SOME EFFECTS OF WEAK SWIRL ON LAMINAR PIPE FLOW

by B. C. Deka, *Department of Mathematics, Gauhati University*

(Communicated by F. C. Auluck, F.N.I.)

(*Received February 19, 1965*)

Here an attempt has been made to solve the problem elaborated in a previous paper by the author (Deka 1963) without the Oseen type of approximation for all Reynolds numbers. Calculations were carried out only for high Reynolds numbers and the results were compared with those of the momentum integral equation. Qualitatively, good agreement was obtained between them.

Introduction

1. In a previous paper by the author (Deka 1963) an attempt was made to study the effects of weak swirl on laminar pipe flow under the approximation $W_0 = 2/3$ in the convective terms of the equations of motion. Here we have tried to study the same without that approximation. The mathematical model is the same; but we have considered case (I) only. Attempt has been made to solve the problem for all Reynolds numbers; but for purposes of calculation only high Reynolds numbers have been considered.

2. Mathematical model and the governing equations in non-dimensional form in cylindrical polar co-ordinates are respectively represented by Fig. 4 and the equations (1.1), (1.2), (1.3), (1.4) of the earlier paper. We have tried to solve the first perturbation, the swirl velocity V_1, and the second perturbations U_2, W_2, P_2.

3. The equation for the swirl velocity V_1 and the boundary conditions are given by (2.1) and (2.2) of the earlier paper. We change from z to x by the relation $z = Rx$ and use two-way Laplace transform :

$$\overline{V}_1 = p \int_{-\infty}^{+\infty} e^{-px} V_1 \, dx$$

$$-c_1 < Rp < 0,$$

where c_1 is the least positive eigenvalue, so that the transformed equation reduces to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{p^2}{R^2} - p(1-r^2) - \frac{1}{r^2}\right] \overline{V}_1 = 0. \qquad \ldots \quad \ldots \quad (3.1)$$

This equation is to be solved under the following boundary conditions :

$$\overline{V}_1 = -1 \text{ at } r = 1,$$

and $\overline{V}_1$ and its derivative are continuous at $r = 0$.

Solution of $\overline{V}_1$, as shown by Talbot (1954), is of the form

$$\overline{V}_1 = -\frac{\phi(a,\ 2;\ \mu r^2)}{\phi(a,\ 2;\ \mu)} re^{\mu(1-r^2)/2},$$

where

$$\mu = \sqrt{-p},\ a = 1-(p^2/R^2-p)/4\mu,$$

and $\phi(a,\ 2;\ \mu r^2)$ is the confluent hypergeometric function.
Hence

$$V_1 = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi(a,\ 2;\ \mu r^2)\, re^{\frac{1}{2}\mu(1-r^2)}}{p\phi(a,\ 2;\ \mu)} e^{px}\, dp, \quad \ldots \quad \ldots \quad (3.2)$$

where

$$-c_1 < c < 0.$$

Integrand of (3.2) has no branch point at the origin and has no singularity except the origin and the zeros of $\phi(a,\ 2;\ \mu) = 0$, which are real.
The eqn. (3.1) suggests that for high Reynolds numbers p^2/R^2 can be neglected in comparison to p for

$$p \ll O(R^2),$$

$$x \gg O(R^{-2}).$$

Hence to get an expression for V_1 for $x > 0$, we neglect p^2/R^2 and have

$$V_1 = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi(b,\ 2;\ \mu r^2)\, re^{\frac{1}{2}\mu(1-r^2)}}{p\phi(b,\ 2;\ \mu)} e^{px} dp, \quad \ldots \quad \ldots \quad (3.3)$$

where

$$b = 1-\mu/4.$$

Lawerier (1950) has shown that the zeros of $\phi(b,\ 2;\ \mu)$ are given by

$$\mu = \beta_n = 4(n+\tfrac{1}{6}) \quad n = 1, 2, 3, \ldots$$

We perform the integration of (3.3) along a large semicircle with $c-i\infty$, $c+i\infty$, as diameter. The integral on the curved path vanishes by Jordan's lemma and we are left with the residues at the poles and hence

$$V_1 = r, \quad x < 0,$$

$$V_1 = \sum_1^{\infty} re^{\beta_n(1-r^2)/2} \frac{\phi(bn,\ 2;\ \beta_n r^2)e^{-\beta_n^2 x}}{\beta_n^2 R_n}, \ x > 0, \quad \ldots \quad \ldots \quad (3.4)$$

where

$$R_n = -\frac{1}{2\beta_n}\left[\frac{d}{d\mu}\,\phi(b,\ 2;\ \mu)\right]_{\mu=\beta_n}.$$

It has been found in Appendix I that

$$R_n = \frac{(-1)^n \Gamma\frac{1}{3} e^{\beta_n/2}}{(4\beta_n)^{1/3} 3^{1/6} \beta_n}(1+O\beta_n{}^{-4/3}). \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

Hence after substitution from (3.5) we have

$$V_1 = r, \quad x < 0,$$
$$= \frac{2^{5/3}3^{1/6}}{\Gamma\frac{1}{3}} \sum_1^{\infty} (-1)^{n+1}\beta_n^{1/3}\psi_n(r)e^{-\beta_n^2 x}, \; x > 0, \quad \ldots \quad \ldots \quad (3.6)$$

where

$$\psi_n(r) = re^{-\beta_n r^2/2}\phi(b_n, 2;\; \beta_n r^2).$$

We have investigated the solution of (3.1) for $x \to 0$ in Appendix II, and it is found to be

$$V_1 = r - \frac{1}{\Gamma\frac{1}{3}r^{\frac{1}{3}}} \Gamma\left(\tfrac{1}{3}, \frac{(1-r^2)^3}{36x}\right) [1+Ox^{\frac{1}{3}}],\; r > 0,\; x \sim O\left(\frac{1}{R}\right), \quad \ldots \quad (3.7)$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function. The solution (3.7) shows that for $x \to 0$, there is solid body rotation outside $1-r^2 \sim O(x^{\frac{1}{3}})$.
Hence

$$V_1 = r, \quad x < 0,$$
$$= \frac{2^{5/3}3^{1/6}}{\Gamma\frac{1}{3}} \sum_1^{\infty} (-1)^{n+1}\beta_n^{1/3}\psi_n(r)e^{-\beta_n^2 x}, \quad x > 0,$$
$$= r - \frac{1}{\Gamma\frac{1}{3}r^{\frac{1}{3}}} \Gamma\left(\tfrac{1}{3}, \frac{(1-r^2)^3}{36x}\right) [1+Ox^{\frac{1}{3}}],\; r > 0,\; x \sim O\left(\frac{1}{R}\right). \quad \ldots \quad (3.8)$$

The curves of V_1 from the expression (3.8) are shown along with the curves of Görtler (Colatz and Görtler 1954) in Figs. 1 and 2. The figures show quite good agreement.

4. The Perturbations U_2, W_2, P_2

The differential equations determining the second perturbations are given by the eqns. (3.1), (3.2) and (3.3) of the earlier paper. Using stream function $\psi_2 = r\chi$ these equations can be transformed to

$$\left[\nabla_1^4 - W_0 R \frac{\partial}{\partial z}\nabla_1^2\right]\chi = R\frac{\partial}{\partial z}\left(\frac{V_1^2}{r}\right), \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}.$$

Now eqn. (4.1) is to be solved under the boundary conditions

$$\chi(1, z) = \frac{\partial}{\partial r}\chi(1, z) = 0,$$

$\chi(r, z)$ and its derivatives continuous at $r = 0$. (4.2)

Changing from z to x by the relation $z = Rx$ and taking two-way Laplace transform for Green's function $\chi_G(r, \rho, \xi)$ at $x = \xi$, $r = \rho$, with the force function

$$h(\rho, \xi) = \frac{\partial}{\partial \xi}\left(\frac{V_1^2}{\rho}\right),$$

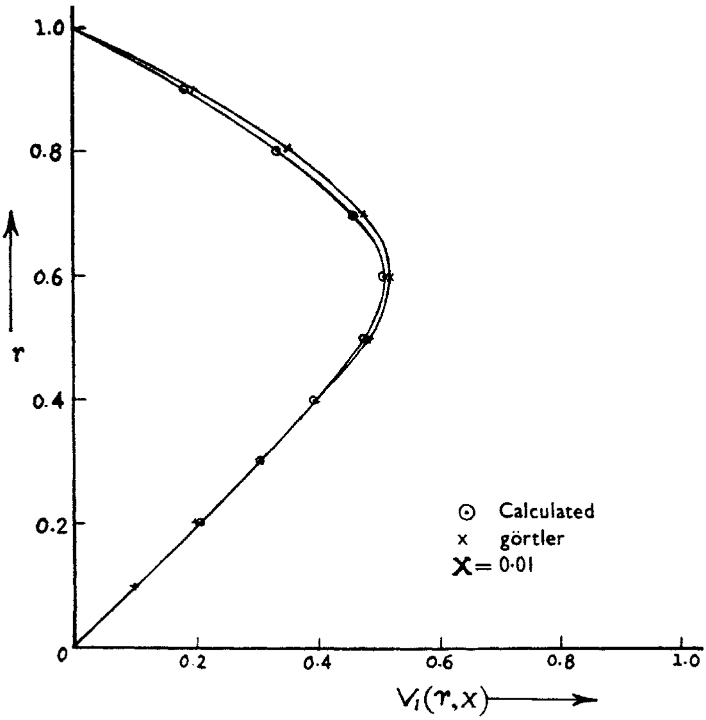

1.0
0.8
0.6
0.4
0.2
0
r
0.2
0.4
0.6
0.8
1.0
$V_1(r, x)$
⊙ Calculated
x görtler
$X = 0{\cdot}01$

Fig. 1.

