SURFACE WAVES IN A MULTI-LAYERED ELASTIC MEDIUM

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In this paper an alternative method has been suggested for evaluating source term in an n-layered elastic medium. The results of the present paper can be utilized to find the displacements for different types of force system at the source representing point source and also line source.

INTRODUCTION

The source problem in an n-layered elastic medium has been studied by Haskell (1964), Harkrider (1964), Ben-Menahem and Harkrider (1964). Harkrider (1964) also evaluated surface displacements associated with surface wave produced by a vertical line source by spatial integration of the solution for a point source. All the previous authors have followed Haskell's matrix analysis in their works and introduced the source as a discontinuity in stress-distribution at the source. In a recent paper Roy (1965) calculated the displacements in a semi-infinite medium associated with a general force system and showed that the displacements in the medium could be separated in two parts, one associated with incident wave and another due to reflected wave in the medium. By analogy, we consider the solution of the wave equation in the source layer in n-layered medium as due to incident wave and reflected wave and utilize the boundary condition in the usual way. Knopoff (1964) gave a different matrix method for an n-layered elastic medium in two-dimensional case and considered the contribution of incident wave in the medium.

BASIC EQUATIONS

The displacements $\vec{u}$ at any point in the medium, consisting of n-layers, satisfy the equation

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \rho \vec{F} + (\lambda + 2\mu) \text{ grad div } \vec{u} - \mu \text{ curl curl } \vec{u}, \quad \ldots \quad \ldots \quad (1)$$

where $\rho \vec{F}$ are the force per unit volume which will be zero except in the layer containing the source.

The displacements $\vec{u}_s$ at the source layers can be written as

$$\vec{u}_s = \vec{u}_{s1} + \vec{u}_{s2}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where $\vec{u}_{s1}$ are the displacements associated with the reflected wave, satisfying the eqn. (1) without the force term $\rho\vec{F}$; in other words, they are the general solution of (1); and $\vec{u}_{s2}$ are the displacements associated with incident wave in the medium satisfying eqn. (1) which, in other words, may be regarded as a particular integral of (1). Therefore (2) is the most general solution of eqn. (1).

For the displacements associated with the incident wave we can directly utilize the results obtained for the semi-infinite medium. If we assume that the forces can be represented by the components $R(r, z) \cos l\phi \exp(i\omega t)$, $X(r, z)$ $\sin l\phi \exp(i\omega t)$ and $z(r, z) \cos l\phi \exp(i\omega t)$ acting in the region $h_1 \leqslant z \leqslant h_2$ the displacements associated with the incident wave can be written (Roy 1965, eqns. 28 and 29) as follows, suppressing the time factor $\exp(i\omega t)$, for $z_{s-1} \leqslant z \leqslant h$,

$$
\begin{aligned}
u'_r(r, \phi, z) = &-\int_0^\infty \left[\frac{\exp(p_{\alpha_s} z)}{4\omega^2 p_{\alpha_s}} \left\{\left(R_{2\alpha_s} + X_{2\alpha_s} + X'_{2\alpha_s} - R'_{2\alpha_s}\right)k^2 + 2Z_{2\alpha_s} k p_{\alpha_s}\right\}\right. \\
&\left. - \frac{\exp(p_{\beta_s} z)}{4\omega^2 p_{\beta_s}} \left\{\left(X_{2\beta_s} + R_{2\beta_s} + X'_{2\beta_s} - R'_{2\beta_s}\right)p_{\beta_s}^2 + 2Z_{2\beta_s} k p_{\beta_s}\right\}\right] \\
&\times \frac{\partial J_l(kr)}{\partial r}\, dk \cos l\phi - \int_0^\infty \frac{\exp(p_{\beta_s} z)}{4\omega^2 p_{\beta_s}} \left(X'_{2\beta_s} - R'_{2\beta_s} - X_{2\beta_s} - R_{2\beta_s}\right) \\
&\times \frac{\omega^2}{\beta_s^2} \frac{lJ_l(kr)}{r}\, dk \cos l\phi;
\end{aligned}
$$

