^2]$. Hence, it is logical to assume that the actual pressure drop required to overcome the friction to be equal to

$$\frac{4\mu K U_m \cdot \Delta x}{r\delta} \left[\frac{\pi r^2 - \pi(r-\delta)^2}{\pi r^2}\right].$$

Hence, the frictional loss in a length $x$,

$$h_f = \frac{1}{\gamma} \int_0^x \frac{4\mu K U_m}{r\delta} \left[\frac{\pi r^2 - \pi(r-\delta)^2}{\pi r^2}\right] dx \quad \ldots \quad \ldots \quad \ldots \quad (18a)$$

$$= \frac{1}{\gamma} \cdot \frac{4\mu U_m}{r^2} \int_0^x \sqrt{6K - 2K^2} \cdot dx. \quad \ldots \quad \ldots \quad \ldots \quad (18b)$$

With this assumption, now, eqn. (11) can be solved for $x$ in terms of $K$. We may proceed as follows for the solution.

Equation (11) reads as

$$\tau_0 \cdot 2\pi r = \left[\frac{\rho U_m^2}{2} \cdot \frac{d\alpha}{dx} + \gamma \frac{d}{dx} h_f\right] \pi r^2$$

$$- \pi \rho U_m^2 \frac{d}{dx}\left[K^2(r-\delta)^2 + \tfrac{2}{15} K^2\left(8r\delta - \tfrac{11}{2}\delta^2\right) - r^2\right] \quad \ldots \quad \ldots \quad (11)$$

where

$$\frac{d\alpha}{dx}=\left[-\tfrac{3}{2}K^2+\tfrac{141}{35}K-\tfrac{9}{35}\left(2K\sqrt{6K-2K^2}+K^2\frac{(3-2K)}{\sqrt{6K-2K^2}}\right)\right]\frac{dk}{dx}.$$

Hence, eqn. (11) can be rewritten, substituting the above value as

$$2\pi r\,.\,\frac{2K^2\mu U_m}{r\left(2K-\sqrt{6K-K^2}\right)}=\pi r^2\frac{\rho U_m^2}{2}\Bigg[-\tfrac{3}{2}K^2+\tfrac{141}{35}K-\tfrac{9}{35}\Bigg\{2K\sqrt{6K-2K^2}$$

$$+K^2\frac{(3-2K)}{\sqrt{6K-2K^2}}\Bigg\}\Bigg]\frac{dk}{dx}+\frac{4\mu U_m}{r^2}\sqrt{6K-2K^2}\,.\,\pi r^2$$

$$-\pi\rho U_m^2\frac{d}{dx}\Bigg[K^2\Bigg\{\left(r-2r+r\sqrt{\left(\frac{6}{K}-2\right)}\right)\Bigg\}^2$$

$$+\tfrac{2}{15}K^2\left\{8r-\tfrac{11}{2}\left(2r-r\sqrt{\frac{6}{K}-2}\right)\right\}\left(2r-r\sqrt{\frac{6}{K}-2}\right)\Bigg]. \quad \text{.. (19)}$$

This reduces to a form

$$\frac{4K^2}{2K-\sqrt{6K-2K^2}}-4\sqrt{6K-2K^2}=r^2\rho U_m\Bigg[-\tfrac{3}{4}K^2+\tfrac{141}{70}K$$

$$-\tfrac{9}{70}\left\{2K\sqrt{6K-2K^2}+K^2\frac{(3-2K)}{\sqrt{6K-2K^2}}\right\}$$

$$-\tfrac{8}{5}+\tfrac{2}{3}K+\tfrac{2}{15}\sqrt{6K-2K^2}+\tfrac{2}{15}\frac{(3-2K)k}{\sqrt{6K-2K^2}}\Bigg]\frac{dk}{dx}. \quad (20)$$