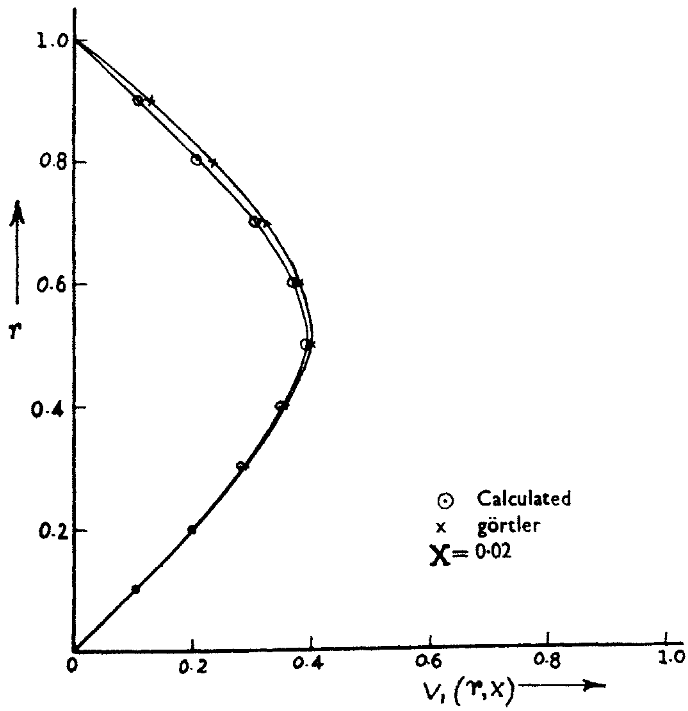

1.0
0.8
0.6
0.4
0.2
0
r
0.2
0.4
0.6
0.8
1.0
$V_1(r, x)$
⊙ Calculated
x görtler
$X = 0{\cdot}02$

Fig. 2.

we have the complete solution :

$$\bar{\chi}_G = -\frac{\pi\rho p^2(1-p/R^2)e^{-p\xi}}{16A(p)}\left[Y_1(\alpha r)\int_0^r \theta J_1(\alpha\theta)\psi(\theta)\,d\theta + J_1(\alpha r)\int_r^1 \theta Y_1(\alpha\theta)\psi(\theta)\,d\theta\right]$$

$$\times\left[\psi(\rho)\int_\rho^1 \theta u(\theta)J_1(\alpha\theta)\,d\theta + u(\rho)\int_0^\rho \theta J_1(\alpha\theta)\psi(\theta)\,d\theta\right]$$

$$+\frac{\pi\rho p^2(1-p/R^2)u(\rho)e^{-p\xi}}{16}\left[Y_1(\alpha r)\int_0^r \theta\psi(\theta)J_1(\alpha\theta)\,d\theta + J_1(\alpha r)\int_r^\rho \theta\psi(\theta)Y_1(\alpha\theta)\,d\theta\right]$$

$$+\frac{\pi\rho p^2(1-p/R^2)\psi(\rho)e^{-p\xi}}{16}J_1(\alpha r)\int_\rho^1 \theta u(\theta)Y_1(\alpha\theta)\,d\theta, \qquad r<\rho, \quad \ldots \quad \ldots \quad (4.3)$$

$$= -\frac{\pi\rho p^2(1-p/R^2)e^{-p\xi}}{16A(p)}\left[Y_1(\alpha r)\int_0^r \theta J_1(\alpha\theta)\psi(\theta)\,d\theta + J_1(\alpha r)\int_r^1 \theta Y_1(\alpha\theta)\psi(\theta)\,d\theta\right]$$

$$\times\left[\psi(\rho)\int_\rho^1 \theta u(\theta)J_1(\alpha\theta)\,d\theta + u(\rho)\int_0^\rho \theta J_1(\alpha\theta)\psi(\theta)\,d\theta\right]$$

$$+\frac{\pi\rho p^2(1-p/R^2)e^{-p\xi}\psi(\rho)}{16}\left[Y_1(\alpha r)\int_\rho^r \theta u(\theta)J_1(\alpha\theta)\,d\theta + J_1(\alpha r)\int_r^1 \theta u(\theta)Y_1(\alpha\theta)\,d\theta\right]$$

$$+\frac{\pi\rho p^2(1-p/R^2)e^{-p\xi}u(\rho)}{16}Y_1(\alpha r)\int_0^\rho \theta J_1(\alpha\theta)\psi(\theta)\,d\theta, \qquad r>\rho, \quad \ldots \quad \ldots \quad (4.4)$$

where

$$\psi(\theta) = \theta e^{-\mu\theta^2/2}\phi(a,\,2\,;\,\mu\theta^2),$$

$$u(\theta) = \theta e^{-\mu\theta^2/2}W(a,\,2\,;\,\mu\theta^2),$$

$$A(p) = \int_0^1 \theta J_1(\alpha\theta)\psi(\theta)\,d\theta, \quad \alpha = p/R,$$

and $\phi(a,\ 2;\ x)$, $W(a,\ 2;\ x)$ are the independent solutions of the confluent hypergeometric equation given by Archibald (1938).

For

$$R\to\infty, \qquad \alpha = \frac{p}{R}\to 0,$$

provided

$$|p|\ll R, \qquad x\gg O(R^{-1}).$$

Hence making this approximation in (4.3) and (4.4) we have

$$\bar{\chi}_G = \frac{\rho p^2 e^{-p\xi}}{16B(p)}\left[\frac{1}{r}\int_0^r \theta^2\psi(\theta)\,d\theta + r\int_r^1 \psi(\theta)\,d\theta\right]\left[\psi(\rho)\int_\rho^1 \theta^2 u(\theta)\,d\theta + u(\rho)\int_0^\rho \theta^2\psi(\theta)\,d\theta\right]$$

$$-\frac{p^2\rho u(\rho)e^{-p\xi}}{16}\left[\frac{1}{r}\int_0^r \theta^2\psi(\theta)\,d\theta + r\int_r^\rho \psi(\theta)\,d\theta\right]$$

$$-\frac{p^2\rho\psi(\rho)e^{-p\xi}}{16}\,r\int_\rho^1 u(\theta)\,d\theta, \qquad r<\rho, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

$$= \frac{p^2\rho e^{-p\xi}}{16B(p)}\left[\frac{1}{r}\int_0^r \theta^2\psi(\theta)\,d\theta + r\int_r^1 \psi(\theta)\,d\theta\right]\left[\psi(\rho)\int_\rho^1 \theta^2 u(\theta)\,d\theta + u(\rho)\int_0^\rho \theta^2\psi(\theta)\,d\theta\right]$$

$$-\frac{p^2\rho\psi(\rho)e^{-p\xi}}{16}\left[\frac{1}{r}\int_\rho^r \theta^2 u(\theta)\,d\theta + r\int_r^1 u(\theta)\,d\theta\right]$$

$$-\frac{p^2\rho u(\rho)e^{-p\xi}}{16}\frac{1}{r}\int_0^\rho \theta^2\psi(\theta)\,d\theta, \qquad r > \rho, \quad \ldots \quad (4.6)$$

where

$$B(p) = \int_0^1 \theta^2\psi(\theta)\,d\theta.$$

Actually (4.5) and (4.6) represent the solutions of Green's function of the parabolic form of the eqn. (4.1).

5. Eigenvalues

Eigenvalues of (4.1) are the characteristic values for the non-zero solution of

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{p^2}{R^2}-p(1-r^2)-\frac{1}{r^2}\right]\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{p^2}{R^2}-\frac{1}{r^2}\right]\phi = 0, \quad \ldots \quad (5.1)$$

under the boundary condition (4.2).

We write (5.1) in the form

$$\left[D^4+\frac{2p^2}{R^2}D^2+\frac{p^4}{R^4}-p(1-r^2)D^2-\frac{p^3}{R^2}(1-r^2)\right]\phi = 0, \quad \ldots \quad (5.2)$$

where

$$D^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}.$$

Multiplying (5.2) by $r\bar{\phi}$, where $\bar{\phi}$ is the complex conjugate of ϕ, and integrating between 0 and 1 we have

$$\int_0^1 r\bar{\phi}\left[D^4+\frac{2p^2}{R^2}D^2+\frac{p^4}{R^4}-p(1-r^2)D^2-\frac{p^3}{R^2}(1-r^2)\right]\phi\,dr = 0. \quad \ldots \quad (5.3)$$

It can be shown that

$$\int_0^1 r\bar{\phi}D^4\phi\,dr = \int_0^1 r\,|D^2\phi|^2\,dr,$$

$$\int_0^1 r\bar{\phi}D^2\phi\,dr = -\int_0^1 r\left|\frac{d\phi}{dr}+\frac{\phi}{r}\right|^2 dr,$$

$$\int_0^1 r\bar{\phi}\phi\,dr = \int_0^1 r\,|\phi|^2\,dr,$$

$$\int_0^1 (1-r^2)r\phi\bar{\phi}\,dr = \int_0^1 r(1-r^2)\,|\phi|^2\,dr,$$

and

$$\int_0^1 r(1-r^2)\bar{\phi}D^2\phi\, dr = -\int_0^1 r(1-r^2)\left|\frac{d\phi}{dr}+\frac{\phi}{r}\right|^2 dr - 2\int_0^1 \bar{\psi}\frac{d\psi}{dr}dr,$$

where $\bar{\psi} = r\bar{\phi}$ and $\int_0^1 \bar{\psi}\frac{d\psi}{dr}dr$ is zero.

Writing the eqn. (5.3) in the form

$$ap^4+4bp^3+6cp^2+4dp+e = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (5.4)$$

where

$$a = \frac{a'}{R^4} = \frac{1}{R^4}\int_0^1 r|\phi|^2 dr,$$

$$b = -\frac{b'}{R^2} = -\frac{1}{4R^2}\int_0^1 r(1-r^2)|\phi|^2 dr,$$

$$c = -\frac{c'}{R^2} = -\frac{1}{3R^2}\int_0^1 r\left|\frac{d\phi}{dr}+\frac{\phi}{r}\right|^2 dr,$$

$$d = d' = \frac{1}{4}\int_0^1 r(1-r^2)\left|\frac{d\phi}{dr}+\frac{\phi}{r}\right|^2 dr,$$

$$e = e' = \int_0^1 r|D^2\phi|^2 dr,$$

such that a', b', c', d', e' are all positive real numbers each of $O(1)$, we apply the condition that (5.4) has all its roots real if

$$\Delta = I^3 - 27J^2 > 0,$$

where

$$I = ae - 4bd + 3c^2$$

$$J = ace + 2bcd - ad^2 - c^3 - eb^2.$$

Expressing I and J in terms of a', b', c', d', e' we have

$$I^3 = \frac{1}{R^6}\left[4b'd' + \frac{a'e'+3c'^2}{R^2}\right]^3 \sim O\left(\frac{1}{R^6}\right)$$

$$J^2 = \frac{1}{R^8}\left[2b'c'd' - a'd'^2 - e'b'^2 + \frac{c'^3 - a'c'e'}{R^2}\right]^2 \sim O\left(\frac{1}{R^8}\right).$$

Hence for large R, $\Delta > 0$.

Besides, if the roots are all imaginary, say $p = ik$, then we have from (5.4)

$$k\left[\frac{4b'k^2}{R^2} + 4d'\right] = 0,$$

which shows that $k = 0$.

So the eigenvalues of (5.1) are all real for large R.

Now (4.5) and (4.6) represent Green's function for the parabolic form of the eqn. (4.1). The force function $h(\rho, \xi)$ vanishes for negative ξ. Therefore the disturbance prevails on the positive side of ξ only. Hence $\bar{\chi}_G$ will have non-zero values on the positive side of the axis of the pipe. Consequently or otherwise it follows that the eigenvalues which are the roots of $B(p) = 0$ are all real and negative.