$$
\begin{aligned}
u'_\phi(r, \phi, z) = &\int_0^\infty \left[\frac{\exp(p_{\alpha_s} z)}{4\omega^2 p_{\alpha_s}} \left\{\left(R_{2\alpha_s} + X_{2\alpha_s} + X'_{2\alpha_s} - R'_{2\alpha_s}\right)k^2 + 2Z_{2\alpha_s} k p_{\alpha_s}\right\}\right. \\
&\left. - \frac{\exp(p_{\beta_s} z)}{4\omega^2 p_{\beta_s}} \left\{\left(R_{2\beta_s} + X_{2\beta_s} + X'_{2\beta_s} - R'_{2\beta_s}\right)p_{\beta_s}^2 + 2Z_{2\beta_s} k p_{\beta_s}\right\}\right] \\
&\times \frac{lJ_l(kr)}{r}\, dk \sin l\phi + \int_0^\infty \frac{\exp(p_{\beta_s} z)}{4\omega^2 p_{\beta_s}} \left\{\left(X'_{2\beta_s} - R'_{2\beta_s} - X_{2\beta_s} - R_{2\beta_s}\right)\right\} \\
&\times \frac{\omega^2}{\beta_s^2} \frac{\partial J_l(kr)}{\partial r}\, dk \sin l\phi;
\end{aligned}
$$

$$
\begin{aligned}
u'_z(r, \phi, z) = &\int_0^\infty \left[-\frac{\exp(p_{\alpha_s} z)}{4\omega^2} \left\{\left(R_{2\alpha_s} + X_{2\alpha_s} + X'_{2\alpha_s} - R'_{2\alpha_s}\right)k + 2Z_{2\alpha_s} p_{\alpha_s}\right\}\right. \\
&\left. + \frac{\exp(p_{\beta_s} z)}{4\omega^2} \left\{\left(R_{2\beta_s} + X_{2\beta_s} + X'_{2\beta_s} - R'_{2\beta_s}\right)k + 2Z_{2\beta_s} k^2 / p_{\beta_s}\right\}\right] \\
&\times kJ_l(kr)\, dk \cos l\phi; \qquad \ldots \qquad (3)
\end{aligned}
$$

and for $h_2 \leqslant z \leqslant z_s$

$$u_r''(r, \phi, z) = -\int_0^\infty \left[\frac{\exp(-p_{\alpha_s}z)}{4\omega^2 p_{\alpha_s}}\left\{\left(\bar{R}_{2\alpha_s}+\bar{x}_{2\alpha_s}+\bar{x}'_{2\alpha_s}-\bar{R}'_{2\alpha_s}\right)k^2-2\bar{Z}_{2\alpha_s}kp_{\alpha_s}\right\}\right.$$
$$\left.-\frac{\exp(-p_{\beta_s}z)}{4\omega^2 p_{\beta_s}}\left\{\left(\bar{R}_{2\beta_s}+\bar{x}_{2\beta_s}+\bar{x}'_{2\beta_s}-\bar{R}'_{2\beta_s}\right)p_{\beta_s}^2-2Z_{2\beta_s}kp_{\beta_s}\right\}\right]$$
$$\times\frac{\partial J_l(kr)}{\partial r}dk\cos l\phi-\int_0^\infty\left(\bar{x}'_{2\beta_s}-\bar{R}'_{2\beta_s}-\bar{R}_{2\beta_s}-\bar{x}_{2\beta_s}\right)\frac{\exp(-p_{\beta_s}z)}{4\omega^2 p_{\beta_s}}$$
$$\times\frac{\omega^2}{\beta_s^2}\frac{lJ_l(kr)}{r}dk\cos l\phi;$$

$$u_\phi''(r, \phi, z) = \int_0^\infty \left[\frac{\exp(-p_{\alpha_s}z)}{4\omega^2 p_{\alpha_s}}\left\{\left(\bar{R}_{2\alpha_s}+\bar{x}_{2\alpha_s}+\bar{x}'_{2\alpha_s}-\bar{R}'_{2\alpha_s}\right)k^2-2\bar{z}_{2\alpha_s}kp_{\alpha_s}\right\}\right.$$
$$\left.-\frac{\exp(-p_{\beta_s}z)}{4\omega^2 p_{\beta_s}}\left\{\left(\bar{x}_{2\beta_s}+\bar{R}_{2\beta_s}+\bar{x}'_{2\beta_s}-\bar{R}'_{2\beta_s}\right)p_{\beta_s}^2-2\bar{z}_{2\beta_s}kp_{\beta_s}\right\}\right]$$
$$\times\frac{lJ_l(kr)}{r}dk\sin l\phi+\int_0^\infty\frac{\exp(-p_{\beta_s}z)}{4\omega^2 p_{\beta_s}}\left(\bar{x}'_{2\beta_s}-\bar{R}'_{2\beta_s}-\bar{R}_{2\beta_s}-\bar{x}_{2\beta_s}\right)\frac{\omega^2}{\beta_s^2}$$
$$\times\frac{\partial J_l(kr)}{\partial r}dk\sin l\phi;$$

