GENERALIZED SPINORS IN A NON-ANALYTIC COMPLEX n-SPACE

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This paper gives an extension of spinor analysis to n-dimensional non-analytic complex space. The topics for discussion are mainly those contained in another paper by the author, 'Generalized theory of spinors', where the treatment is restricted to two-dimensional non-analytic complex space.

Introduction

In a recent paper (Ghosh 1962) a generalized theory of spinors has been developed in a non-analytic two-dimensional complex space S_2 having an auxiliary real space R_4 associated with it. It has been shown that there exists a (1–1) correspondence between similar entities in S_2 and R_4, the relevant connection formulae being preserved if either S_2 or R_4 is specialized (Ghosh 1964). In the present paper the earlier treatment is modified so as to be applicable to an n-dimensional non-analytic complex space S_n. After discussing the transformation properties of generalized spinors in S_n under a general coordinate frame one can introduce in S_n a metrical spinor, metrical affinities with the associated covariant derivative of spinors leading up to the definition of curvature spinor in much the same way as in one of my previous papers (Ghosh 1950) and can build up a generalized theory of spinors in S_n. The connecting relations between similar entities in S_n and in the associated real space R_{2n} are expressible by means of a set of two-index spin-tensors with characteristic structure. As a special case in this general theory we have considered a $2n$-dimensional Lorentz transformation in R_{2n} with its spinor representation in S_n having (1–1) correspondence.

1. Non-analytic Coordinate Transformation and Definition of Generalized Spinors

We consider an n-dimensional complex space S_n in which a point is located by n coordinates X^α:

$$X^\alpha = x^{\alpha-1}+ix^{2n-\alpha}(\alpha = 1, 2, \dots n), \qquad (1.1)$$

where the x^k's are $2n$ real quantities. Associated with S_n we have the real auxiliary space R_{2n} with coordinates x^k. Let ξ_α denote differential operators corresponding to (1.1) so that

$$\xi_\alpha = \tfrac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\,\frac{\partial}{\partial x^{2n-\alpha}}\right)(\alpha = 1, 2, \dots n). \quad \dots \quad \dots \quad (1.2)$$

To denote the complex conjugates of (1.1) and (1.2), we use the dotted notations $X^{\dot\alpha}$ and $\xi_{\dot\alpha}$.

Let us next define the general non-analytic complex coordinate transformation from the system X^α to another system X'^α. We set up a general transformation scheme of real coordinates x^k in R_{2n} given by the system of reversible transformation equations

$$x'^k = \phi^k(x^0, x^1, \dots x^{2n-1})(k = 0, 1, \dots 2n-1) \quad \dots \quad \dots \quad (1.3)$$

and regard them as effecting a non-analytic complex transformation in S_n from the system X^α to the new system X'^α. The differentials dX^α are then transformed (Ghosh 1950) as

$$\left.\begin{aligned} dX'^\mu &= dX^\alpha(\xi_\alpha X'^\mu) \\ dX'^{\dot\mu} &= dX^\alpha(\xi_\alpha X'^{\dot\mu}) \end{aligned}\right\} (\mu = 1, 2, \dots n), \quad \dots \quad \dots \quad (1.4)$$

where the dummy index α runs over the values $1, 2 \dots n, \dot 1, \dot 2, \dots \dot n$.

We now define a contravariant spinor of rank one with $2n$ components $(\psi^\alpha, \psi^{\dot\alpha})$ and a covariant spinor of rank one with $2n$ components $(\psi_\alpha, \psi_{\dot\alpha})$ by the respective transformation formulae

$$\left.\begin{aligned} \psi'^\mu &= \psi^\alpha(\xi_\alpha X'^\mu) \\ \psi'_\mu &= \psi_\alpha(\xi'_\mu X^\alpha) \end{aligned}\right\} (\alpha = 1, 2, \dots n;\ \dot 1, \dot 2, \dots \dot n). \quad \begin{aligned} \dots \quad \dots \quad (1.5) \\ \dots \quad \dots \quad (1.6) \end{aligned}$$

It is easily verified that coefficients of transformation in (1.5) (1.6) satisfy the relations

$$\left.\begin{aligned} (\xi_\alpha X'^\mu)(\xi'_\mu X^\beta) &= \delta^\beta_\alpha, \\ (\xi_\alpha X'^\mu)(\xi'_\mu X^{\dot\beta}) &= 0. \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (1.7)$$

Transformation formulae with regard to spinors of higher rank can be written down following the usual rule. The process of contraction holds good in accordance with the extended summation convention.

