ELLIPTIC INCLUSION IN ORTHOTROPIC STRESSED MATRIX

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An explicit analytical solution has been obtained for the plane strain problem of an elliptic inclusion undergoing spontaneous dimensional changes in an infinite orthotropic medium. This infinite medium is under the action of a uniform tension in a direction inclined at an angle α to the axis of the ellipse. The solution has been obtained by using Castigliano's minimum energy principle, along with complex variable methods and superposition principle of linear elasticity. Many particular cases, such as when the inclusion material is either cubic or isotropic or rigid, different from, or same as, that of the matrix, may be derived from the solution. Further, the inclusion may be a circular or a straight line segment which may be used to obtain the solution of a crack problem.

INTRODUCTION

Inclusion problems were first studied by Mott and Nabarro (1940), and by Frenkel (1946) for some simple cases. Eshelby (1957, 1959) introduced the concept of point-force to solve inclusion problems and it was extended to ellipsoidal inclusion by Sneddon (1961). This method was used to solve some two-dimensional problems by Jaswon and Bhargava (1961). Bhargava (1961) introduced the principle of minimum strain energy for such problems of circular and spherical shape. This principle was successfully applied by Bhargava and Radhakrishna to solve more general problems of inclusion (Bhargava and Radhakrishna 1963*a*, *b*).

The papers cited dealt with only isotropic materials. Bhargava and Radhakrishna extended the application of minimum strain energy to inclusion problems of orthotropic materials (Bhargava and Radhakrishna 1964). The materials of the inclusion and matrix were assumed to be different by them (Bhargava 1961; Bhargava and Radhakrishna 1963*a*, *b*, 1964). Radhakrishna (1964) solved the problem of circular orthotropic inclusion in a stressed infinite matrix. In this paper the more general elliptic inclusion is solved.

Statement of the Problem

Inclusion, as defined in the papers cited, is a region capable of undergoing spontaneous dimensional changes within an infinite medium called *matrix*. The elastic properties of the materials have been assumed to be different and orthotropic in symmetry. The matrix is under the action of a uniform tension T at infinity, inclined at an angle α to the axis of the ellipse. The presence of the inclusion changes the elastic field due to tension in the matrix at infinity. Further locked-up accommodation elastic field is developed in both the media due to the presence of inclusion. To find the elastic field everywhere is the problem of this paper.

Brief Review

For a brief review of the theory and notations used, the reader is referred to Sec. 2 of our paper entitled *Elliptic Inclusion in Orthotropic Medium* (Bhargava and Radhakrishna 1964). Suffice it to say that the stresses and the displacements depend upon the two functions $\phi(z_1)$ and $\psi(z_2)$:

$$\left.\begin{aligned} p_{xx} &= 2Re[a_1^2\phi'(z_1)+a_2^2\psi'(z_2)] \\ p_{yy} &= 2Re[\phi'(z_1)+\psi'(z_2)] \\ p_{xy} &= -2Re[a_1\phi'(z_1)+a_2\psi'(z_2)] \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and

$$\left.\begin{aligned} u_x &= 2Re[m_1\phi(z_1)+m_2\psi(z_2)] \\ u_y &= 2Re[n_1\phi(z_1)+n_2\phi(z_2)] \end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (1A)$$

The quantities a_1, a_2, a_3, a_4 are the roots of the characteristic equation

$$\alpha_{11}a^4+(2\alpha_{12}+\alpha_{66})a^2+\alpha_{22} = 0.$$

The governing equation is

$$\alpha_{22}\frac{\partial^4 x}{\partial x^4}+(2\alpha_{12}+\alpha_{66})\frac{\partial^4 x}{\partial x^2\partial y^2}+\alpha_{11}\frac{\partial^4 x}{\partial y^4} = 0, \quad \ldots \quad \ldots \quad (2)$$

a_1, a_2, a_3, a_4 are complex (Green and Zerna 1954; Savin 1961),

$$\left.\begin{aligned} z_1 &= x+a_1y,\ z_2 = x+a_2y \\ e_{xx} &= \alpha_{11}p_{xx}+\alpha_{12}p_{yy} \\ e_{yy} &= \alpha_{22}p_{yy}+\alpha_{12}p_{xx} \\ e_{xy} &= \alpha_{66}p_{xy} \end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