$$u_z''(r, \phi, z) = \int_0^\infty \left[\frac{\exp(-p_{\alpha_s}z)}{4\omega^2}\left\{\left(\bar{R}_{2\alpha_s}+\bar{x}_{2\alpha_s}+\bar{x}'_{2\alpha_s}-\bar{R}'_{2\alpha_s}\right)k-2\bar{z}_{2\alpha_s}p_{\alpha_s}\right\}\right.$$
$$\left.-\frac{\exp(-p_{\beta_s}z)}{4\omega^2}\left\{\left(\bar{R}_{2\beta_s}+\bar{x}_{2\beta_s}+\bar{x}'_{2\beta_s}-\bar{R}'_{2\beta_s}\right)k-2\bar{z}_{2\beta_s}k^2/p_{\beta_s}\right\}\right]$$
$$\times kJ_l(kr)dk\cos l\phi. \quad \ldots \quad (4)$$

For any point $h_1 \leqslant z_1 \leqslant h_2$, the displacements associated with the incident wave are given by

$$\left.\begin{aligned} u_r'''(r, \phi, z_1) &= u_r'^{(1)}(r, \phi, z_1)+u_r''^{(2)}(r, \phi, z_1)\\ u_\phi'''(r, \phi, z_1) &= u_\phi'^{(1)}(r, \phi, z_1)+u_\phi''^{(2)}(r, \phi, z_1)\\ u_z'''(r, \phi, z_1) &= u_z'^{(1)}(r, \phi, z_1)+u_z''^{(2)}(r, \phi, z_1)\end{aligned}\right\} \quad \ldots \quad (5)$$

where $u_r'^{(1)}(r, \phi, z_1)$ is obtained from $u_r'(r, \phi, z)$ in (3) by writing z_1 for z and (writing the force system, for example, $R_{2\alpha_s}$ with a superscript 1, namely $R^1_{2\alpha_s}$) $u_r''^{(2)}(r, \phi, z_1)$ is obtained from $u_r''(r, \phi, z)$ in (4) by writing $\bar{R}^2_{2\alpha_s}$ for $\bar{R}_{2\alpha}$

and z_1 for z where

$$R_1 = R_1(k, z) = \int_0^\infty R(r, z) r J_{l+1}(kr)\, dk \qquad \chi_1 = \int_0^\infty \chi(r, z) r J_{l+1}(kr)\, dk$$

$$R_1' = R_1'(k, z) = \int_0^\infty R(r, z) r J_{l-1}(kr)\, dk \qquad \chi_1' = \int_0^\infty \chi(r, z) r J_{l-1}(kr)\, dk$$

$$Z_1 = Z_1(k, z) = \int_0^\infty Z(r, z) r J_l(kr)\, dk \qquad \ldots \qquad (6)$$

and

$$R_2 = \int_0^\infty R_1(k, z) \exp(-pz)\, dz. \qquad \ldots \qquad (7)$$

Similarly R_2', Z_2, χ_2, etc. are defined.

R_2^1 is the transform defined by (6) and (7) on $R(r, z)[H(z-h_1)-H(z-z_1)]$ and R_2^2 that on $R(r, z)[H(z-z_1)-H(z-h_2)]$

$$R_{2\alpha_s} = R_2(k, p)|_{p=p_{\alpha_s}} \qquad R_{2\beta_s} = R_2(k, p)|_{p=p_{\beta_s}}$$

$$\bar{R}_{2\alpha_s} = R_2(k, p)|_{p=-p_{\alpha_s}} \qquad \bar{R}_{2\beta_s} = R_2(k, p)|_{p=p_{\beta_s}}$$

$$p_{\alpha_s} = \sqrt{k^2-\omega^2/\alpha_s^2} = k\sqrt{1-c^2\,\alpha_s^2} = ikr_{\alpha_s}$$

$$p_{\beta_s} = \sqrt{k^2-\omega^2/\beta_s^2} = k\sqrt{1-c^2/\beta_s^2} = ikr_{\beta_s}.$$

Similarly $\chi_{2\alpha_s}$, $\chi'_{2\alpha_s}$, etc. are defined.

It is easy to verify that the eqns. (3), (4) and (5) satisfy the eqn. (1).

The conditions of continuity of displacements and stresses at $z = z_{m-1}$ can be expressed in vector form following Harkrider (1964, eqns. 9*a* and 9*b*) as

$$B_{R_m}(z_{m-1}) = B_{R_{m-1}}(z_{m-1})$$

$$B_{L_m}(z_{m-1}) = B_{L_{m-1}}(z_{m-1}). \qquad \ldots \qquad (8)$$

where $B_{R_m}(z)$ are the column matrix with elements $\left(c^{-1}\dot{u}_{R_m}(z), c^{-1}\dot{w}_{R_m}(z), \sigma_{R_m}(z), \tau_{R_m}(z)\right)$ and $B_{L_m}(z)$ are the column matrix with elements $\left(c^{-1}\dot{v}_{L_m}(z), \tau_{L_m}(z)\right)$. Complete expressions for $B_{R_m}(z)$ and $B_{L_m}(z)$ are given in Harkrider (1964).