Infinitesimal transformation formulae in S_n and R_{2n} are all obtained from the system of infinitesimal transformation equations

$$x'^k = x^k + \epsilon f^k(x^0, x^1, \dots x^{2n-1}) \ (\epsilon, \text{ infinitely small}) \quad \dots \quad \dots \quad (1.8)$$

following up the procedure in the paper (Ghosh 1962) with due regard to the extended summation convention. Thus in the formulae of transformation coefficients

$$\left.\begin{aligned} \xi_\alpha X'^\mu &= \delta^\mu_\alpha + \epsilon\xi_\alpha F^\mu, \quad \xi_\alpha X'^{\dot\mu} = \epsilon\xi_\alpha F^{\dot\mu} \\ \xi'_\alpha X^\mu &= \delta^\mu_\alpha - \epsilon\xi_\alpha F^\mu, \quad \xi'_\alpha X^{\dot\mu} = -\epsilon\xi_\alpha F^{\dot\mu} \end{aligned}\right\} (\alpha, \mu = 1, 2, \dots n) \quad \dots \quad (1.9)$$

we take now $F^{\mu} = f^{\mu-1} + if^{2n-\mu}$, $F^{\dot{\mu}}$ denoting its complex conjugate. Infinitesimal transformation formulae of spinors and tensors of different ranks are all similar in form to those already given (Ghosh 1962).

For purposes of developing an analytical theory of spinors in S_n we note the following two modifications required in the transformation scheme (1.3) in two ways:

First postulate all transformation coefficients like $\xi_{\alpha}X^{'\dot{\mu}}$ to be zero, the set of transformation equations (1.3) then takes the complex form

$$X^{'\mu} = \phi^{\mu}(X^1, X^2, \dots X^n), \qquad \dots \quad (1.10)$$

next postulate all transformation coefficients like $\xi_{\alpha}X^{'\mu}$ to be zero, the set of transformation equations (1.3) then takes the complex form

$$X^{'\mu} = \phi^{\mu}(X^{\dot{1}}, X^{\dot{2}}, \dots X^{\dot{n}}). \qquad \dots \quad (1.11)$$

Further development must proceed from these two sets of transformation equations of which the first set possesses group property while the second does not.

2. Connection between Spinors in S_n and Tensors in R_{2n}

Let $A_0, A_1, \dots A_{2n-1}$ be the components of a vector in R_{2n} obeying the transformation law

$$A'_k = A_l \frac{\partial x^l}{\partial X'^k} (k, l = 0, 1, \dots 2n-1) \qquad \dots \quad (2.1)$$

and $\psi_1, \psi_2, \dots \psi_n$; $\psi_{\dot{1}}, \psi_{\dot{2}}, \dots \psi_{\dot{n}}$ be the components of a spinor of rank one in S_n obeying the transformation law

$$\psi'_{\mu} = \psi_{\alpha}(\xi'_{\mu}X^{\alpha})(\alpha, \mu = 1, 2 \dots n; \dot{1}, \dot{2}, \dots \dot{n}). \qquad \dots \quad (2.2)$$

Comparing the above for equivalence with (2.1) we get

$$\psi_{\alpha} = \tfrac{1}{2}(A_{\alpha-1} - iA_{2n-\alpha})(\alpha = 1, 2, \dots, n). \qquad \dots \quad (2.3)$$