For simply-connected finite regions, these functions may be determined comparatively easily. For infinite regions, however, the internal boundary and the region is mapped by a suitable mapping function

$$z = \omega(\zeta)$$

to the boundary and interior of the unit circle in the ζ-plane. In addition the z_1, z_2 planes obtained by an affine transformation

$$z_1 = x+a_1y,\ z_2 = x+a_2y$$

are also transformed to the ζ-plane. These transforming functions $z_1 = \omega_1(\zeta)$ and $z_2 = \omega_2(\zeta)$ are found by evaluating the boundary values of z_1 and z_2 in the ζ-plane by making use of $z = \omega(\zeta)$ and eqn. (1). We write

$$\left.\begin{aligned} \phi(z_1) &= \phi\{\omega_1(\zeta)\} \equiv F(\zeta) \\ \psi(z_2) &= \psi\{\omega_2(\zeta)\} = G(\zeta) \end{aligned}\right\} . \qquad (4)$$

The boundary conditions when stresses are given, are

$$\left.\begin{aligned} 2Re\{F(\sigma)+G(\sigma)\} &= f_1(\theta) \\ 2Re\{a_1F(\sigma)+a_2G(\sigma)\} &= f_2(\theta) \end{aligned}\right\}, \qquad (5)$$

and when the displacements are given, they are

$$\left.\begin{aligned} 2Re\{m_1F(\sigma)+m_2G(\sigma)\} &= u_x(\sigma) \\ 2Re\{n_1F(\sigma)+n_2G(\sigma)\} &= u_y(\sigma) \end{aligned}\right\}, \qquad (6)$$

where $\sigma(\equiv e^{i\theta})$ is the boundary value of ζ. Application of Schwarz formulae will enable the determination of $F(\zeta)$ and $G(\zeta)$.

The problem is solved by the following imaginary operations: First the inclusion is removed; then the matrix is stressed at infinity. An elastic field is created everywhere and in particular a displacement at the inner boundary. This has been named as the free state of the matrix. Analogous to matrix, the inclusion is allowed to attain its spontaneous deformation which, however, is not elastic. This is the free state of inclusion. Next a state of equilibrium boundary is guessed thus introducing some suitable parameters. Energy is then calculated in the matrix and the inclusion. It is then minimized. If the solution satisfies all the conditions, e.g. the field equations, the constitutive equations and the boundary conditions, then it is the solution.

The complex functions $F(\zeta)$ and $G(\zeta)$ for the case of an elliptic hole in an infinite region under the action of tension T are known to be (Savin 1961)

$$\left.\begin{aligned} F(\zeta) &= B^*\left\{\frac{a-h_1b}{2}\,\zeta+\frac{a+h_1b}{2}\cdot\frac{1}{\zeta}\right\}+F_0(\zeta), \\ G(\zeta) &= (B'^*+ic'^*)\left\{\frac{a-ih_2b}{2}\,\zeta+\frac{a+ih_2b}{2}\cdot\frac{1}{\zeta}\right\}+G_0(\zeta) \end{aligned}\right\}, \qquad (7)$$

where

$$\left.\begin{aligned} B^* &= \frac{\left\{(1+h_2^2)+(1-h_2^2)\cos 2\alpha\right\}T}{4(h_2^2-h_1^2)} \\ B'^* &= \frac{-\left\{(1+h_1^2)+(1-h_1^2)\cos 2\alpha\right\}T}{4(h_2^2-h_1^2)} \\ c'^* &= \frac{T\sin 2\alpha}{4h_2} \\ F_0(\zeta) &= \frac{-T\zeta}{4i(h_1-h_2)}\left\{ih(ih_2\sin 2\alpha+\cos 2\alpha+1)-a(ih_1-ih_1\cos 2\alpha+\sin 2\alpha)\right\} \\ G_0(\zeta) &= \frac{T\zeta}{4i(h_1-h_2)}\left\{ih(ih_1\sin 2\alpha+\cos 2\alpha+1)-a(ih_1-ih_1\cos 2\alpha+\sin 2\alpha)\right\} \end{aligned}\right\}. \qquad (7A)$$