Let us take the roots of $B(p) = 0$, as given by

$$p = -t_m^2 \quad m = 1, 2, 3, \ldots$$

Now $B(p) = 0$ can be written as

$$\int_0^1 xe^{-t_m x/2} \phi\left(1 - \frac{t_m}{4}, 2;\ t_m x\right) dx = 0. \qquad \ldots \qquad (5.5)$$

Using Tricomi's expansion

$$\phi(a, b;\ x) = \Gamma b e^{x/2} (kx)^{(1-b)/2} \sum_0^\infty A_n(k, \tfrac{1}{2}b) \left(\frac{x}{4k}\right)^{n/2} J_{b+n-1}(2\sqrt{kx}),$$

where $A_n(k, \frac{1}{2}b)$ are the coefficients of z^n in the expression

$$e^{kz/2}(1-z)^{k-b/2}(1+z)^{-k-b/2} = \sum_0^\infty A_n(k, \tfrac{1}{2}b) z^n,$$

and $k = 1-a$, $|z| < 1$,

eqn. (5.5) can be simplified to

$$\sum_0^\infty A_n(k, 1) J_{n+2}(t_m) = 0. \qquad \ldots \qquad (5.6)$$

It will be recalled that the eigenvalues in our previous paper were given by the first term of (5.6) as

$$J_2(t_m) = 0.$$

As the series (5.6) is not convenient for the calculation of t_m, we break up $\phi(1 - t_m/4,\ 2;\ t_m x)$ in series and convert (5.5) into a series of incomplete gamma function $\nu(\alpha, x)$ such that

$$\nu(t_m/2, 2) + b_m \nu(t_m/2, 3) + \frac{b_m(b_m+1)}{3!\,2!}\, 2^2 \nu(t_m/2, 4) + \ldots = 0$$

where $b_m = 1 - t_m/4$.

Using the tables of incomplete gamma function we have by graphical construction the first five roots as

$$t_1 = 5{\cdot}7,\ t_2 = 9{\cdot}8,\ t_3 = 13{\cdot}8,\ t_4 = 17{\cdot}9,\ t_5 = 22{\cdot}1. \qquad \ldots \qquad (5.7)$$

This graphical construction shows that the roots are large and are symmetrically located with approximate difference of four. Hence we try to find the roots by an analytical method.

Now

$$xe^{-\frac{1}{2}x}\phi(b, 2; x) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\frac{ux}{2}}\left(\frac{u-1}{u+1}\right)^{-b}}{(u+1)^2} du, \quad \ldots \quad \ldots \quad (5.8)$$

where $b = 1-\frac{\mu}{4}$, Real $c > 1$.

The eqn. (5.8) holds for all x and b. We integrate (5.8) with respect to x between 0 and t_m and put $b = b_m$, so that

$$\int_0^{t_m} xe^{-\frac{1}{2}x}\phi(b_m, 2; x)\, dx = \frac{2}{\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\frac{1}{2}ut_m}\left(\frac{u-1}{u+1}\right)^{t_m/4}}{u(u^2-1)} du. \quad \ldots \quad (5.9)$$

We have evaluated this integral in Appendix III to get the values of t_m and it comes to

$$-\left[\frac{8}{3}\cos\frac{\pi t_m}{4}+\frac{4\sqrt{3}\Gamma\frac{2}{3}\,.\,2^{2/3}}{\pi 3^{1/3}t_m^{2/3}}\cos\left(\frac{\pi t_m}{4}+\frac{2\pi}{3}\right)+O(t_m^{-4/3})\right]. \quad \ldots \quad (5.10)$$

For large t_m we can approximate the zeros given by $\cos \pi t_m/4 = 0$, neglecting terms of $O(t_m^{-2/3})$.

Hence

$$t_m = 2(2m+1),$$

$$m = 1, 2, 3, \ldots$$

This gives

$$t_1 = 6,\ t_2 = 10,\ t_3 = 14,\ t_4 = 18,\ t_5 = 22$$

and graphical values are

$$t_1 = 5{\cdot}7,\ t_2 = 9{\cdot}8,\ t_3 = 13{\cdot}8,\ t_4 = 17{\cdot}9,\ t_5 = 22{\cdot}1.$$

Actually the first root is not corresponding well. If we take into account the second term of (5.10), we have $t_1 = 5{\cdot}94$, and perhaps it is not possible to get more accurate values than this from (5.10). For our calculation we have taken values of t_m obtained from (5.10).

6. Axial Velocity

Now (4.5) and (4.6) are single-valued at $p = 0$ and have no other singularity except at the zeros of $B(p) = 0$. So we take the Bromwich integral along a semicircle with the line $c-i\infty$, $c+i\infty$ as diameter, $-t_1 < Rc$. The integral on the curved path can be shown to vanish by Jordan's lemma. Hence

$$\chi_G(r, \rho, x-\xi) = -\sum_1^\infty \frac{\rho t_m^2 e^{-t_m^2(x-\xi)}}{16R_m}\left[\frac{1}{r}\int_0^r \theta^2\psi_m(\theta)d\theta + r\int_r^1 \psi_m(\theta)d\theta\right] I_m(\rho)$$

$$x-\xi > 0,$$

$$= 0, \qquad x-\xi < 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.1)$$

where R_m = residue at the mth pole

$$= \left[\frac{\partial}{\partial p} B(p)\right]_{p=-t_m^2}$$

$$I_m(\rho) = \psi_m(\rho)\int_\rho^1 \theta^2 u_m(\theta)d\theta - u_m(\rho)\int_0^\rho \theta^2\psi_m(\theta)d\theta.$$

It has been found in Appendix IV that

$$R_m = \frac{(-1)^{m+1}}{2t_m^3}\left[\frac{\pi}{3} - \frac{3^{1/6}\Gamma\frac{2}{3}}{2^{4/3}t_m^{2/3}} + O\left(t_m^{-4/3}\right)\right]. \quad \ldots \quad \ldots \quad (6.2)$$

Now

$$\chi(r, x) = \int_0^x\int_0^1 \chi_G(r, \rho, x-\xi)h(\rho, \xi)\, d\rho\, d\xi$$

and

$$W_2(r, x) = \frac{\partial\chi}{\partial r} + \frac{\chi}{r}.$$

Therefore to get $W_2(r, x)$ for $x > 0$ we evaluate $h(\rho, \xi)$ from (3.6) and get

$$W_2(r, x) = -\frac{4\,.\,6^{\frac{1}{3}}}{(\Gamma\frac{1}{3})^2}\sum_1^\infty\sum_1^\infty\sum_1^\infty \frac{(-1)^{i+m+n+1}t_m^5\int_r^1 \psi_m(\theta)\, d\theta}{\left(\frac{\pi}{3} - \frac{3^{1/6}\Gamma 2/3}{2^{4/3}t_m^{2/3}}\right)}(\beta_n\beta_i)^{1/3}(\beta_n^2+\beta_i^2)$$

$$\times \frac{e^{-t_m^2 x} - e^{-(\beta_i^2+\beta_n^2)x}}{\beta_i^2+\beta_n^2-t_m^2}\int_0^1 \psi_i(\rho)\psi_n(\rho)I_m(\rho)\, d\rho, \qquad x > 0. \quad \ldots \quad (6.3)$$

But it has been shown in Appendix V that the values of

$$\int_0^1 \psi_i(\rho)\psi_n(\rho)I_m(\rho)\, d\rho$$

$$= \frac{8}{t_m^4}\left[(-1)^{n+i}\left(1+\frac{20}{t_m^2}\right)\frac{6^{4/3}(\Gamma 2/3)^2}{\pi^2\beta_n^2(\beta_n\beta_i)^{2/3}}\right][1+O\beta_i^{-4/3}] \qquad n > i$$

$$= \frac{8}{t_m^4}\left[\left(1+\frac{8}{t_m^2}\right)\frac{1}{\beta_n^3} + \left(1+\frac{20}{t_m^2}\right)\frac{6^{4/3}(\Gamma 2/3)^2}{\pi^2\beta_n^{10/3}}\right][1+O\beta_i^{-4/3}] \qquad n = i.$$

Hence $W_2(r, x)$ comes to

$$= -\frac{32\,.\,6^{\frac{1}{3}}}{(\Gamma\frac{1}{3})^2}\sum_1^\infty\sum_1^\infty\sum_1^\infty \frac{e^{-t_m^2 x} - e^{-(\beta_i^2+\beta_n^2)x}}{\beta_i^2+\beta_n^2-t_m^2}\;\frac{(-1)^{m+1}t_m\int_r^1 \psi_m(\theta)\, d\theta}{\left(\frac{\pi}{3} - \frac{3^{1/6}\Gamma 2/3}{2^{4/3}t_m^{2/3}}\right)}$$

$$\times\left[\left\{\left(1+\frac{8}{t_m^2}\right)\frac{1}{\beta_n^{\frac{1}{3}}} + \left(1+\frac{20}{t_m^2}\right)\frac{6^{4/3}(\Gamma 2/3)^2}{\pi^2\beta_n^{2/3}}\right\}\delta_{mn}\right.$$

$$\left. + \left(1+\frac{\beta_i^2}{\beta_n^2}\right)\left(1+\frac{20}{t_m^2}\right)\frac{6^{4/3}(\Gamma 2/3)^2}{\pi^2(\beta_n\beta_i)^{\frac{1}{3}}}\right], \qquad x > 0, \quad \ldots \quad \ldots \quad \ldots \quad (6.4)$$

where δ_{mn} is the Kronecker delta.

Again in (6.4) $\int_r^1 \psi_m(\theta)d\theta$ can be calculated numerically, otherwise it is found in Appendix VI that

$$\int_r^1 \psi_m(\theta)d\theta = \frac{2}{t_m^2}\left[(-1)^{m+1}\frac{6^{2/3}\Gamma 2/3}{2\pi t_m^{2/3}} + e^{-t_m r^2/2}\{\phi(b_m, 1;\ t_m r^2) + \tfrac{1}{2}t_m r^2\phi(b_m, 2;\ t_m r^2)\} - \sum_0^\infty A_n(k, 1)r^{n+2}J_{n+2}(t_m r)\right][1+Ot_m^{-4/3}]. \qquad (6.5)$$

The eqns. (6.4) and (6.5) give the velocity profile of $W_2(r, x)$ for $x > 0$. We have drawn the profile at $x = 0{\cdot}02$ in Fig. 3. Besides, we have calculated the values of

$$W_2(0, x) = -\frac{64 . 6^{1/3}}{(\Gamma\frac{1}{3})^2}\sum_1^\infty\sum_1^\infty\sum_1^\infty \frac{\left[e^{-t_m^2 x} - e^{-(\beta_i^2+\beta_n^2)x}\right](-1)^{m+1}}{(\beta_i^2+\beta_n^2-t_m^2)t_m\left[1+\frac{(-1)^{m+1}6^{2/3}\Gamma 2/3}{2\pi t_m^{2/3}}\right]}$$

$$\times \frac{1}{\left(\frac{\pi}{3} - \frac{3^{1/6}\Gamma 2/3}{2^{4/3}t_m^{2/3}}\right)}\left[\left\{\left(1+\frac{8}{t_m^2}\right)\frac{1}{\beta_n^{\frac{1}{3}}} + \left(1+\frac{20}{t_m^2}\right)\frac{6^{4/3}(\Gamma 2/3)^2}{\pi^2\beta_n^{2/3}}\right\}\delta_{mn}\right.$$

$$\left. + \left(1+\frac{\beta_i^2}{\beta_n^2}\right)\left(1+\frac{20}{t_m^2}\right)\frac{6^{4/3}(\Gamma 2/3)^2}{\pi^2(\beta_n\beta_i)^{1/3}}\right], \qquad x > 0, \qquad (6.6)$$