Also we have for the mth layer

$$B_{R_m}(z_m) = D_{R_m} E_{R_m}^{-1} B_{R_m}(z_{m-1}). \qquad \ldots \qquad (9)$$

The elements of the layer-matrix

$$a_{R_m} = D_{R_m} E_{R_m}^{-1} \qquad \ldots \qquad (10)$$

are the same as given in Harkrider (1964). For the source layer, the conditions of continuity of stress and displacements at $z = z_{s-1}$ and $z = z_s$ take the form

$$B_{R_{s-1}}(z_{s-1}) = B_{R_s}(z_{s-1}) = B_{R_{s1}}(z_{s-1}) + B'_{R_s}(z_{s-1}) \quad \dots \quad \dots \quad (11)$$

$$B_{R_{s+1}}(z_s) = B_{R_s}(z_s) = B_{R_{s1}}(z_s) + B''_{R_s}(z_s), \quad \dots \quad \dots \quad (12)$$

where $B_{R_{s1}}$ is the part associated with reflected wave and hence satisfies the condition

$$B_{R_{s1}}(z_s) = a_{R_s} B_{R_{s1}}(z_{s-1}) \quad \dots \quad \dots \quad \dots \quad \dots \quad (13)$$

$B'_{R_s}(z)$ and $B''_{R_s}(z)$ are the parts associated with incident wave.
By a process of iteration we have

$$B_{R_{s-1}}(z_{s-1}) = a_{R_{s-1}} a_{R_{s-2}} \dots a_1 B_{R_1}(o), \quad \dots \quad \dots \quad \dots \quad (14)$$

where $B_{R_1}(o)$ are the column matrix with elements $(c^{-1}u_R(o),\ c^{-1}\omega_R(o),\ o,\ o)$.
Similarly

$$B_{R_{n-1}}(z_{n-1}) = a_{R_{n-1}} a_{R_{n-2}} \dots a_{R_{s+1}} B_{R_{s+1}}(z_s). \quad \dots \quad \dots \quad (15)$$

We define a new column matrix C_R with elements $(W,\ X,\ Y,\ Z)$

$$C_R = A_{R_s}^{-1} B_{R_{s+1}}(z_s), \quad \dots \quad \dots \quad \dots \quad \dots \quad (16)$$

where

$$A_{R_s} = a_{R_s} a_{R_{s-1}} \dots a_1. \quad \dots \quad \dots \quad \dots \quad (17)$$

Then on utilizing (11) and (12)

$$\begin{aligned} C_R &= A_{R_s}^{-1} B_{R_{s1}}(z_s) + A_{R_s}^{-1} B''_{R_s}(z_s) \\ &= A_{R_s}^{-1} a_{R_s} B_{R_{s1}}(z_s) + A_{R_s}^{-1} B''_{R_s}(z_s) \\ &= A_{R_s}^{-1} a_{R_s} B_{R_{s1}}(z_s) - A_{R_s}^{-1} a_{R_s} B'_{R_s}(z_{s-1}) + A_{R_s}^{-1} B''_{R_s}(z_s). \quad \dots \quad (18) \end{aligned}$$

Assuming a force system distributed over a finite domain of depth $h_2 - h_1$ and independent of z, we can take

$$R(r, z) = R(r)[H(z-h_1) - H(z-h_2)]$$

$$\chi(r, z) = \chi(r)[H(z-h_1) - H(z-h_2)]$$

$$Z(r, z) = Z(r)[H(z-h_1) - H(z-h_2)].$$

It can be shown

$$B'_{R_s}(h_1) = E_{R_s} M \quad \dots \quad \dots \quad \dots \quad \dots \quad (19)$$

and

$$B'_{R_s}(z_{s-1}) = D'_{R_s} M, \quad \dots \quad \dots \quad \dots \quad \dots \quad (20)$$

where M is a column matrix with elements $(P, -P, P', -P')$ and

$$P = \frac{(R_1+\chi_1+\chi_1'-R_1')+2iZ_1 r_{\alpha_s}}{4\alpha_s^2 r_{\alpha_s}^2}\left[\exp\{-ikr_{\alpha_s}(h_2-h_1)\}-1\right]$$