The displacement components u^m_{xT}, u^m_{yT}, due to tension T, at the inner boundary of matrix are

$$\left.\begin{aligned} u^m_{xT} &= \delta^m_a x + \gamma^m_a y \\ u^m_{yT} &= \gamma^m_b x + \delta^m_b y \end{aligned}\right\}, \qquad (8)$$

where

$$\left.\begin{aligned} \delta^m_a &= \alpha_{11}\{b(h_1+h_2)\cos^2\alpha + a(\cos^2\alpha - h_1h_2\sin^2\alpha)\}\frac{T}{a} \\ \delta^m_b &= \frac{\alpha_{22}}{h_1h_2}\{b(h_1h_2\sin^2\alpha - \cos^2\alpha) + a(h_1+h_2)\sin^2\alpha\}\frac{T}{b} \\ \gamma^m_a &= \frac{(a+bh_2)(h_1+h_2)y\sin 2\alpha\,\alpha_{11}}{2}\frac{T}{b} \\ \gamma^m_b &= \frac{(a+bh_2)(h_1+h_2)x\sin 2\alpha\,\alpha_{22}}{2}\frac{T}{a} \end{aligned}\right| \qquad (8A)$$

We are writing superscripts m for quantities pertaining to matrix and i for inclusion. For tension T parallel to x-axis, $\alpha = 0$, and for that parallel to y-axis, $\alpha = \frac{\pi}{2}$.

Suppose that due to further action of inclusion, the boundary undergoes a displacement given by

$$\left.\begin{aligned} u^m_x &= (\epsilon_1 - \delta^m_a)\,x + (\gamma_1 - \gamma^m_a)y \\ u^m_y &= (\epsilon_2 - \delta^m_b)\,x + (\gamma_2 - \gamma^m_b)y \end{aligned}\right\}. \qquad (9)$$

Here we have introduced the unknown parameters ϵ_1, ϵ_2, γ_1, γ_2, which we shall determine subsequently. It may be easily seen that the equilibrium boundary between inclusion and matrix would be an ellipse obtained by replacing x by $x+\epsilon_1 x+\gamma_1 y$ and y by $y+\epsilon_2 y+\gamma_2 x$ in the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

We may replace x by $\frac{z+\bar{z}}{2}$ and y by $\frac{z-\bar{z}}{2i}$ in the above equations.

The function $z = \omega(\zeta) = \frac{a+b}{2}\frac{1}{\zeta}+\frac{a-b}{2}\zeta$ maps the region outside the ellipse into a circle $|\zeta| \leqslant 1$. The corresponding transforming functions from z_1 and z_2 planes are

$$z_1 = \omega_1(\zeta) = \frac{a+ia_1b}{2}\zeta+\frac{a-ia_1h}{2}\cdot\frac{1}{\zeta},$$

$$z_2 = \omega_2(\zeta) = \frac{a+ia_2b}{2}\zeta+\frac{a-ia_2b}{2}\cdot\frac{1}{\zeta}.$$

The boundary conditions, by eqns. (6) and (9), are

$$\left.\begin{aligned} 2Re\{m_1F(\sigma)+m_2G(\sigma)\} &= \frac{\epsilon-\delta_a^m}{2}\left(\sigma+\frac{1}{\sigma}\right)a+\frac{i(\gamma_1-\gamma_a^m)}{2}\left(\sigma-\frac{1}{\sigma}\right)b \\ 2Re\{n_1F(\sigma)+n_2G(\sigma)\} &= \frac{\gamma_2-\gamma_b^m}{2}\left(\sigma+\frac{1}{\sigma}\right)a+i(\epsilon_2-\gamma_b^m)\left(\sigma-\frac{1}{\sigma}\right)b \end{aligned}\right\}. \quad (10)$$

By using Schwarz formula we may solve for $F(\zeta)$ and $G(\zeta)$. We may, however, simplify these values if the coordinate axes are taken to be the lines of intersection of planes of elastic symmetry of the material. In that case $a_1 = ih_1$, $a_2 = ih_2$ (Savin 1961); and therefore

$$m_1 = \alpha=_{11}h_1^2+\alpha_{12},\ \ m_2 = -\alpha_{11}h_2^2+\alpha_{12}\,;\ \ n_1 = \frac{\alpha_{22}-\alpha_{12}h_1^2}{ih_1},\ \ n_2 = \frac{\alpha_{22}-\alpha_{12}h_2^2}{ih_2}.$$