(The term within the square bracket is denoted as $[\Delta]$)

$$=\frac{Rr}{2}[\Delta]\frac{dK}{dx} \quad \text{.. .. .. .. .. .. .. (20a)}$$

$$\therefore\ \frac{dx}{rR}=\tfrac{1}{8}\left[\frac{2K-\sqrt{6K-2K^2}}{K^2-\left[2K-\sqrt{6K-2K^2}\right]\sqrt{6K-2K^2}}\right][\Delta]\,dK. \quad \text{.. (20b)}$$

Integrating both sides between the limits 0 to $x$, we get

$$\frac{x}{rR}=\tfrac{1}{8}\int_1^K\left[\frac{2K-\sqrt{6K-2K^2}}{K^2-\sqrt{6K-2K^2}\left[2K-\sqrt{6K-2K^2}\right]}\right][\Delta]\,dK \quad \text{.. (20c)}$$

$$=\int_1^K f(K)\,dK$$

$$\therefore\ \frac{x}{RD}=\tfrac{1}{2}\int_1^K f(K)\,dK. \quad \text{.. .. .. .. .. .. (21)}$$

The right-hand side cannot be integrated mathematically and hence graphical integration is resorted to as has been suggested by Asthana.

## CALCULATIONS

Figure (2) gives a plot of $f(k)$ versus $K$. By graphical integration, $\frac{x}{RD}$ values have been computed and tabulated as below:

| $K$ | $\frac{x}{R.D}$ |
|---|---|
| 1 | 0 |
| 1·1 | 0·0000611 |
| 1·2 | 0·000336 |
| 1·3 | 0·0009855 |
| 1·4 | 0·0022205 |
| 1·5 | 0·004423 |
| 1·6 | 0·0081505 |
| 1·7 | 0·0146250 |
| 1·8 | 0·027925 |
| 1·9 | 0·067613 |
| 1·95 | 0·140113 |
| 1·96 | 0·1991 |
| 1·97 | 02·6035 |
| 2 | $\infty$ |

### *Calculation of Velocity Profiles*

For any value of $k$, $\delta$ is known in terms of $r$ from eqn. (5). If the core velocity is $KU_m$ at a point in the boundary layer, $y$ from the centre, the velocity, $U$, is given by

$$U = KU_m \left[\frac{2}{\delta}(r-y) + \left(\frac{r-y}{\delta}\right)^2\right]$$

$y$ is given values of 0, 0·2$r$, 0·4$r$, 0·6$r$, 0·7$r$, 0·8$r$ and 0·9$r$ and the corresponding values of $U$ are calculated. For this $K$, $\frac{x}{RD}$ is known from eqn. (21).

Calculations are repeated for various values of $K$. The computed velocities are shown non-dimensionally in Fig. 4.

### *Calculation of* $\alpha$

A value of $K$ is assumed and $\alpha$ is calculated from eqn. (16) and for the same $K$, $\frac{x}{RD}$ is also computed. Figure 5 shows a plot of computed $\alpha$ versus $\frac{x}{RD}$.