$$P' = \frac{(R_1+\chi_1+\chi_1'-R_1')ir_{\beta_s}+2Z_1}{4i\gamma_s r_{\beta_s}^2 c^2}\left[\exp\{-ikr_{\beta_s}(h_2-h_1)\}-1\right]$$

$$D'_{R_s} = \begin{bmatrix} -\left(\frac{\alpha_s}{c}\right)^2\cos P_{s1} & -i\left(\frac{\alpha_s}{c}\right)^2\sin P_{s1} & -\gamma_s r_{\beta_s}\cos Q_{s1} & -i\gamma_s r_{\beta_s}\sin Q_{s1} \\ i\left(\frac{\alpha_s}{c}\right)^2 r_{\alpha_s}\sin P_{s1} & -\left(\frac{\alpha_s}{c}\right)^2 r_{\alpha_s}\cos P_{s1} & i\gamma_s\sin Q_{s1} & \gamma_s\cos Q_{s1} \\ -\rho_s\alpha_s^2(\gamma_s-1)\cos P_{s1} & i\rho_s\alpha_s^2(\gamma_s-1)\sin P_{s1} & -\rho_s c^2\gamma_s^2 r_{\beta_s}\cos Q_{s1} & -i\rho_s\gamma_s^2 r_{\beta_s}\sin Q_{s1} \\ i\rho_s\alpha_s^2\gamma_s r_{\alpha_s}\sin P_{s1} & \rho_s\alpha_s^2 r_{\alpha_s}\gamma_s\cos P_{s1} & -i\rho_s c^2\gamma_s(\gamma_s-1)\sin Q_{s1} & -\rho_s c^2\gamma_s(\gamma_s-1)\cos Q_{s1} \end{bmatrix}$$

$$P_{s1} = kr_{\alpha_s}(h_1-z_{s-1}) \qquad Q_{s1} = kr_{\beta_s}(h_1-z_{s-1}).$$

From (19) and (20) we get

$$B'_{R_s}(z_{s-1}) = D'_{R_s}E_{R_s}^{-1}B'_{R_s}(h_1). \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

Using the relation $\left(a_{R_s}^{-1}\right)_{jk} = (-1)^{j+k}\left(a_{R_s}\right)_{lp}$ (Harkrider, 1964, eqn. 29), where $l = n+1-k$, $p = n+1-j$ and a_{R_s} is a $n\times n$ matrix, it can be shown that

$$D'_{R_s}E_{R_s}^{-1} = a_{R_{s1}}^{-1}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Complete expression for $a_{R_{s1}}^{-1}$ has been given in Harkrider (1964).

In a similar way it can be shown that

$$B''_{R_s}(z_s) = D_{R_s}N \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

$$B''_{R_s}(h_1) = E_{R_s}N, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where N is the column matrix with elements $(Q, Q, Q'. Q')$ and

$$Q = \frac{(R_1+\chi_1+\chi_1'-R_1')-2iZ_1 r_{\beta_s}}{4\alpha_s^2 r_{\alpha_s}^2}\left[1-\exp\{ikr_{\alpha_s}(h_2-h_1)\}\right]$$

$$Q' = \frac{(R_1+\chi_1+\chi_1'-R_1')ir_{\beta_s}-2Z_1}{4ic^2\gamma_s r_{\beta_s}^2}\left[1-\exp\{ikr_{\beta_s}(h_2-h_1)\}\right].$$

Therefore

$$\begin{aligned} B''_{R_s}(z_s) &= D_{R_s}E_{R_s}^{-1}B''_{R_s}(h_1) \\ &= a_{R_{s2}}B''_{R_s}(h_1), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25) \end{aligned}$$

$a_{R_{s2}}$ are defined similar to Harkrider (1964).

The equation (18) on utilizing (14), (20), (21) and (25) reduces to

$$C_R = A_{R_s}^{-1} a_{R_s} a_{R_{s-1}} \dots a_1 B_{R_1}(0) - A_{R_s}^{-1} a_{R_s} a_{R_{s1}}^{-1} B'_{R_s}(h_1) + A_{R_s}^{-1} a_{R_{s2}} B''_{R_s}(h_1) \qquad \dots \quad \dots \quad (26)$$

utilizing the relation

$$A_{R_s} = a_{R_s} a_{R_{s-1}} \dots a_1, \; A_{R_s}^{-1} A_{R_s} = I \qquad I = \text{unit matrix}$$

$$A_{R_s}^{-1} = A_{R_{s-1}}^{-1} a_{R_s}^{-1} \qquad a_{R_s} = a_{R_{s2}} a_{R_{s1}} \qquad a_{R_s}^{-1} = a_{R_{s1}}^{-1} a_{R_{s2}}^{-1}$$