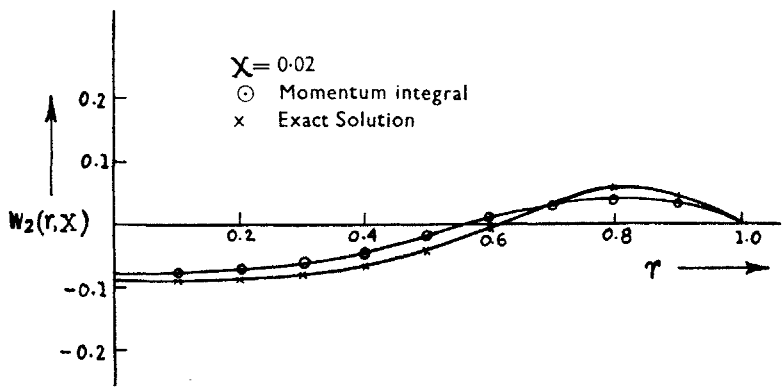


FIG. 3.

for different values of x and it is shown in Fig. 4. Since (6.6) is not suitable to calculate $W_2(0, x)$ for $x \to 0$, therefore we have calculated in Appendix VII an expression of $W_2(0, x)$

$$= -\frac{3^{2/3}x^{1/3}E}{2^{1/6}(\Gamma\frac{1}{3})^3}\left(6 - \frac{0{\cdot}38473}{z}\right)(1+Ox^{\frac{1}{3}}), \qquad z > 0$$

$$= 0, \qquad z = 0, \qquad (6.7)$$

where

$$E = \left[-\tfrac{3}{4}(\Gamma 7/6)^2+\frac{3\sqrt{\pi}}{4}\Gamma\frac{11}{6}-\Gamma 7/6\sum_{0}^{\infty}\frac{\Gamma(n+1+1/6)}{n!\,(2n+1)2^{n+1+1/6}}\right.$$
$$\left.+\frac{\sqrt{\pi}}{2}\sum_{0}^{\infty}\frac{\Gamma(n+1+5/6)}{n!\,(n+1+1/6)2^{n+1+5/6}}\right].$$

Both branches (6.6) and (6.7) meet near $x = 0{\cdot}01$ and these are shown in Fig. 4.

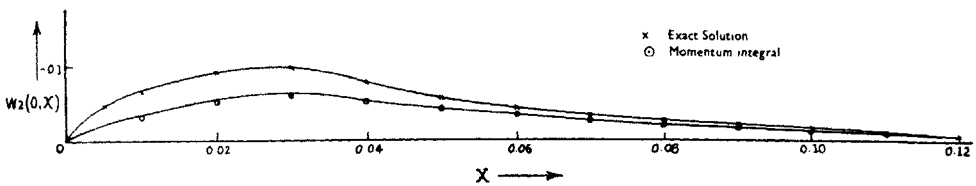


FIG. 4.

7. MOMENTUM INTEGRAL EQUATION

We transform the momentum integral equation (4.1) of the previous paper from z to x by the relation $z = Rx$; and by following the same procedure we arrive at the equation determining $a(x)$ of $W_2(r, x) = a(x)\ (1-6r^2+5r^3)$

$$\frac{3}{7}\frac{da}{dx}+15a = \frac{\partial}{\partial x}\int_0^1 rdr\int_0^r \frac{V_1^2}{t}\,dt. \qquad \ldots \qquad (7.1)$$

In Appendices VIII and IX it is shown that

$$\frac{\partial}{\partial x}\int_0^1 rdr\int_0^r\frac{V_1^2}{t}\,dt = -\frac{2^{10/3}3^{1/3}}{(\Gamma\frac{1}{3})^2}\sum_1^{\infty}\sum_1^{\infty}(-1)^{m+n}\beta_m^{1/3}\beta_n^{1/3}e^{-(\beta_m^2+\beta_n^2)x}$$
$$\times\left[\left(1-\frac{1}{\beta_m}+\frac{6^{4/3}(\Gamma 2/3)^2}{2\pi^2\beta_m^{4/3}}\right)\beta_m^{2/3}e^{-2\beta_m^2 x}\delta_{mn}+\left\{1+\frac{(-1)^{m+n}6^{4/3}(\Gamma 2/3)^2}{2\pi^2(\beta_m\beta_n)^{2/3}}\right\}\right.$$
$$\left.\times\left(1+\frac{\beta_n^2}{\beta_m^2}\right)\right], \qquad x>0, \qquad \ldots \qquad (7.2)$$

and

$$= -\frac{9\,.\,2^{1/6}3^{5/6}\Gamma 11/6}{2\sqrt{\pi}(\Gamma\frac{1}{3})^2x^{\frac{1}{3}}}\left(\beta^{2/3}+\frac{23}{64}\beta^{8/3}\right)+\left(1\,.\,5-\frac{243\pi}{4\,.\,2^{5/6}(\Gamma\frac{1}{3})^3}\right)\left(\beta^{2/3}-\frac{5}{8}\beta^{8/3}\right)$$
$$\times[1+O(x^{\frac{1}{3}})], \qquad x\to 0, \qquad \ldots \qquad (7.3)$$

where

$$\beta = \sin^2\frac{\pi}{8}.$$

Introducing (7.2) and (7.3) in (7.1) we have after integration

$$a(x) = -\frac{63\,.\,2^{1/6}\,.\,3^{5/6}\Gamma 11/6}{4\sqrt{\pi}(\Gamma\frac{1}{3})^2}\left(\beta^{2/3}+\frac{23}{64}\beta^{8/3}\right)x^{\frac{2}{3}}$$

$$+\left[1-\frac{16{\cdot}2\;\pi}{4\,.\,2^{5/6}(\Gamma\frac{1}{3})^3}\left(\beta^{2/3}-\frac{5}{8}\beta^{8/3}\right)\right][1-e^{-35x}][1+O(x^{\frac{1}{3}})], \qquad x \to 0, \qquad (7.4)$$

$$= -\frac{7\,.\,2^{10/3}}{3^{2/3}(\Gamma\frac{1}{3})^2}\sum_1^\infty\sum_1^\infty \frac{e^{-2\beta_m^2 x}-e^{-35x}}{35-2\beta_m^2}\left(1-\frac{1}{\beta_m}+\frac{6^{4/3}(\Gamma 2/3)^2}{2\pi^2\beta_m^{4/3}}\right)\beta_m^{\frac{2}{3}}\delta_{mn}$$

$$+(-1)^{m+n}\frac{e^{-\beta_m^2 x-\beta_n^2 x}-e^{-35x}}{35-\beta_m^2-\beta_n^2}\beta_m^{1/3}\beta_n^{1/3}\left(1+\frac{\beta_n^2}{\beta_m^2}\right)\left(1+\frac{(-1)^{m+n}6^{4/3}(\Gamma 2/3)^2}{2\pi^2(\beta_m\beta_n)^{2/3}}\right),$$

$$x > 0. \qquad \ldots \qquad (7.5)$$

Both branches of (7.4) and (7.5) meet near $x = 0{\cdot}005$ and we have shown them in Fig. 4.

Appendix I

Here values of R_n given in (3.5) are found.

We have

$$\mu e^{-\mu/2}\phi(b, 2\,;\,\mu) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\mu u/2}\left(\frac{u-1}{u+1}\right)^{\mu/4}}{(u^2-1)}\,du,$$

$$Rc > 1. \qquad \ldots \qquad (\text{I.1})$$

Differentiating (I.1) with respect to μ and putting $\mu = \beta_n$ we have

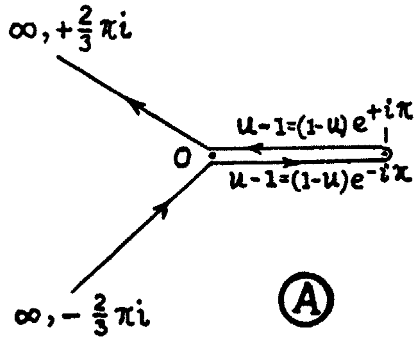
$$\beta_n e^{-\beta_n/2}\frac{d}{d\mu}\phi(b_n, 2\,;\,\beta_n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\beta_n f(u)/2}\,\frac{f(u)}{u^2-1}\,du, \qquad \ldots \qquad (\text{I.2})$$

where

$$f(u) = u+\tfrac{1}{2}\log(u-1)-\tfrac{1}{2}\log(u+1).$$

Now β_n is a large positive number. Integrand has branch points at $u = \pm 1$ and $f(u)$ has a saddle point at $u = 0$. We take Lawerier's path represented by (A) to evaluate the integral.

The path $c-i\infty$, $c+i\infty$ can be converted to the path (A) with a branch cut at $u = +1$ having arguments of $u-1$ as $-\pi i$ and $+\pi i$ below and above the cut respectively.



Since the main contribution to the integral is obtained from the neighbourhood of the origin, therefore we express the integrand in increasing powers of u, such that

$$f(u) = \tfrac{1}{2}\log(1-u) - \tfrac{1}{2}\log(1+u) + u \simeq -\frac{u^3}{3},$$

$$(u^2-1)^{-1}e^{\beta_n f(u)/2} \simeq -(1+u^2+u^4+\ldots)e^{-\beta_n u^3/6}. \qquad \ldots \qquad \text{(I.3)}$$

Hence (I.2) becomes

$$\frac{1}{2\pi i}\Bigg[\int_{\infty}^{0}(1+u^2e^{-\frac{4}{3}\pi i}+\ldots)\left(\frac{u^3}{3}+\frac{i\pi}{2}+\ldots\right)e^{-\frac{1}{6}\beta_n u^3-\frac{1}{4}\beta_n\pi i-\frac{2}{3}\pi i}\,du$$

$$+\int_{0}^{\infty}(1+u^2e^{+\frac{4}{3}\pi i}+\ldots)\left(\frac{u^3}{3}-\frac{i\pi}{2}+\ldots\right)e^{-\frac{1}{6}\beta_n u^3+\frac{1}{4}\beta_n\pi i+\frac{2}{3}\pi i}\,du$$

$$+\int_{0}^{1}(1+u^2+\ldots)\left(\frac{i\pi}{2}+\frac{u^3}{3}+\ldots\right)e^{-\frac{1}{4}\beta_n\pi i-\frac{1}{6}\beta_n u^3}\,du$$

$$+\int_{1}^{0}(1+u^2+\ldots)\left(-\frac{i\pi}{2}+\frac{u^3}{3}+\ldots\right)e^{+\frac{1}{4}\beta_n\pi i-\frac{1}{6}\beta_n u^3}\,du\Bigg]. \qquad \ldots \qquad \text{(I.4)}$$

For $\beta_n \to \infty$, the upper limits of the third and the fourth integrals can be extended to infinity and hence we have after simplification (I.4)

$$= \tfrac{1}{2}\left[\cos\tfrac{1}{4}\pi\beta_n - \cos\left(\frac{\pi}{4}\beta_n+\frac{2\pi}{3}\right)\right]\frac{\Gamma\frac{1}{3}\,.\,2^{1/3}}{3^{\frac{2}{3}}\beta_n^{\frac{1}{3}}}[1+O(\beta_n^{-\frac{4}{3}})]. \qquad \ldots \qquad \text{(I.5)}$$

Putting the values of β_n we have

$$\beta_n e^{-\beta_n/2}\frac{d}{d\mu}\phi(b_n, 2\,;\beta_n) = \frac{(-1)^n\Gamma\frac{1}{3}}{(4\beta_n)^{1/3}3^{1/6}}[1+O(\beta_n^{-4/3})].$$

Appendix II

Here an expression for swirl velocity for small x is found.