## EXPERIMENTAL DATA

1. Nikuradse's and Asthana's measured velocity profiles are shown in Fig. 4.

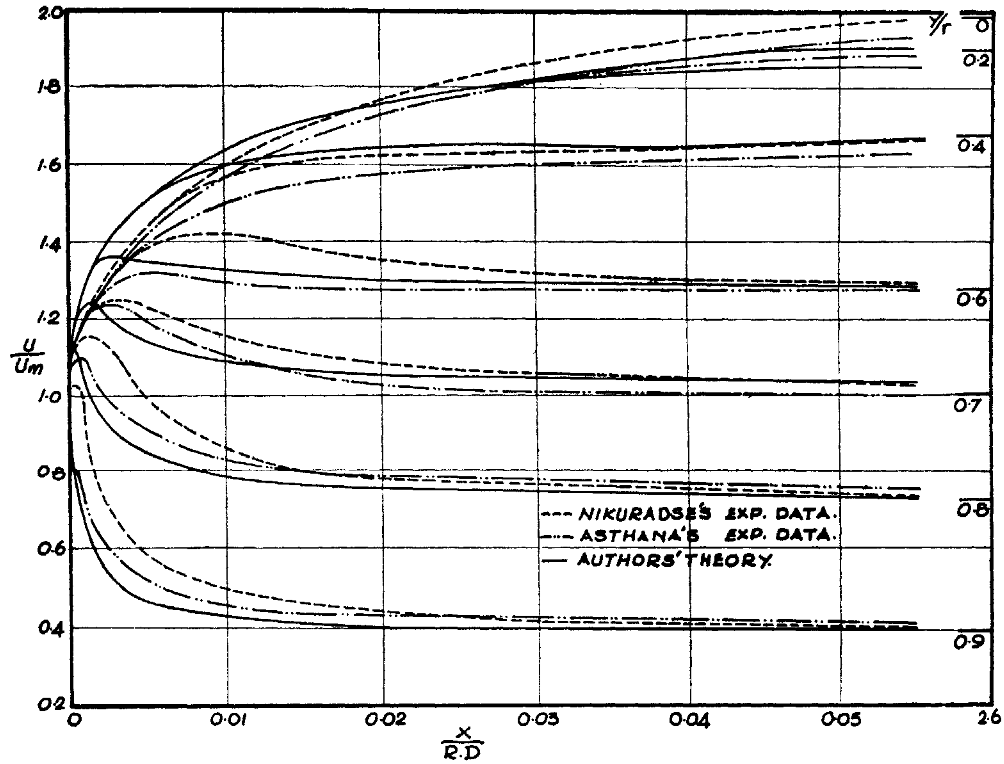


FIG. 4. Authors' velocity profiles and experimental curves.

2. From the measured velocity profiles $\alpha$ is calculated by numerical integration at any section within the zone of transition. These values are shown in Fig. 5.

3. Shapiro's experimental data for the mean apparent friction for $\frac{x}{RD} < 10^{-3}$ is shown in Fig 7.

*Comparison of the Theoretical Solution with the Experimental Data*

1. *Velocity Profiles.*—It can be seen from Fig. 4 that for values of $\frac{x}{RD} > 0{\cdot}02$ authors' curves agree very closely with the experimental data of both except that for the centre-line velocity.

For values of $\frac{x}{RD} < 0{\cdot}02$, the theoretical curves are in good agreement with Asthana's data but not with those of Nikuradse. It may be pointed out here that Nikuradse's values are reproduced from a small-scale graph in Prandtl-Tietjans 'Hydro- und Aero-Mechanik' 2 (Berlin 1931). Further details do not appear to have been published.

It may also be pointed out that when a swirl exists at the entrance, Talbot (1960) suggests that it takes about 40 diameters for complete decay when the Reynolds Number is 1,000. This may be the reason for the deviation of the experimental data from the theoretical curves very near the entry region.

The centre-line velocity curve, even though it shows a very good agreement near the entry with the experimental data of Nikuradse, deviates from his values at higher $\frac{x}{RD}$ values, showing a much more gradual transition. But the theoretical centre-line curve is in very good agreement with that of Asthana throughout the region.

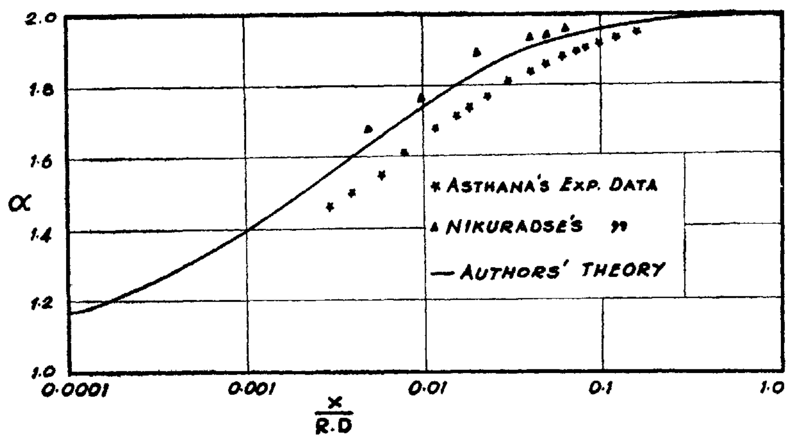


FIG. 5. Theoretical α values compared with other data.