Also from characteristic equation $\alpha_{11}h_1^2h_2^2 = \alpha_{22}$. Using these, the values of $F(\zeta)$ and $G(\zeta)$ are found to be

$$F(\zeta) = \frac{c_1-ic_1'}{2(h_1-h_2)}\frac{\zeta}{d} \quad \text{and} \quad G(\zeta) = \frac{c_2-ic_2'}{2(h_1-h_2)}\frac{\zeta}{d}, \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where

$$\left.\begin{aligned} c_1 &= -a(\alpha_{22}-\alpha_{12}h_2^2)h_1(\epsilon_1-\delta_a^m)-bh_1h_2(\alpha_{12}-\alpha_{11}h_2^2)(\epsilon_2-\delta_b^m) \\ c_1' &= ah_2(\alpha_{22}-\alpha_{12}h_1^2)(\epsilon_1-\delta_a^m)+bh_1h_2(\alpha_{12}-\alpha_{11}h_1^2)(\epsilon_2-\delta_b^m) \\ c_2 &= ah_2(\alpha_{22}-\alpha_{12}h_1^2)(\epsilon_1-\delta_a^m)+bh_1h_2(\alpha_{12}-\alpha_{11}h_1^2)(\epsilon_2-\delta_b^m) \\ c_2' &= bh_2(\alpha_{22}-\alpha_{12}h_1^2)(\gamma_1-\gamma_a^m)-ah_1h_2(\alpha_{12}-\alpha_{11}h_1^2)(\gamma_2-\gamma_b^m) \\ \text{and} \quad d &= \alpha_{11}\alpha_{22}(h_1^2+h_1h_2+h_2^2)-\alpha_{12}^2h_1h_2-2\alpha_{12}\alpha_{22} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (11A)$$

The stresses in the matrix then can be easily found by substituting these values of $F(\zeta)$, $G(\zeta)$ in eqn. (1).

To evaluate strain energy in the matrix, it is easier first to find the boundary values of the stress components p_{xx}^{mb}, p_{yy}^{mb}, p_{xy}^{mb} and then to use Claperon's theorem (Sokolnikoff 1956) which states that the strain energy

$$w_M = \tfrac{1}{2}\oint (P_{nx}\,u_x+P_{ny}\,u_y)\,ds.$$

Here

$$\text{and} \quad \left.\begin{aligned} P_{nx} &= P_{xx}^{mb}\cos(x,n)+P_{xy}^{mb}\cos(y,n) \\ P_{ny} &= P_{xy}^{mb}\cos(x,n)+P_{yy}^{mb}\cos(y,n) \end{aligned}\right\}. \quad \ldots \quad \ldots \quad (12)$$

Using the eqns. (1), (9), (11), (12), we get

$$\begin{aligned} w_M = \frac{\pi}{2d}\Big\{ & a^2(h_1+h_2)\left[\alpha_{22}(\epsilon_1-\delta_a^m)^2+\alpha_{11}h_1h_2(\gamma_2-\gamma_b^m)^2\right] \\ & +2ab(\alpha_{22}+\alpha_{12}h_1h_2)\left[(\epsilon_1-\delta_a^m)(\epsilon_2-\delta_b^m)-(\gamma_1-\gamma_a^m)(\gamma_2-\gamma_b^m)\right] \\ & +b^2(h_1+h_2)\left[\alpha_{11}h_1h_2(\epsilon_2-\delta_b^m)^2+\alpha_{22}(\gamma_1-\gamma_a^m)^2\right]\Big\}. \quad \ldots \quad (12A) \end{aligned}$$

This gives the energy in the matrix.

We find below the energy in the inclusion. The inclusion undergoes a spontaneous deformation, given by the displacements $(\delta^i_a x+\gamma^i_a y,\ \gamma^i_b x+\delta^i_b y)$. This is the free state of the inclusion. In equilibrium position, its displacement from this free state is therefore

$$u^i_x = (\epsilon_1-\delta^i_a)x+(\gamma_1-\gamma^i_a)y,$$
$$u^i_y = (\gamma_2-\gamma^i_b)x+(\epsilon_2-\delta^i_b)y$$

whence the strains are

$$e^i_{xx} = (\epsilon_1-\delta^i_a),\quad e^i_{yy} = (\epsilon_2-\delta^i_b),\quad e^i_{xy} = (\gamma_1+\gamma_2-\gamma^i_a-\gamma^i_b).$$