Expression (3.2) for V_1 can be written as

$$V_1 = r - \frac{1}{2\pi i}\int_N \frac{re^{\mu(1-r^2)/2}\phi(a, 2\,;\mu r^2)}{p\phi(a, 2\,;\mu)}e^{px}\,dp, \qquad \ldots \qquad \ldots \text{(II.1)}$$

where the path N is along the imaginary axis indented at the origin.

Now, with the substitution

$$y\,\frac{e^{\xi/2}}{\xi} = \phi(a, 2\,;\xi)$$

equation (3.1) can be transformed to

$$\frac{d^2y}{d\theta^2}+\lambda^2\left(1-\frac{1}{\theta}\right)y+\frac{4\lambda^4 y}{R^2\theta} = 0, \qquad \ldots \qquad \ldots \qquad \ldots \text{(II.2)}$$

where

$$\lambda^2 = \frac{p}{4}, \quad \xi = \mu r^2 = \mu\theta.$$

Following Erdelyi (1954) we have the asymptotic expansion of (II.2) for $\lambda \to \infty$, $\theta > 0$, as

$$y = \frac{A_i\{-\lambda^{\frac{2}{3}}K(\theta)\}}{[K'(\theta)]^{\frac{1}{2}}}[1+O(\lambda^{-1})], \quad \ldots \quad \ldots \quad \ldots \text{(II.3)}$$

where A_i is the Airy function of first kind,

$$K(\theta) = -(3/2)^{2/3}\beta^{2/3},$$

$$[K'(\theta)]^{-1/2} = (\tfrac{3}{2})^{1/6}\frac{\theta^{1/4}\beta^{1/6}}{(1-\theta)^{\frac{1}{4}}},$$

$$\beta = [\cos^{-1}\sqrt{\theta}-\sqrt{\theta-\theta^2}] \approx \tfrac{2}{3}(1-\theta)^{3/2}.$$

Hence

$$y = (\tfrac{3}{2})^{1/6}\frac{\theta^{1/4}\beta^{1/6}}{(1-\theta)^{1/4}} A_i\left[\left(\frac{3\lambda\beta}{2}\right)^{2/3}\right][1+O(\lambda^{-1})],$$

$$= \frac{1}{\Gamma 2/3}\,\frac{1}{3^{2/3}}[1+O(\lambda^{-1})], \quad \theta = 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(II.4)}$$

Introducing (II.4) in (II.1) and expressing $A_i(z)$ in terms of $K_{\frac{1}{3}}(\frac{2}{3}z^{3/2})$ and $K_{\frac{1}{3}}(\eta)$ in its integral form we have

$$V_1 = r - \frac{\Gamma 2/3\,.\,3^{2/3}(3/2)^{1/6}\beta^{1/6}}{r^{\frac{1}{2}}(1-r^2)^{\frac{1}{4}}\pi^{\frac{1}{2}}\Gamma 1/6\,.\,3^{1/6}}\frac{1}{2\pi i}\int_N \frac{e^{px}}{p}\,dp\int_1^\infty e^{-\lambda\beta t}(t^2-1)^{-5/6}\,dt[1+O(\lambda^{-1})]. \quad \ldots \text{(II.5)}$$

We change the order of integration in (II.5) and integrate with respect to p with the result

$$V_1 = r - \frac{\Gamma 2/3\,.\,3^{2/3}\beta^{1/6}}{\pi^{\frac{1}{2}}r^{\frac{1}{2}}(1-r^2)^{\frac{1}{4}}2^{\frac{1}{6}}\Gamma 1/6}\int_1^\infty (t^2-1)^{-5/6}erf_c\,\frac{\beta t}{4x^{\frac{1}{2}}}\,dt\,[1+O(x^{1/2})]. \quad \text{(II.6)}$$

Again expressing erf_c in its integral form and integrating directly we have

$$V_1 = r - \frac{\Gamma 2/3\,.\,3^{2/3}\beta^{1/6}}{2\pi r^{\frac{1}{2}}(1-r^2)^{\frac{1}{4}}2^{\frac{1}{6}}}\Gamma\left(\tfrac{1}{3}, \frac{\beta^2}{16x}\right)[1+O(x^{1/2})], \quad \ldots \text{(II.7)}$$

where

$$\Gamma\left(\tfrac{1}{3}, \frac{\beta^2}{16x}\right)$$

is the incomplete gamma function.

Substituting the value of β, (II.7) can be reduced to

$$V_1 = r - \frac{1}{\Gamma\frac{1}{3}r^{\frac{1}{2}}}\Gamma\left(\tfrac{1}{3}, \frac{(1-r^2)^3}{36x}\right)[1+O(x^{\frac{1}{2}})], \quad r>0, \quad x \sim O\left(\frac{1}{R}\right). \quad \ldots \text{(II.8)}$$

Appendix III

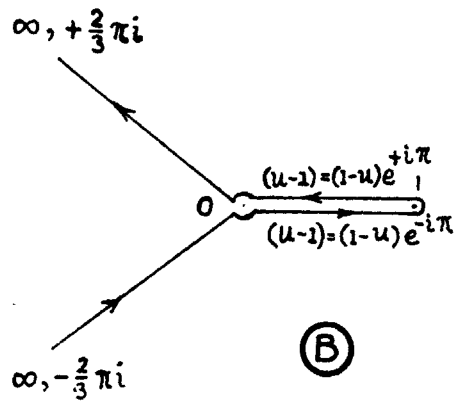
The complex integral in (5.9) can be written as

$$\frac{2}{\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\frac{1}{2}t_m f(u)}}{u(u^2-1)}\,du, \qquad \ldots \quad \ldots \quad \ldots \text{(III.1)}$$

where

$$f(u) = u+\tfrac{1}{2}\log(u-1)-\tfrac{1}{2}\log(u+1), \quad Rc > 1.$$

Integrand in (III.1) has a pole at $u = 0$ and has branch points at $u = \pm 1$. Again it has saddle point at $u = 0$. For $t_m \to \infty$, main contribution to the integral comes from the pole and from the part near $u = 0$. So we take the path (A) in the form (B) indenting at the origin by a small circle of radius ϵ such that $\epsilon \to 0$.



Expressing the integrand in increasing powers of u as in (I.3) we have (III.1),

$$= \frac{2}{\pi i}\Bigg[\int_{\infty}^{\epsilon} e^{-\frac{t_m u^3}{6}-\frac{t_m \pi i}{4}}\left\{\left(-\frac{1}{u}-u-u^3\ldots\right)du\right\}_{u=ue^{-\frac{2}{3}\pi i}}$$

$$+\int_{\epsilon}^{\infty} e^{-\frac{t_m u^3}{6}+\frac{t_m \pi i}{4}}\left\{\left(-\frac{1}{u}-u-u^3\ldots\right)du\right\}_{u=ue^{+\frac{2}{3}\pi i}}$$

$$+\int_{\epsilon}^{1} e^{-\frac{t_m u^3}{6}-\frac{t_m \pi i}{4}}\left(-\frac{1}{u}-u-u^3\ldots\right)du$$

$$+\int_{1}^{\epsilon} e^{-\frac{t_m u^3}{6}+\frac{t_m \pi i}{4}}\left(-\frac{1}{u}-u-u^3\ldots\right)du$$

$$+\int_{-2\pi/3}^{0} -i d\theta e^{-\frac{t_m \pi i}{4}}+\int_{0}^{\frac{2}{3}\pi} -i d\theta e^{\frac{t_m \pi i}{4}}\Bigg]. \qquad \ldots \quad \ldots \quad \ldots \text{(III.2)}$$

For $t_m \to \infty$, $e^{-t_m u^3/6}$ is asymptotically small and hence the upper limits of the third and fourth integrals can be extended to infinity. Making $\epsilon \to 0$, after simplification, we find that (III.2)

$$= -\tfrac{8}{3}\cos\frac{\pi t_m}{4} - \frac{4\sqrt{3}\,\Gamma 2/3\,.\,2^{2/3}}{\pi\,.\,3^{\frac{1}{3}} t_m^{2/3}}\cos\left(\frac{\pi t_m}{4}+\frac{2\pi}{3}\right)+O(t_m^{-4/3}).$$

Appendix IV

$$R_m = \left[\frac{\partial B(p)}{\partial p}\right]_{p=-t_m^2}$$

can be expressed as

$$= -\frac{1}{4t_m^3}\left[\frac{d}{d\mu}\int_0^\mu xe^{-\frac{1}{2}x}\phi(b, 2\,;\, x)\,dx\right]_{\mu=t_m}. \qquad \ldots \qquad \ldots \text{(IV.1)}$$

Differentiating the relation

$$\int_0^\mu xe^{-\frac{1}{2}x}\phi(b, 2\,;\, x)\,dx = \frac{2}{\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\frac{1}{2}\mu f(u)}}{u(u^2-1)}\,du, \quad Rc > 1,$$

with respect to μ and introducing it to (IV.1) we have

$$R_m = -\frac{1}{2t_m^3}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{\frac{t_m}{2}f(u)}f(u)}{u(u^2-1)}\,du, \qquad \ldots \qquad \ldots \qquad \ldots \text{(IV.2)}$$

where $f(u)$ is given by (III.1). Bromwich path in (IV.2) is broken into the path (B) and the integral is evaluated like (III.1) with the result

$$R_m = \frac{(-1)^{m+1}}{2t_m^3}\left[\frac{\pi}{3} - \frac{3^{1/6}\,.\,\Gamma 2\,3}{2^{4/3}t_m^{2/3}} + O(t_m^{-4/3})\right]. \qquad \ldots \qquad \ldots \text{(IV.3)}$$

Appendix V

Here

$$\int_0^1 \psi_n(\rho)\psi_i(\rho)I_m(\rho)\,d\rho$$

is evaluated.