2. *Kinetic Energy Correction Factor* α.—The theoretical α curve shown in Fig. 5 is in between the experimental values of Nikuradse and Asthana and fits better with those of Nikuradse.

It can be seen that Nikuradse obtains a value of $\alpha = 1{\cdot}96$ for his 100 per cent velocity profile transition. This means only 98 per cent transition in α is obtained. According to the authors' solution, for the same value of $\frac{x}{RD} = 0{\cdot}0625$, α is found to be as 1·93, i.e. about 96·5 per cent transition in α, whereas Asthana's values show about 94 per cent transition. It can be seen that the theoretical value is within $\pm 2{\cdot}5$ per cent of the experimental data.

It may also be noticed that Nikuradse's α curve shows a much more gradual transition than the velocity profile curves, clearly indicating that the profile at $\frac{x}{RD} = 0{\cdot}0625$ is not a true parabola. To substantiate this, the authors have computed the value of α for an ideal parabola with the same method of numerical integration and found α to be 1·999.

So, even a slight deviation of any velocity distribution from the true parabola can be detected from the value of α it gives. The authors feel that the transition should be characterized better with $\alpha$ criterion rather than the central velocity criterion.

3. *Length of Transitions.*—The theoretical length of transition is computed to be $\infty$. The theory gives 94 per cent transition for velocity profile at $\frac{x}{RD} = 0{\cdot}0625$ where Nikuradse shows 100 per cent transition. But Asthana's experimental values show that 98 per cent transition is obtained at a value of $\frac{x}{RD} = 0{\cdot}1925$. The authors' solution gives the same amount of transition for $\frac{x}{RD} = 0{\cdot}2$.

Based on $\alpha$ curve, 98 per cent transition is achieved at $\frac{x}{RD} = 0{\cdot}23$.

It is concluded in the light of the calculations based on $\alpha$, the actual length of transition should be much more than those that have been reported earlier.

4. *Pressure Drop.*—The pressure drop from the cistern to any section at a distance $x$ from the entry is given by

$$\frac{p_0 - p_x}{\frac{1}{2}\rho U_m^2} = \frac{1}{\frac{1}{2}\rho U_m^2}\int_0^x \frac{4\mu K U_m}{r\delta}\left(\frac{2r\delta - \delta^2}{r^2}\right) dx + \alpha$$

$$= \frac{64}{RD}\int_0^x K\left(1 - \frac{\delta}{2r}\right) dx + \alpha. \qquad \ldots \quad (22)$$

In order to compare with Shapiro's experimental data the above pressure drop equation can be written as follows:

$$f = -\frac{dp}{d\left(\frac{x}{D}\right)} \cdot \frac{1}{\frac{1}{2}\rho U_m^2} \qquad \ldots \quad (23)$$

and

$$f = \frac{p_1 - p_x}{\frac{1}{2}\rho U_m^2} \text{ (as defined by Shapiro)} \qquad \ldots \quad (24)$$

$$-\frac{dp}{dx} = \gamma\frac{dh_f}{dx} + \tfrac{1}{2}\rho U_m^2 \cdot \frac{d\alpha}{dx} \qquad \ldots \quad (25)$$

$$f = D\frac{1}{\frac{1}{2}\rho U_m^2}\left[\left(\gamma \cdot \frac{dh_f}{dx}\right) + \tfrac{1}{2}\rho U_m^2 \frac{d\alpha}{dx}\right] \qquad \ldots \quad (26)$$