we get from (26)

$$C_R = B_{R_1}(o) - A_{R_{s-1}}^{-1} a_{R_s}^{-1} a_{R_s} a_{R_{s1}}^{-1} B'_{R_s}(h_1) + A_{R_{s-1}}^{-1} a_{R_s}^{-1} a_{R_{s2}} B''_{R_s}(h_1)$$

$$= B_{R_1}(o) - A_{R_{s-1}}^{-1} a_{R_{s1}}^{-1} B'_{R_s}(h_1) + A_{R_{s-1}}^{-1} a_{R_{s1}}^{-1} a_{R_{s2}}^{-1} a_{R_{s2}} B''_{R_s}(h_1)$$

$$= B_{R_1}(o) + A_{R_{s1}}^{-1}\left(B''_{R_s}(h_1) - B'_{R_s}(h_1)\right) \qquad \dots \quad \dots \quad \dots \quad \dots \quad (27)$$

or

$$\begin{bmatrix} W \\ X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \frac{\dot{u}_{R_0}}{c} \\ \frac{\dot{\omega}_{R_0}}{c} \\ O \\ O \end{bmatrix} + A_{R_{s1}}^{-1} \begin{bmatrix} \delta\left(\frac{\dot{u}_{R_s}}{c}\right) \\ \delta\left(\frac{\dot{\omega}_{R_s}}{c}\right) \\ \delta\sigma_{R_s} \\ \delta\tau_{R_s} \end{bmatrix} \qquad \dots \quad \dots \quad \dots \quad (28)$$

where

$$A_{R_{s1}}^{-1} = A_{R_{s-1}}^{-1} A_{R_{s1}}^{-1} = (a_{R_{s1}} a_{R_{s-1}} \dots a_1)^{-1}; \quad \frac{\dot{u}_{R_0}}{c} = \frac{\dot{u}_R(o)}{c} \quad \frac{\dot{\omega}_{R_0}}{c} = \frac{\dot{\omega}_{R_0}}{c}$$

$$\delta\left(\frac{\dot{u}_{R_s}}{c}\right) = \frac{\dot{u}''_{R_s}(h_1)}{c} - \frac{\dot{u}'_{R_s}(h_1)}{c} \qquad \delta\left(\frac{\dot{\omega}_{R_s}}{c}\right) = \frac{\dot{\omega}''_{R_s}(h_1)}{c} - \frac{\dot{\omega}'_{R_s}(h_1)}{c}$$

$$\delta\sigma_{R_s} = \sigma''_{R_s}(h_1) - \sigma'_{R_s}(h_1) \qquad \delta\tau_{R_s} = \tau''_{R_s}(h_1) - \tau'_{R_s}(h_1).$$

Also if k_n denotes column matrix with elements $\left(\hat{\Delta}'_n + \hat{\Delta}'_n,\ \hat{\Delta}'_n - \hat{\Delta}''_n,\ \hat{\omega}'_n - \hat{\omega}''_n,\ \hat{\omega}'_n + \hat{\omega}'_n\right)$ we have, following Harkrider (1964),

$$k_n = E_{R_n}^{-1} B_{R_n}(z_{n-1})$$

$$= E_{R_n}^{-1} a_{R_{n-1}} \dots a_{R_{s+1}} B_{R_{s+1}}(z_s) = E_{R_n}^{-1} a_{R_{n-1}} \dots a_{R_{s+1}} A_{R_s} A_{R_s}^{-1} B_{R_{s+1}}(z_s)$$

$$= JC_R, \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (29)$$

where

$$J = E_{R_n}^{-1} a_{R_{n-1}} \dots a_{R_{s+1}} A_{R_s} = E_{R_n}^{-1} a_{R_{n-1}} \dots a_1 = E_{R_n}^{-1} A_R.$$

The boundary condition at the half space requires

$$\hat{\Delta}_n'' = 0 \text{ and } \hat{\omega}_n'' = 0.$$

Thus evaluating W, X, Y, Z from (29) and substituting in (28), the surface displacements can be easily obtained replacing $A_{R_{s1}}^{-1}$ by their $A_{R_{s1}}$ equivalents.