$I_m(\rho)$ satisfies the differential equation

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+t_m^2(1-\rho^2)-\frac{1}{\rho^2}\right]I_m(\rho) = \frac{8\rho}{t_m^2}. \qquad \ldots \qquad \ldots \text{(V.1)}$$

Asymptotic solution of (V.1) in inverse power of t_m^2, within the range $0 < \rho < 1$, is of the form

$$I_m(\rho) = \frac{8}{t_m^2}\left[\frac{\rho}{t_m^2(1-\rho^2)}+\frac{8\rho}{t_m^4(1-\rho^2)^4}+\ldots\right]. \qquad \ldots \qquad \ldots \text{(V.2)}$$

Introducing (V.2) in the integral and using the orthogonal property of $\psi_i(\rho)$ such that

$$\int_0^1 \rho(1-\rho^2)\psi_n(\rho)\psi_i(\rho)\,d\rho = 0 \qquad n \neq i$$

$$= e^{-\beta_n}\phi_n'(b_n, 2\,;\, \beta_n)\frac{\partial\phi}{\partial\beta_n}(b_n, 2\,;\, \beta_n), \qquad n = i,$$

we have after simplification

$$\int_0^1 \psi_n(\rho)\psi_i(\rho)I_m(\rho)\,d\rho = \frac{16}{t_m^4}\left(1+\frac{20}{t_m^2}+\dots\right)\int_0^1 \rho^3\psi_n(\rho)\psi_i(\rho)\,d\rho \qquad n\neq i$$

$$= \frac{8}{t_m^2}\left[\left(1+\frac{8}{t_m^2}+\dots\right)e^{-\beta_n}\phi_n'(b_n, 2;\ \beta_n)\frac{\partial\phi(b_n, 2, \beta_n)}{\partial\beta_n}\right.$$

$$\left.+2\left(1+\frac{20}{t_m^2}+\dots\right)\int_0^1 \rho^3\psi_n(\rho)\psi_i(\rho)\,d\rho\right], \qquad n=i. \quad \text{(V.3)}$$

Again to evaluate

$$\int_0^1 \rho^3\psi_n(\rho)\psi_i(\rho)\,d\rho$$

it is converted into the complex integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{4}{\beta_i\beta_n}\,\frac{e^{\frac{u\beta_n}{2}}\left(\frac{u-1}{u+1}\right)^{\frac{\beta_n}{4}}}{u^2-1}\cdot\frac{e^{\frac{v\beta_i}{2}}\left(\frac{v-1}{v+1}\right)^{\frac{\beta_i}{4}}}{v^2-1}\,\frac{du\,dv}{\beta_n u+\beta_i v}$$

Rc and Rc_1 both > 1. .. .. (V.4)

Since u and v are completely independent, therefore we consider u-integral and v-integral to be separate integrals, one containing the other variable as a parameter or constant having its modulus exceeding unity.

Let us consider

$$\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{e^{v\beta_i/2}\left(\frac{v-1}{v+1}\right)^{\beta_i/4}}{(v^2-1)(\beta_n u+\beta_i v)}\,dv \quad \dots \quad \dots \quad \dots \quad \text{(V.5)}$$

when $\beta_n \geqslant \beta_i$. Now u can be considered to be some constant such that $|u| > 1$. So, for $\beta_i \to \infty$, the integral (V.5) can be integrated along the path (A) by expanding the integrand in the form

$$e^{-\beta_i v^3/6}\{-1-v^2-v^4\dots\}\left\{1-\frac{\beta_i v}{\beta_n u}-\frac{\beta_i^2 v^2}{\beta_n^2 u^2}\dots\right\}\frac{1}{\beta_n u}$$

and investigating the main contribution in the neighbourhood of the origin with the result

$$\frac{(-1)^{i+1}\Gamma 2/3\beta_i^{1/3}6^{2/3}}{2\pi\beta_n^2 u^2}\left[1+O\beta_i^{-4/3}\right]. \quad \dots \quad \dots \quad \text{(V.6)}$$

Hence (V.4) reduces to

$$\frac{(-1)^{i+1}.4.\Gamma 2/3\beta_i^{1/3}3^{2/3}}{2^{\frac{1}{3}}\pi\beta_n^3\beta_i}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{u\beta_n/2}\left(\frac{u-1}{u+1}\right)^{\beta_n/4}}{u^2(u^2-1)}\,du\left[1+O\beta_i^{-4/3}\right]. \quad \text{(V.7)}$$

Again, we integrate (V.7) along the path (B) as in Appendix III and take into consideration the main contribution at its double pole at $u = 0$, and in its neighbourhood so that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{u\beta_n/2}\left(\frac{u-1}{u+1}\right)^{\beta_n/4}}{u^2(u^2-1)}\,du = (-1)^{n+1}\beta_n^{\frac{1}{3}}\frac{3\Gamma 2/3}{2\pi 6^{\frac{1}{3}}}\left(1+O\beta_n^{-4/3}\right). \qquad \text{(V.8)}$$

Besides

$$\phi_n'(b_n, 2;\ \beta_n) = \frac{1}{\beta_n}\phi(b_n, 1;\ \beta_n)$$

is evaluated by the same method as in Appendix I to

$$\frac{(-1)^n e^{\beta_n/2} 6^{2/3}\Gamma 2/3}{2\pi\beta_n^{5/3}}\left(1+O\beta_n^{-4/3}\right). \qquad \ldots \qquad \ldots \qquad \text{(V.9)}$$

Hence (V.7), (V.8), (V.9) give

$$\int_0^1 \psi_n(\rho)\psi_i(\rho)I_m(\rho)d\rho = \frac{8}{t_m^4}\left(1+\frac{20}{t_m^2}+\ldots\right)\frac{(-1)^{n+i}6^{4/3}(\Gamma 2/3)^2}{\pi^2\beta_n^2(\beta_n\beta_i)^{2/3}}\left[1+O\beta_i^{-4/3}\right], \qquad n > i,$$

$$= \frac{8}{t_m^4}\left[\left(1+\frac{8}{t_m^2}+\ldots\right)\frac{1}{\beta_n^3}+\frac{6^{4/3}(\Gamma 2/3)^2}{\pi^2\beta_n^{\frac{10}{3}}}\left(1+\frac{20}{t_m^2}+\ldots\right)\right]$$

$$\times\left[1+O\beta_n^{-4/3}\right] \qquad n = i.$$

Appendix VI

$$\int_r^1 \psi_m(r)\,dr = \frac{1}{2t_m}\left[\int_0^{t_m}\frac{M_{k,\frac{1}{2}}(y)}{y}\,dy - \int_0^{t_m r^2}\frac{M_{k,\frac{1}{2}}(y)}{y}\,dy\right],$$

where $M_{k,\frac{1}{2}}(y)$ is the Whittakar's function and

$$k = \frac{t_m}{4}.$$

From Whittakar's differential equation we find

$$\frac{1}{2t_m}\int_0^{t_m}\frac{M_{k,\frac{1}{2}}(y)\,dy}{y} = \int_0^1 \psi_m(r)\,dr = \frac{2}{t_m^2}\left[1-\left\{\frac{d}{dy}M_{k,\frac{1}{2}}(y)\right\}_{y=t_m}\right]. \ldots \text{(VI.1)}$$

Again, if we put $b = b_m$ in (I.1) and differentiate the relation with respect to μ and integrate along the path (A) as in Appendix I, we have

$$\left[\frac{d}{dy}M_{k,\frac{1}{2}}(y)\right]_{y=t_m} = (-1)^m\frac{2}{t_m^2}\frac{6^{2/3}\Gamma 2/3}{2\pi t_m^{2/3}}\left(1+Ot_m^{-4/3}\right),$$

so that

$$\int_0^1 \psi_m(r)\,dr = \frac{2}{t_m^2}\left[1+(-1)^{m+1}\frac{6^{2/3}\Gamma 2/3}{2\pi t_m^{2/3}}\right]\left[1+Ot_m^{-4/3}\right]. \qquad \ldots \text{(VI.2)}$$

Again

$$\int_0^{t_m r^2} \frac{M_{k,\frac{1}{2}}(y)}{y}\, dy = \frac{4}{t_m}\left[1-\left\{\frac{d}{dy} M_{k,\frac{1}{2}}(y)\right\}_{y=t_m r^2}\right]+\frac{1}{t_m}\int_0^{t_m r^2} M_{k,\frac{1}{2}}(y)dy.$$

Expressing $M_{k,\frac{1}{2}}(y)$ in Tricomi's expansion we have after simplification

$$\int_r^1 \psi_m(r)dr = (-1)^{m+1}\frac{2}{t_m^2}\frac{6^{2/3}\Gamma 2/3}{2\pi t_m^{2/3}}+\frac{2}{t_m^2}e^{-\frac{1}{2}t_m r^2}\{\phi(b_m, 1\,;\, t_m r^2)$$

$$+\tfrac{1}{2}t_m r^2\phi(b_m, 2\,;\, t_m r^2)\}-\frac{2}{t_m^2}\sum_0^\infty A_n(k, 1)r^{n+2}J_{n+2}(t_m r).$$

Appendix VII

Here an expression for axial velocity for small x is found.

We take two-way Laplace transform of (4.1) and solving for $\bar{\chi}$ under the boundary conditions (4.2) find

$$\overline{W}_2(o, p) = \left[\frac{\partial\bar{\chi}}{\partial r}+\frac{\bar{\chi}}{r}\right]_{r=0}$$

in the form

$$\overline{W}_2(o, p) = \frac{\pi p}{2R}\int_0^1 \rho\psi(\rho)\bar{h}(p, \rho)\int_\rho^1 \eta Y_1\left(\frac{p\eta}{R}\right)\psi(\eta)\int_\rho^\eta \frac{dt}{t\psi^2(t)}\,d\eta d\rho$$

$$-\frac{\pi p}{2R}\int_0^1 \rho\psi(\rho)\bar{h}(p, \rho)\int_\rho^1 \eta J_1\left(\frac{p\eta}{R}\right)\psi(\eta)\int_\rho^\eta \frac{dt d\eta d\rho}{t\psi^2(t)}\,\frac{\int_0^1 \eta Y_1\left(\frac{p\eta}{R}\right)\psi(\eta)d\eta}{\int_0^1 \eta J_1\left(\frac{p\eta}{R}\right)\psi(\eta)d\eta}, \qquad \text{(VII.1)}$$

where

$$\bar{h}(p, \rho) = p\int_{-\infty}^{+\infty}\frac{\partial}{\partial x}\left(\frac{V_1^2}{\rho}\right)e^{-px}\,dx.$$