$$\therefore \quad f\left(\frac{x}{D}\right) = \frac{1}{\frac{1}{2}\rho U_m^2}\int_0^x \gamma\frac{dh_f}{dx} \cdot dx + \int_0^x \frac{d\alpha}{dx} \cdot dx \qquad \ldots \quad (27)$$

$$f\left(\frac{x}{D}\right) = \frac{1}{\frac{1}{2}\rho U_m^2}\int_0^x \frac{4\mu U_m}{r^2}\sqrt{6K - 2K^2}\, dx + \int_0^x \frac{d\alpha}{dx} \cdot dx \qquad \ldots \quad (28)$$

from equation (21)

$$\frac{x}{RD} = \frac{1}{2}\int_1^K f(K)\,dk \quad \ldots \quad (21)$$

$$\therefore \quad \bar{f}\left(\frac{x}{D}\right) = \int_1^K 16\sqrt{6K-2K^2}\,.f(K)\,dk + (\alpha - 1). \quad \ldots \quad (29)$$

The values of $\bar{f}\left(\frac{x}{D}\right)$ are calculated for values of $K$ and plotted against $\frac{x}{RD}$ in Fig. 6.

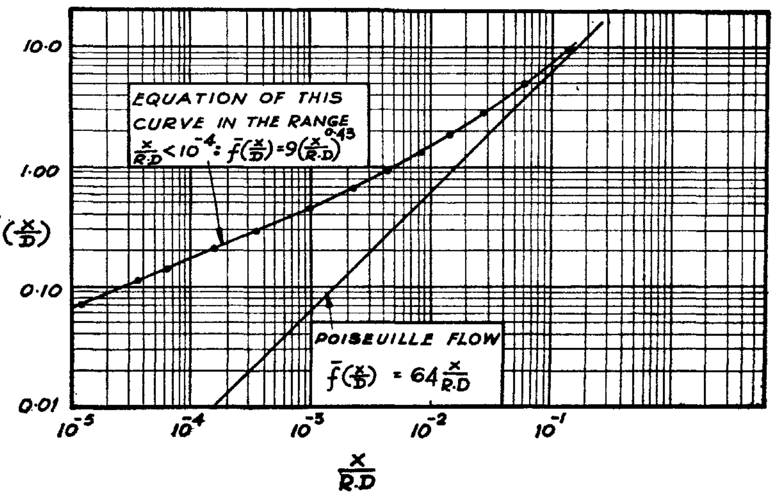


FIG. 6. Theoretical non-dimensional pressure-drop profile.

For $\frac{x}{RD} < 10^{-3}$, the pressure-drop equation is

$$\bar{f}\left(\frac{x}{D}\right) = 9\left(\frac{x}{RD}\right)^{0.43}.$$

From this curve the values of $f$ can be calculated by differentiating equation No. (28) and the values of $f.(R)$ are plotted in Fig. 7 for $\frac{x}{RD} < 10^{-3}$, the equation of the curve is found to be

$$f.(R) = 3.87\left(\frac{x}{RD}\right)^{-.57}.$$

The experimental points fit this curve very well for values of $\frac{x}{RD} > 10^{-4}$.