The components of surface displacements for the jth mode Rayleigh wave are given by

$$u_r^R(r, \phi, o) = -\pi i\left(\frac{\dot{u}_{R_0}}{c}\right)_{k=k_{R_j}} \frac{1}{k_{R_j}^2} \frac{d}{dr} H_l^2(k_{R_j} r) \cos l\phi$$

$$u_\phi^R(r, \phi, o) = \pi i\left(\frac{\dot{u}_{R_0}}{c}\right)_{k=k_{R_j}} \frac{l}{k_{R_j} r} H_l^2(k_{R_j} r) \sin l\phi$$

$$u_z^R(r, \phi, o) = -\pi\left(\frac{\dot{w}_{R_0}}{c}\right)_{k=k_{R_j}} \frac{1}{k_{R_j}} H_l^2(k_{R_j} r) \cos l\phi, \qquad \ldots \quad \ldots \quad (30)$$

where K_{R_j} are the jth mode or root of $F_R = NK - LM = 0$ and

$$\frac{\dot{\omega}_{R_0}}{c} = \frac{GN-LH}{\frac{\partial F_R}{\partial k}}\Bigg[\delta\left(\frac{\dot{u}_{R_s}}{c}\right)\left\{(A_{R_{s1}})_{42} - \frac{K}{L}(A_{R_{s1}})_{41}\right\}$$

$$+\delta\left(\frac{\omega_{R_s}}{c}\right)\left\{-(A_{R_{s1}})_{32} + \frac{K}{L}(A_{R_{s1}})_{31}\right\} + \delta\sigma_{R_s}\left\{(A_{R_{s1}})_{22}\right.$$

$$\left. - \frac{K}{L}(A_{R_{s1}})_{21}\right\} + \delta\tau_{R_s}\left\{-(A_{R_{s1}})_{12} + \frac{K}{L}(A_{R_{s1}})_{11}\right\}\Bigg]$$

$$= \frac{i(G^*N - L^*H)}{\frac{\partial F_R^*}{\partial k}}\Bigg[\delta\left(\frac{\dot{u}_{R_s}}{c}\right)\left[\frac{\tau_s(h_1)}{\frac{\dot{\omega}_0}{c}}\right]_{H_j} - i\delta\left(\frac{\omega_{R_s}}{c}\right)\left[\frac{\sigma_s^*(h_1)}{\frac{\dot{\omega}_0}{c}}\right]_{H_j}$$

$$+\delta\sigma_{R_s}\left[\frac{\dot{\omega}_s(h_1)}{\dot{w}_0}\right]_{H_j} + i\delta\tau_{R_s}\left[\frac{\dot{u}_s^*(h_1)}{\dot{\omega}_0}\right]_{H_j}\Bigg] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

using homogeneous solution notation following Harkrider (1964)

$$\frac{\dot{u}_{R_0}}{c} = \frac{K}{L}\left(\frac{\dot{\omega}_{R_0}}{c}\right) = -i\frac{K}{L^*}\left(\frac{\dot{w}_{R_0}}{c}\right) \quad F_R^* = NK + L^*M^*.$$

In a similar way we can find the surface displacements associated with Love wave components in the following form:

$$u_r^L(r, \phi, o) = -\pi\left(\frac{\dot{v}_{L_0}}{c}\right)_{K=K_{L_j}} \frac{lH_l^2(K_{L_j} r)}{K_{L_j} r} \cos l\phi$$

$$u_\phi^L(r, \phi, o) = \pi\left(\frac{\dot{v}_{L_0}}{c}\right)_{K=K_{L_j}} \frac{1}{K_{L_j}} \frac{d}{dr} H_l^2(K_{L_j} r) \sin l\phi, \qquad \ldots \quad (32)$$

where K_{Lj} is the jth mode or root of $F_L = -(A_L)_{12}\mu_n r_{\beta_n} - (A_L)_{21} = 0$

$$\frac{\dot{v}_{L_0}}{c} = \frac{1}{(A_L)_{11}\dfrac{\partial F_2}{\partial k}}\left[-(A_{L_{s1}})_{21}\delta\left(\frac{\dot{v}_{L_s}}{c}\right)+(A_{L_{s1}})_{11}\delta\tau_{L_s}\right]$$

$$= \frac{i}{(A_L)_{11}\dfrac{\partial F_L^*}{\partial k}}\left[-i\delta\left(\frac{\dot{v}_{L_s}}{c}\right)\left[\frac{\tau^*_{L_s}(h_1)}{\dfrac{\dot{v}_0}{c}}\right]_{H_j}+\delta\tau_{L_s}\left[\frac{\dot{v}_s(h_1)}{\dot{v}_1}\right]_{H_j}\right] \quad .. \quad (33)$$

$$\delta\left(\frac{\dot{v}_{L_s}}{c}\right) = \frac{\dot{v}''_{L_s}(h_1)}{c} - \frac{\dot{v}'_{L_s}(h_1)}{c} \qquad \delta\tau_{L_s} = \tau''_{L_s}(h_1) - \tau'_{L_s}(h_1)$$