(3.7) shows that $\bar{h}(p, \rho)$ is asymptotically small outside $1-\rho^2 \sim O(p^{-\frac{1}{2}})$. Therefore the integrand in (VII.1) can be approximated for $\rho < 1$. Using multiplication theorem for $J_1\left(\frac{p\eta}{R}\right)$ and $Y_1\left(\frac{p\eta}{R}\right)$ for $\eta > 0$ and expanding asymptotically the Airy function in (II.4) in the form

$$\psi(\eta) = \frac{2^{1/6}e^{-\theta}}{2\sqrt{\pi}3^{1/6}\eta^{1/2}\theta^{1/6}}[1+Op^{-1/2}], \qquad \text{(VII.2)}$$

we integrate

$$\int_0^1 \eta J_1\left(\frac{p\eta}{R}\right)\psi(\eta)d\eta = \frac{2^{1/6}}{2\sqrt{\pi}(3p)^{\frac{1}{3}}}\left[\sqrt{\pi}J_1\left(\frac{p}{R}\right) - \frac{3^{2/3}}{2p^{\frac{1}{3}}}\frac{p}{R}\Gamma 7/6\, J_1'\left(\frac{p}{R}\right) + \ldots\right]$$
$$\times[1+Op^{-\frac{1}{2}}], \quad \ldots \quad (\text{VII.3})$$

$$\int_0^1 \eta Y_1\left(\frac{p\eta}{R}\right)\psi(\eta)d\eta = \frac{2^{1/6}}{2\sqrt{\pi}(3p)^{\frac{1}{3}}}\left[\sqrt{\pi}Y_1\left(\frac{p}{R}\right) - \frac{3^{2/3}}{2p^{\frac{1}{3}}}\frac{p}{R}\Gamma 7/6\, Y_1'\left(\frac{p}{R}\right) + \ldots\right]$$
$$\times[1+Op^{\ 1/2}], \quad \ldots \quad (\text{VII.4})$$

$$\int_\rho^1 \eta J_1\left(\frac{p\eta}{R}\right)\psi(\eta)\int_\rho^\eta \frac{dtd\eta}{t\psi^2(t)} = \frac{\sqrt{\pi}}{2^{\frac{1}{3}}3^{\frac{1}{2}}p^{\frac{2}{3}}}\left[(2e^{2\theta_1} erf\ \theta_1^{1/2} - 2\ erfi\ \theta_1^{1/2})J_1\left(\frac{p}{R}\right)\right.$$
$$+\frac{3^{\frac{2}{3}}}{2p^{\frac{1}{3}}}\left\{-\frac{p}{R}J_1'\left(\frac{p}{R}\right)e^{2\theta_1}\int_0^{\theta_1} e^{-\theta}\theta^{\frac{1}{6}}d\theta + 2J_1\left(\frac{p}{R}\right)\theta_1^{\frac{2}{3}}e^{2\theta_1} erf\ \theta_1^{\frac{1}{2}}\right.$$
$$\left.-\frac{p}{R}J_2\left(\frac{p}{R}\right)\int_0^{\theta_1} e^{\theta}\theta^{1/6}d\theta\right][1+Op^{-\frac{1}{2}}], \quad \ldots \quad (\text{VII.5})$$

and

$$\int_\rho^1 \eta Y_1\left(\frac{p\eta}{R}\right)\psi(\eta)\int_\rho^\eta \frac{dtd\eta}{t\psi^2(t)} = \frac{\sqrt{\pi}}{2^{1/6}3^{1/2}p^{2/3}}\left[(2e^{2\theta_1} erf\ \theta_1^{1/2} - 2\ erfi\ \theta_1^{\ 1/2})\, Y_1\left(\frac{p}{R}\right)\right.$$
$$+\frac{3^{2/3}}{2p^{1/3}}\left\{-\frac{p}{R}Y_1'\left(\frac{p}{R}\right)e^{2\theta_1}\int_0^{\theta_1} e^{-\theta}\theta^{1/6}d\theta + 2Y_1\left(\frac{p}{R}\right)\theta_1^{2/3}e^{2\theta_1} erf\ \theta_1^{\frac{1}{2}}\right.$$
$$\left.\left.-\frac{p}{R}Y_2\left(\frac{p}{R}\right)\int_0^{\theta_1} e^{\theta}\theta^{1/6}\,d\theta\right\}\right][1+Op^{-\frac{1}{2}}], \quad \ldots \quad (\text{VII.6})$$

where

$$\theta = \frac{p^{1/2}(1-\eta^2)^{3/2}}{3},\ \theta_1 = \frac{p^{\frac{1}{2}}(1-\rho^2)^{3/2}}{3},$$

and

$$erf\,\frac{p^{\frac{1}{2}}}{3} \approx \frac{\sqrt{\pi}}{2}.$$

Substituting the expressions (VII.2), (VII.3), (VII.4), (VII.5), (VII.6) in (VII.1) we simplify $\overline{W}_2(o, p)$ with the result

$$\overline{W}_2(o, p) = \frac{\sqrt{\pi}}{3^{1/2}\cdot 2^{1/6}R\left[\sqrt{\pi}J_1\left(\frac{p}{R}\right) - \dfrac{3^{2/3}\,\Gamma 7/6\, pJ_1'\left(\frac{p}{R}\right)}{2Rp^{1/3}} + \ldots\right]}$$
$$\int_0^1 \rho\psi(\rho)\bar{h}(p,\rho)\left[3^{2/3}\Gamma 7/6\,(e^{2\theta_1} erf\ \theta_1^{\frac{1}{2}} - erfi\ \theta_1^{\frac{1}{2}})\right.$$
$$\left.+\frac{3^{\frac{2}{3}}\sqrt{\pi}}{2}\left\{-e^{+2\theta_1}\int_0^{\theta_1} e^{-\theta}\theta^{\frac{1}{6}}d\theta + \int_0^{\theta_1} e^{\theta}\theta^{\frac{1}{6}}d\theta\right\}\right]d\rho[1+Op^{-\frac{1}{2}}]. \quad (\text{VII.7})$$

Again, if we put $V_1 = r - V_0$, such that

$$V_0 = \frac{1}{\Gamma\frac{1}{3}r^{1/2}}\Gamma\left(\frac{1}{3}, \frac{(1-r^2)^3}{36x}\right)[1+Ox^{\frac{1}{3}}],$$

then

$$\bar{h}(r, p) = -2p\int_{-\infty}^{+\infty}\left[\frac{\partial V_0}{\partial x} - \frac{V_0}{r}\frac{\partial V_0}{\partial x}\right]e^{-px}\,dx. \quad \ldots \quad \ldots \text{(VII.8)}$$

Using the result of Laplace transform for the product of two functions and expressing

$$\bar{V}_0 = \frac{2(1-r^2)^{1/2}p^{1/6}}{\Gamma 1/3 r^{\frac{1}{2}}6^{\frac{1}{3}}}K_{1/3}\left\{\frac{p^{1/2}(1-r^2)^{3/2}}{3}\right\}[1+Op^{-\frac{1}{2}}].$$

in its integral form (VII.8) can be reduced to

$$\bar{h}(r. p) = 2p\Bigg[-\frac{2(1-r^2)^{1/2}p^{1/6}}{\Gamma\frac{1}{3}r^{1/2}6^{1/3}}K_{\frac{1}{3}}\left\{\frac{p^{1/2}(1-r^2)^{3/2}}{3}\right\}$$

$$+\frac{4\pi}{r^{\frac{1}{2}}(\Gamma\frac{1}{3}\Gamma\frac{1}{6})^2}\int_1^\infty (t^2-1)^{-5/6}dt\int_1^\infty(\theta^2-1)^{-5/6}d\theta\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-\frac{(1-r^2)^{3/2}}{3}(s^{\frac{1}{2}}t+\sqrt{p-s}\,\theta)}}{s}\,ds\Bigg]$$

$$\times[1+Op^{-\frac{1}{2}}]. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(VII.9)}$$

Performing the integration of s with a branch cut along the negative real axis and taking the pole at the origin into consideration and putting $s = pu$, we have (VII.9) in the form

$$\bar{h}(r. p) = \frac{2p}{r^{\frac{1}{2}}}\Bigg[\frac{3.6^{2/3}\theta K_{1/3}(\theta)}{4\Gamma\frac{1}{3}p^{1/3}} - \frac{2^{1/3}\theta^{2/3}}{(\Gamma\frac{1}{3})^2}\left\{1+\frac{3^{5/3}\theta^{2/3}}{4p^{1/3}}+\ldots\right\}\int_0^\infty u^{1/6}(1+u)^{1/6}$$

$$\times K_{1/3}(\theta\sqrt{1+u})J_{1/3}(\theta\sqrt{u})\frac{du}{u}\Bigg][1+Op^{-\frac{1}{2}}], \quad \ldots \quad \ldots \text{(VII.10)}$$

where

$$\theta = \frac{p^{1/2}(1-r^2)^{3/2}}{3}.$$

Again, expanding $k_{\frac{1}{3}}(\theta\sqrt{1+u})$ asymptotically for $\theta\to\infty$ and putting $\phi = \theta\sqrt{1+u}$, the integral in (VII.10) can be simplified to

$$\frac{\sqrt{2\pi}}{\theta^{2/3}}\int_\theta^\infty\frac{e^{-\phi}J_{\frac{1}{3}}(\sqrt{\phi^2-\theta^2})\phi^{5/6}}{(\phi^2-\theta^2)^{5/6}}\,d\phi. \quad \ldots \quad \ldots \text{(VII.11)}$$

Since θ is large, the integrand in (VII.11) will be asymptotically small for $\phi-\theta > 0$ and hence we expand $J_{1/3}(\phi^2-\theta^2)^{1/2}$ in series and perform the integration in (VII.11) and introduce the result in (VII.10) so that

$$\bar{h}(r, p) = -\frac{3\sqrt{\pi}2^{5/6}pe^{-\theta}\theta^{1/6}}{(\Gamma\frac{1}{3})^2 r^{1/2}}\left[1+\frac{3^{5/3}\theta^{2/3}}{4p^{1/3}}-\frac{\Gamma\frac{1}{3}\theta^{1/3}6^{2/3}}{4\,.\,2^{\frac{1}{3}}p^{\frac{1}{3}}}+\ldots\right]. \qquad \text{(VII.12)}$$

After putting the values of (VII.12) in (VII.7) integration is performed with the result

$$\overline{W}_2(o, p) = -\frac{(3p)^{2/3}E_1}{R\,.\,2^{1/6}(\Gamma\frac{1}{3})^2 J_1\left(\frac{p}{R}\right)}[1+Op^{-\frac{1}{3}}],$$

where

$$E_1 = \left[-\frac{3}{4}(\Gamma 7/6)^2+\frac{3\sqrt{\pi}\Gamma 11/6}{4}-\Gamma 7/6\sum_0^\infty\frac{\Gamma(n+7/6)}{n\,!\,(2n+1)2^{n+7/6}}\right.$$
$$\left.+\frac{\sqrt{\pi}}{2}\sum_0^\infty\frac{\Gamma(n+11/6)}{n\,!\,(n+7/6)2^{n+11/6}}\right].$$

We now expand

$$\frac{1}{J_1\left(\frac{p}{R}\right)}$$

in series and take inverse Laplace transform so that

$$W_2(o, x) = -\frac{3^{2/3}E_1}{2^{1/6}(\Gamma\frac{1}{3})^3}\left[6x^{1/3}+\sum_1^\infty\frac{e^{-j_n z}\int_0^{j_n z}e^{\phi}\phi^{2/3}\,d\phi}{J_0(j_n)(Rj_n)^{1/3}}\right][1+Ox^{\frac{1}{3}}] \qquad z > 0,$$
$$= 0, \qquad z = 0. \qquad \text{(VII.13)}$$

For $J_n z \to \infty$ (VII.13) can be written as

$$W_2(o, x) = -\frac{3^{2/3}x^{1/3}E_1}{2^{1/6}(\Gamma\frac{1}{3})^3}\left(6-\frac{\cdot 38479}{z}\right)(1+Ox^{1/3}), \qquad z > 0$$
$$= 0, \qquad z = 0, \qquad \text{(VII.14)}$$

where

$$\sum_1^\infty\frac{1}{j_n J_0(j_n)} = -\cdot 38479.$$

Appendix VIII

Here an expression for

$$\frac{\partial}{\partial x}\int_0^1 r\,dr\int_0^r\frac{V_1^2}{r}\,dr$$

is found for $x > 0$.