## CONCLUSIONS

1. By assuming a rate of frictional loss which is varying, a solution has been obtained to study the zone of transition.

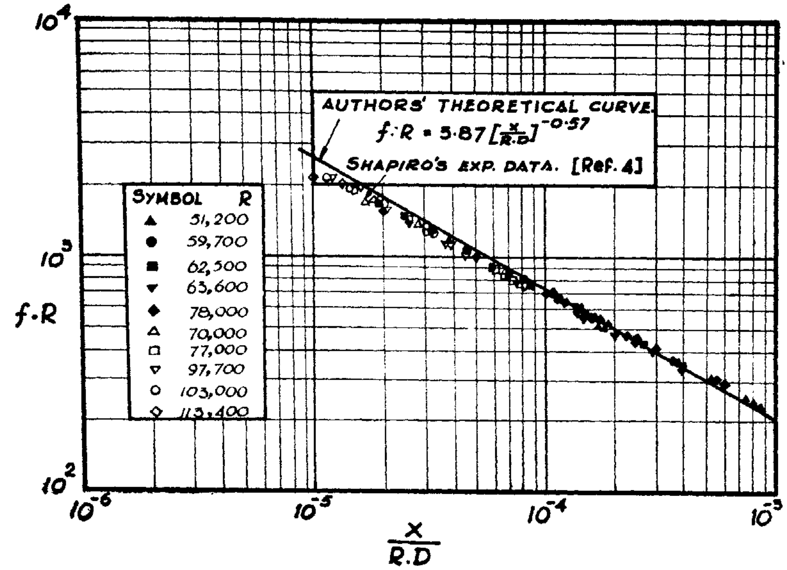


FIG. 7. Theoretical apparent friction factor compared with experimental data.

2. This has been found to check very well with the experimental data available. Hence, the assumption made for the frictional loss has been verified to be appropriate.

3. Values of $\alpha$ are better indicators of deviation from the true parabolic distribution of velocity than the ratio of core velocity to the average velocity and the transition may be characterized better by the $\alpha$ values approaching the limiting values.

4. It is felt that the 'length of transition' has to be more than that reported by others.

## ACKNOWLEDGEMENT

The authors wish to thank Dr. Asthana for sending his thesis for their reference.

## REFERENCES

Asthana, K. C. (1951). Study of the zone of transition in laminar flow near the entrance to a smooth pipe. Ph.D. Thesis submitted to the Graduate School, Cornell University, Ithaca, N.Y.

Atkinson and Goldstein, S. (1938). Modern Developments in Fluid Mechanics, Vol. 1. Oxford University Press.

Boussinesq. J. (1960). Same as reference Schlichting (1960).

Langhaar, H. L. (1951). Same as reference Asthana (1951).

Prandtl, L., and Tietjans, O. G. (1934). Applied Hydro- and Aero-mechanics. McGraw-Hill Book Company Inc., N.Y., 27.

Schillar, L. (1938). Same as reference Atkinson and Goldstein (1938).

Schlichting, H. (1960). Boundary Layer Theory. McGraw-Hill Book Company Inc., N.Y.

Shapiro, A. M., Seigel, R., and Kline, S. J. (1954). Friction factor in the laminar entry region of a smooth tube. Proc. 2nd U.S. natn. congr. appl. mech. held at the University of Michigan, pp. 733-41.
Talbot, L. (1960). Same as reference Schlichting (1960).

## APPENDIX

### *Notation*

$D$ = Diameter of the pipe
$F$ = Resultant of the external forces in the direction of motion
$f$ = Apparent friction factor
$\bar{f}$ = Mean apparent friction factor
$h_f$ = Energy losses in a length $x$
$K$ = The ratio of core velocity to the average velocity
$L$ = Length of transition
$p_0$ = Pressure in the cistern where velocity is zero
$p_1$ = Pressure at the entry of the pipe
$p_2$ = Pressure at the end of the transition
$p_x$ = Pressure at any length $x$ from the entry
$r$ = Radius of the pipe
$R$ = Reynolds' Number of the flow
$t$ = Time
$U$ = Velocity at a radius $(r-Z)$
$U_m$ = Average velocity
$x$ = Distance from the entrance of the pipe
$y$ = Radial distance from the centre of the pipe
$z$ = Radial distance from the wall
$\alpha$ = Kinetic energy correction factor
$\gamma$ = Unit weight of fluid
$\delta$ = Boundary layer thickness
$\mu$ = Viscosity of the fluid
$\rho$ = Mass density of fluid
$\tau_0$ = Shear stress at the wall of the pipe