$$F^*_L = (A_L)_{21} + (A_L)_{11}\mu_n r^*_{\beta_n}.$$

For force system given by

$$\rho R = f(r)\cos\phi\,\delta(z-h)\exp(iwt) \qquad \rho\chi = -f(r)\sin\phi\,\delta(z-h)\exp(iwt)$$

or

$$\rho X = f(r)\delta(z-h)\exp(iwt) \qquad \rho Y = 0,$$

ρX, ρY being the components of forces in the direction of $\phi = 0$ and $\phi = \pi/2$; if we replace $f(r)$ by a δ-function such that

$$\int_0^\infty f(r) r J_0(kr) = \bar{L}/2\pi,$$

$\bar{L}$ being the strength of the source, we get the case of point source in the direction of $\phi = 0$.

In this case

$$\delta\tau_{R_s} = ik\bar{L}/2\pi,\ \delta\tau_{L_s} = -k\bar{L}/2\pi,\ \delta\left(\frac{\dot{u}_{R_s}}{c}\right) = 0,\ \delta\left(\frac{\omega_{R_s}}{c}\right) = 0,\ \delta\left(\frac{\dot{v}_{L_s}}{c}\right) = 0,\ \delta\left(\frac{\sigma_{R_s}}{c}\right) = 0$$

and surface displacements as obtained from (30) and (32) agree exactly with those for Harkrider (1964, eqns. 134, 137).

Similarly a force system given by

$$\rho R = f(r)\cos\phi\exp(iwt) \qquad [H(z-h_1)-H(z-h_2)]$$
$$\rho\chi = -f(r)\sin\phi\exp(iwt) \qquad [H(z-h_1)-H(z-h_2)]$$

such that

$$\int_0^\infty f(r) r J_0(kr) = \bar{L}/2\pi\,\Delta h,\ \Delta h = h_2 - h_1$$

corresponds to a vertical line source of total strength $\bar{L}$, pointing in the direction $\phi = 0$.

In this case, we have, from (21) and (25),

$$\delta\left(\frac{\dot{u}_{R_s}}{c}\right) = \frac{\bar{L}}{2\pi\rho_s c^2 \Delta h}\left[\frac{1}{r_{\alpha_s}^2}(1-\cos\Delta P_s)+(1-\cos\Delta Q_s)\right]$$

$$\delta\left(\frac{\dot{w}_{R_s}}{c}\right) = \frac{i\bar{L}}{2\pi\rho_s c^2 \Delta h}\left[\frac{\sin\Delta P_s}{r_{\alpha_s}} - \frac{\sin\Delta Q_s}{r_{\beta_s}}\right]$$

$$\delta\sigma_{R_s} = \frac{\bar{L}}{2\pi\Delta h}\left[(\gamma_s-1)\frac{1}{r_{\alpha_s}^2}(1-\cos\Delta P_s)+\gamma_s(1-\cos\Delta Q_s)\right]$$

$$\delta\tau_{R_s} = -\frac{i\bar{L}}{2\pi\Delta h}\left[\gamma_s\frac{\sin\Delta P_s}{r_{\alpha_s}} - (\gamma_s-1)\frac{\sin\Delta Q_s}{r_{\beta_s}}\right]$$

$$\delta\left(\frac{\dot{v}_{L_s}}{c}\right) = \frac{i\bar{L}}{2\pi\Delta h}\frac{\sin\Delta Q_s}{r_{\beta_s}} \qquad \delta\tau_{L_s} = \frac{\bar{L}(1-\cos\Delta Q_s)}{2\pi\mu_s r_{\beta_s}\Delta h}$$

$$\Delta P_s = kr_{\alpha_s}\Delta h \qquad \Delta Q_s = kr_{\beta_s}\Delta h$$

and surface displacements as obtained from (30) and (32) agree exactly with those of Harkrider (1964, eqns. 141, 144, 145).

For a force system not in cylindrical system, one can directly use the incident wave given by eqns. (28) and (29) of the author's previous paper (Roy 1965) for finding displacements representing point and line sources.

Conclusion

Burridge and Knopoff (1964) had given an explicit expression for the body force that would have to be applied in the absence of fault to produce the same radiation as a given dislocation. Roy (1965) calculated the displacements in a semi-infinite elastic medium produced by body forces acting inside the medium. The present paper gives the surface displacements associated with body forces of all types acting inside an n-layered medium. Therefore by calculating the body-force equivalents of any dislocation in a fault, following Burridge and Knopoff (1964), one can find the displacements on the surface for all types of dislocation in an n-layered medium by utilizing the results of the present paper.

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