For $x > 0$, using (3.6)

$$\int_0^1 rdr \int_0^r \frac{V_1^2}{r}\,dr = \frac{2^{10/3}\,.\,3^{1/3}}{(\Gamma\frac{1}{3})^2}\sum_1^\infty \sum_1^\infty (-1)^{m+n}\beta_m^{1/3}\beta_n^{1/3}e^{-(\beta_m^2+\beta_n^2)x}\int_0^1 rdr \times \int_0^r \frac{\psi_m(\theta)\psi_n(\theta)}{\theta}\,d\theta. \quad \text{(VIII.1)}$$

But

$$\int_0^1 rdr \int_0^r \frac{\psi_m(\theta)\psi_n(\theta)}{\theta}\,d\theta = \tfrac{1}{2}\int_0^1 \frac{\psi_m(\theta)\psi_n(\theta)}{\theta}\,d\theta - \tfrac{1}{2}\int_0^1 r\psi_m(r)\psi_n(r)\,dr. \quad \text{(VIII.2)}$$

Using orthogonal properties of $\psi_n(r)$ and (V.4) it can be shown that

$$\int_0^1 r\psi_m(r)\psi_n(r)dr = \frac{(-1)^{m+n}3\,.\,6^{1/3}(\Gamma 2/3)^2}{\pi^2\beta_m^2(\beta_m\beta_n)^{2/3}}\left[1+O\beta_n^{-4/3}\right], \qquad m > n,$$

$$= \frac{1}{\beta_n^3}+\frac{3\,.\,6^{1/3}(\Gamma 2/3)^2}{\pi^2\beta_n^{10/3}}\left[1+O\beta_n^{-4/3}\right], \qquad m = n. \quad \text{(VIII.3)}$$

To evaluate the first integral in (VIII.2) we use the relation

$$\phi(b_n, 2\,;\,\beta_n x) = -\frac{2}{\beta_n}e^{1/2\beta_n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{u-1}{u+1}\right)^{\beta_n/4}e^{1/2\beta_n x u}\,du, \qquad Rc > 1,$$

and convert it into a complex integral of the form

$$-\frac{4}{\beta_m\beta_n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\left(\frac{u-1}{u+1}\right)^{\beta_n/4}\left(\frac{v-1}{v+1}\right)^{\beta_m/4}\frac{du\,dv}{\beta_m v+\beta_n u}$$

$$+\frac{4}{\beta_m\beta_n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\left(\frac{u-1}{u+1}\right)^{\beta_n/4}\left(\frac{v-1}{v+1}\right)^{\beta_m/4}\frac{e^{(\beta_n u+\beta_m v)/2}}{\beta_m v+\beta_n u}\,du\,dv. \quad \text{(VIII.4)}$$

For $\beta_m \geqslant \beta_n$ the first integral in (VIII.4) can be shown to be $1/\beta_m^2$ and the second integral can be evaluated in the same way as (V.4).
So that

$$\int_0^1 \frac{\psi_m(r)\psi_n(r)}{r}\,dr = \frac{1}{\beta_m^2}+\frac{(-1)^{m+n}6^{4/3}(\Gamma 2/3)^2}{\pi^2\beta_m^{8/3}\beta_n^{2/3}}\left[1+O\beta_n^{-4/3}\right], \qquad m \geqslant n. \quad \text{(VIII.5)}$$

Inserting (VIII.3), (VIII.5) in (VIII.1) we have finally

$$\frac{\partial}{\partial x}\int_0^1 rdr\int_0^r \frac{V_1^2}{r}dr = -\frac{2^{10/3}3^{1/3}}{(\Gamma\frac{1}{3})^2}\sum_1^\infty\sum_1^\infty (-1)^{m+n}\beta_m^{1/3}\beta_n^{1/3}e^{-(\beta_m^2+\beta_n^2)x} \times\left[\left(1-\frac{1}{\beta_m}+\frac{6^{4/3}(\Gamma 2/3)^2}{2\pi^2\beta_m^{4/3}}\right)\beta_m^{2/3}e^{-2\beta_m^2 x}\delta_{mn} +\left(1+\frac{\beta_m^2}{\beta_n^2}\right)\left\{1+\frac{(-1)^{m+n}6^{4/3}(\Gamma 2/3)^2}{2\pi^2(\beta_m\beta_n)^{2/3}}\right\}\right].$$

Appendix IX

Here an expression for

$$\frac{\partial}{\partial x}\int_0^1 r\,dr\int_0^r \frac{V_1^2}{r}\,dr$$

is found for $x \to 0$.

Let

$$I(x) = \frac{\partial}{\partial x}\int_0^1 r\,dr\int_0^r \frac{V_1^2}{r}\,dr. \qquad \text{(IX.1)}$$

Taking Laplace transform and simplifying it can be reduced to

$$\bar{I}(p) = \tfrac{1}{2}\int_0^1 (1-r^2)\bar{h}(r, p)\,dr, \qquad \text{(IX.2)}$$

where $\bar{h}(r, p)$ is defined in (VII.10).

Introducing $\bar{h}(r, p)$ from (VII.10) in (IX.2) and putting asymptotic expression of $K_{\frac{1}{3}}(\theta\sqrt{1+u})$ for $\theta \to \infty$, (IX.2) can be transformed to

$$\frac{2\sqrt{\pi}\,(3p)^{1/3}}{\Gamma 1/6\,\Gamma 1/3}\int_1^\infty (t^2-1)^{-5/6}\,dt\int_0^{\sqrt{p}/3} e^{-\theta t}\left(\frac{3^{5/3}\theta}{4p^{1/3}}+\dots\right)d\theta - \frac{(6p)^{1/3}\sqrt{\pi}}{\sqrt{2}(\Gamma\frac{1}{3})^2}$$

$$\times\int_0^\infty \frac{du}{u^{5/6}(1+u)^{\frac{1}{12}}}\int_0^{\sqrt{p}/3} e^{-\theta\sqrt{1+u}}\,J_{\frac{1}{3}}(\theta\sqrt{u})\left(\theta^{\frac{1}{2}}+\frac{3^{5/3}\theta^{7/6}}{2p^{\frac{1}{3}}}+\dots\right)d\theta. \qquad \text{(IX.3)}$$

In (IX.3) integrand is asymptotically small for $p \to \infty$, so that the integral can be transformed to real axis with the upper limit extending to infinity. Performing the integral with respect to θ and t, (IX.3) can be reduced to

$$\frac{3}{2} - \frac{(6p)^{1/3}\sqrt{\pi}}{\sqrt{2}(\Gamma\frac{1}{3})^2}\int_0^\infty \frac{du}{u^{5/6}(1+u)^{1/12}}\left[\frac{\Gamma 11/6}{(1+2y)^{3/4}}P_{1/2}^{-1/3}\left(\frac{1+y}{1+2y}\right)^{1/2}\right.$$

$$\left.+\frac{3^{5/3}\Gamma 5/2}{2p^{\frac{1}{3}}(1+2y)^{\frac{13}{12}}}P_{7/6}^{-1/3}\left(\frac{1+y}{1+2y}\right)^{\frac{1}{2}}+\dots\right], \qquad \text{(IX.4)}$$

where $P_\nu^{-\mu}(\theta)$ is the Legendre's associated function.

Again, using the expansion

$$P_\nu^{-\mu}(\cos\theta) = \frac{1}{\left[\left(\nu+\frac{1}{2}\right)\cos\frac{\theta}{2}\right]^\mu}\left\{J_\mu(a)+\sin^2\frac{\theta}{2}\left[\frac{a}{6}J_{\mu+3}(a)-J_{\mu+2}(a)\right.\right.$$

$$\left.\left.+\frac{1}{2a}J_{\mu+1}(a)\right]+O\left(\sin\frac{\theta}{2}\right)^4\right\},$$

where $a = (2\nu+1)\sin(\theta/2)$,

for small θ, and raising the integrand in power of x where

$$x = \sin\frac{\theta}{2}$$

$$\cos^2\theta = \frac{1+y}{1+2y},$$

and performing the integration we have

$$\bar{I}(p) = \left\{ -\frac{9 \cdot 2^{1/6}(3p)^{1/3}\sqrt{\pi}\,\Gamma 11/6}{(\Gamma\frac{1}{3})^3}\left(\beta^{2/3}+\frac{23}{64}\beta^{8/3}+\ldots\right) + \frac{3}{2} - \frac{243\pi}{4 \cdot 2^{5/6}(\Gamma\frac{1}{3})^3}\left(\beta^{2/3}-\frac{5}{8}\beta^{8/3}\ldots\right)\right\}(1+Op^{-\frac{1}{3}}). \quad \text{(IX.5)}$$

Taking the inverse transform we have

$$I(x) = \left\{ -\frac{9 \cdot 2^{1/6} \cdot 3^{5/6}\Gamma 11/6}{2\sqrt{\pi}(\Gamma\frac{1}{3})^2 x^{\frac{1}{3}}}\left(\beta^{2/3}+\frac{23}{64}\beta^{8/3}\ldots\right) + 1 \cdot 5 - \frac{243\pi}{4 \cdot 2^{5/6}(\Gamma\frac{1}{3})^3}\left(\beta^{2/3}-\frac{5}{8}\beta^{8/3}\ldots\right)\right\}[1+Ox^{\frac{1}{3}}],$$

where

$$\beta = \sin\frac{\pi}{8}.$$

Conclusion

We have made an attempt to study the problem rigorously at high Reynolds numbers. If we change swirl velocity V_1 to $r-V_1$ we have the solution for case II. Figs. 1, 2 and 4 show that there are some discrepancies with the previous results of Figs. 1, 2 and 3. It is, perhaps, because of our Oseen type approximation used at high Reynolds number. However, to study this simple-looking problem at all Reynolds numbers is a good task. We expect to study further on this topic which is quite new in the line.

Acknowledgements

The author is grateful to Mr. L. E. Fraenkel for suggesting the problem and to Professor J. N. Kapur for kindly going through the manuscript.

References

Archibald, W. J. (1938). Confluent hypergeometric equation and its solution. *Phil. Mag.*, **26**, 415–419.

Colatz, L., and Görtler, H. (1954). Rohrstömung mit schwachem drall. *Z. angew. Math. Phys.*, **5**, 95–110.

Deka, B. C. (1963). Some effects of weak swirl on laminar pipe flow. *Proc. natn. Inst. Sci. India*, **A29**, 251–270.

Erdelyi, A. (1954). Tables of Integral Transforms, Vol. I.

————. Higher Transcendental Functions, Vols. 1-2.

————. Asymptotic Expansions. McGraw-Hill.

Lawerier, H. A. (1950). The use of confluent hypergeometric function in mathematical physics and the solution of an eigen-value problem. *Appl. scient. Res.*, **A2**, 184.

Talbot, L. (1954). Laminar swirling pipe flow. *J. appl. Mech.*, **21**, 1–7.

Van der Pol, B., and Bremmer, H. (1955). Operational Calculus based on Two-sided Laplace Integral. Cambridge University Press.

Watson, G. N. (1944). Theory of Bessel Functions. Cambridge University Press.