The stresses, therefore, are

$$p^i_{xx} = \frac{(\epsilon_1-\delta^i_a)\beta_{22}-(\epsilon_2-\delta^i_b)\beta_{12}}{\beta_{11}\beta_{22}-\beta^2_{12}},$$
$$p^i_{yy} = \frac{(\epsilon_2-\delta^i_b)\beta_{11}-(\epsilon_1-\delta^i_b)\beta_{12}}{\beta_{11}\beta_{22}-\beta^2_{12}},$$
$$p^i_{xy} = \frac{\gamma_1+\gamma_2-\gamma^i_a-\gamma^i_b}{\beta_{66}},$$

where β_{ij} are elastic constants of the inclusion material with the same meanings as α_{ij} in eqn. (3). The strain energy is

$$w_I = \frac{\pi ab}{2}\left[\frac{(\epsilon_1-\delta^i_a)^2\beta_{22}+(\epsilon_2-\delta^i_b)^2\beta_{11}-2(\epsilon_1+\delta^i_a)(\epsilon_2-\delta^i_b)\beta_{12}}{\beta_{11}\beta_{22}-\beta^2_{12}}+\frac{(\gamma_1+\gamma_2-\gamma^i_a-\gamma^i_b)^2}{\beta_{66}}\right].$$

The sum total of energy in the inclusion and in the matrix is $w_M+w_I \equiv w$. Minimizing w with respect to the parameters ϵ_1, ϵ_2, γ_1, γ_2, i.e. putting $\frac{\partial\omega}{\partial\epsilon_1} = \frac{\partial\omega}{\partial\epsilon_2} = \frac{\partial\omega}{\partial\gamma_1} = \frac{\partial\omega}{\partial\gamma_2} = 0$, we get

$$\epsilon_1 = \frac{\begin{aligned}&\{ab[\delta^i_a d+h_1h_2(\beta_{11}\beta_{22}-\beta^2_{12})\delta^m_a]+ab\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)(\delta^i_a+\delta^m_a)\\&+\alpha_{11}h_1h_2(h_1+h_2)(b^2\beta_{22}\delta^i_a+a^2h_1h_2\beta_{11}\delta^m_a)\\&-b[a\beta_{11}(\alpha_{22}+\alpha_{12}h_1h_2)+b\beta_{12}\alpha_{11}h_1h_2(h_1+h_2)](\delta^i_b-\delta^m_b)\}\end{aligned}}{\begin{aligned}&\{ab[d+(\beta_{11}\beta_{22}-\beta^2_{12})h_1h_2]+2ab\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)\\&+a^2\beta_{11}\alpha_{22}(h_1+h_2)+b^2\beta_{22}\alpha_{11}h_1h_2(h_1+h_2)\}\end{aligned}},$$

$$\epsilon_2 = \frac{\begin{aligned}&ab[d\delta^i_b+h_1h_2(\beta_{11}\beta_{22}-\beta^2_{12})\delta^m_b]+ah\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)(\delta^i_b+\delta^m_b)\\&+\alpha_{11}h_1h_2(h_1+h_2)(a^2h_1h_2\beta_{11}\delta^i_b+b^2\beta_{22}\delta^m_b)-a[b\beta_{22}(\alpha_{22}+\alpha_{12}h_1h_2)\\&+a\beta_{12}\alpha_{22}(h_1+h_2)](\delta^i_a-\delta^m_b)\end{aligned}}{\begin{aligned}&ab[d+(\beta_{11}\beta_{22}-\beta^2_{12})h_1h_2]+2ab\beta_{12}(\alpha_{22}+\alpha_{12}h_1h_2)+a^2\beta_{11}\alpha_{22}(h_1+h_2)\\&+b^2\beta_{22}\alpha_{11}h_1h_2(h_1+h_2)\end{aligned}},$$

$$\gamma_1 = \frac{a[a\alpha_{11}(h_1+h_2)+b(\alpha_{11}h_1h_2+\alpha_{12})](\gamma_a^i+\gamma_b^i-\gamma_b^m)+\gamma_a^m b[b\alpha_{11}h_1h_2(h_1+h_2) + a(\alpha_{11}h_1h_2+\alpha_{12}+\beta_{66})]}{(a^2+b^2h_2h_1)\alpha_{11}(h_1+h_2)+ab(2\alpha_{11}h_1h_2+2\alpha_{12}+\beta_{66})},$$

$$\gamma_2 = \frac{b[b\alpha_{11}h_1h_2(h_1+h_2)+a(\alpha_{11}h_1h_2+\alpha_{12})](\gamma_a^i+\gamma_b^i-\gamma_a^m)+a\gamma_b^m[a\alpha_{11}(h_1+h_2) + b(\alpha_{12}+\alpha_{11}h_1h_2+\beta_{66})]}{(a^2+b^2h_2h_1)\alpha_{11}(h_1+h_2)+ah(2\alpha_{11}h_1h_2+2\alpha_{12}+\beta_{66})}.$$

The solution has been derived from eqn. (2) which itself is derived from the equations of equilibrium. Thus the solution satisfies the equations of equilibrium. They also satisfy Hooke's Law. It may be verified that the normal stress p_{nn}^b and tangential stress p_{ns}^b at the boundary are continuous. They are in fact

$$p_{nn}^b = \frac{-[(b+ah_1)c_1-(b+ah_2)c_2]+[(b-ah_1)c_1-(b-ah_2)c_2]\cos 2\beta}{2ab(h_1-h_2)d}$$
$$-\frac{ab(\gamma_a^i+\gamma_b^i-\gamma_a^m-\gamma_b^m)\sin 2\beta}{\alpha_{11}(a^2+b^2h_1h_2)+ah(2\alpha_{11}h_1h_2+2\alpha_{12}+\beta_{66})},$$

$$p_{ns}^b = \frac{-[(b-ah_1(c_1-(b-ah_2)c_2]\sin 2\beta}{2ab(h_1-h_2)d}$$
$$-\frac{ah(\gamma_a^i+\gamma_b^i-\gamma_a^m-\gamma_b^m)\cos 2\beta}{\alpha_{11}(a^2+b^2h_1h_2)(h_1+h_2)+ah(2\alpha_{11}h_1h_2+2\alpha_{12}+\beta_{66})},$$

where c_1, c_2 are given by eqn. (11A); and β is the angle between the normal and the x-axis.

As regards the hoop stress in the matrix at the equilibrium interface, it may be seen that it already had a hoop stress due to a tension T at infinity. Thus the hoop stress in the matrix at the equilibrium interface is

$$p_{ss}^b = \frac{-[b+ah_1)c_1-(b+ah_2)c_2]-[(b-ah_1)c_1-(b-ah_2)c_2]\cos 2\beta}{2ab(h_1-h_2)d}$$
$$+\frac{ab(\gamma_a^i+\gamma_b^i-\gamma_a^m-\gamma_b^m)\sin 2\beta}{(a^2+b^2h_1h_2)\alpha_{11}(h_1+h_2)+ab(2\alpha_{11}h_1h_2+2\alpha_{12}+\beta_{66})}$$
$$+\frac{2}{ab(h_1-h_2)d}\left\{\frac{h_1(a+h_1b)c_1}{(1+h_1^2)-(1-h_1^2)\cos 2\beta}-\frac{h_2(a+h_2b)c_2}{(1+h_2^2)-(1-h_2^2)\cos 2\beta}\right\}$$
$$+\frac{2\sin 2\beta}{ab(h_1-h_2)d}\left\{\frac{h_1(b-ah_1)c_1'}{(1+h_1^2)-(1-h_1^2)\cos 2\beta}-\frac{h_2(b-ah_2)c_2'}{(1+h_2^2)-(1-h_2^2)\cos 2\beta}\right\},$$

while in the inclusion, it is

$$p_{ss}^b = \frac{-[(b+ah_1)c_1-(b+ah_2)c_2]-[(b-ah_1)c_1-(b-ah_2)c_2]\cos 2\beta}{2ab(h_1-h_2)d}$$
$$+\frac{ab(\gamma_a^i+\gamma_b^i-\gamma_a^m-\gamma_b^m)\sin 2\beta}{(a^2+b^2h_1h_2)\alpha_{11}(h_1+h_2)+ab(2\alpha_{11}h_1h_2+2\alpha_{12}+\beta_{66})}.$$

It may be remarked that to find the stress field in the matrix, one has to superpose the stress field obtained from the tension at infinity over the one obtained by the complex functions $F(\zeta)$, $G(\zeta)$.

Quite a few important cases, and some of them physically important, can be derived from the above results, e.g. when the materials are the same, when one is cubic and another is isotropic, etc. Further, all the results derived by Jaswon and Bhargava (1961), Bhargava and Radhakrishna (1963*b*, 1964) and some of the results of Eshelby (1957, 1959) can be deduced as particular cases.

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