Introducing spin-tensors T^k_{μ} defined by the $4n$ non-vanishing components

$$T^{\alpha-1}_{\alpha} = \tfrac{1}{2}, T^{2n-\alpha}_{\alpha} = -\frac{i}{2}, T^{\alpha-1}_{\dot{\alpha}} = \tfrac{1}{2}, T^{2n-\alpha}_{\dot{\alpha}} = \frac{i}{2} (\alpha = 1, 2, \dots n) \qquad \dots \quad (2.4)$$

we can express (2.3) in the form

$$\psi_{\mu} = T^k_{\mu} A_k. \qquad \dots \quad (2.5)$$

To invert the relation (2.5) let us introduce spin-tensors S^{μ}_k defined by the non-vanishing components

$$S^{\alpha}_{\alpha-1} = 1, S^{\alpha}_{2n-\alpha} = i, S^{\dot{\alpha}}_{\alpha-1} = 1, S^{\dot{\alpha}}_{2n-\alpha} = -i(\alpha = 1, 2, \dots n) \qquad \dots \quad (2.6)$$

and obtain

$$A_k = S^{\mu}_{k}\psi_{\mu}. \qquad \ldots \quad (2.7)$$

It is easily verified that the spin-tensors S and T satisfy the relations

$$S^{\mu}_{k}T^{k}_{\nu} = \delta^{\mu}_{\nu},\ S^{\mu}_{l}T^{k}_{\mu} = \delta^{k}_{l},\ S^{\mu}_{k}T^{k}_{\dot{\nu}} = 0. \qquad \ldots \quad (2.8)$$

For contravariant spinors and tensors of rank one we have the connecting relations

$$\psi^{\mu} = S^{\mu}_{k}A^{k},\ A^{k} = T^{k}_{\mu}\psi^{\mu}. \qquad \ldots \quad (2.9)$$

Spinors and tensors of higher rank are similarly correlated (Ghosh 1962). We remark further that for any change of coordinates of the type (1.3) both in spin-space S_n and in auxiliary space R_{2n}, all such relations will hold good, the T's and S's remaining unchanged. We give below the formulae connecting the coefficients of transformations under the general coordinate system in S_n and R_{2n}:

$$\left.\begin{aligned} \xi'_{\mu}X^{\alpha} &= T^{k}_{\mu}S^{\alpha}_{l}\frac{\partial x^{l}}{\partial x'^{k}},\ \frac{\partial x^{l}}{\partial x'^{k}} = T^{l}_{\alpha}S^{\mu}_{k}\xi'_{\mu}X^{\alpha} \\ \xi_{\alpha}X'^{\mu} &= T^{k}_{\alpha}S^{\mu}_{l}\frac{\partial x'^{l}}{\partial x^{k}},\ \frac{\partial x'^{l}}{\partial x^{k}} = T^{l}_{\mu}S^{\alpha}_{k}\xi_{\alpha}X'^{\mu} \end{aligned}\right\} \qquad \ldots \quad (2.10)$$

The equations of transformation (1.3) being the same in both S_n and R_{2n}, in the present theory entities of a new kind occur which can behave either as spinors in S_n or as tensors in R_{2n} with characteristic connecting relations.

3. Metric Spinors and Metrical Affinities

Let us now introduce in S_n a Hermitian covariant metric spinor $g_{\mu\nu}$ with components

$$\left.\begin{aligned} g_{\alpha\beta} &= g_{\beta\alpha},\ g_{\dot{\alpha}\beta} = g_{\beta\dot{\alpha}} = \text{conj}\ g_{\dot{\beta}\alpha} \\ g_{\dot{\alpha}\dot{\beta}} &= \text{conj}\ g_{\alpha\beta},\ g_{\alpha\dot{\beta}} = \text{conj}\ g_{\dot{\alpha}\beta} \end{aligned}\right\} \qquad \ldots \quad (3.1)$$

Making use of the Hermitian determinant

$$g = \begin{vmatrix} g_{\dot{1}1} & g_{\dot{1}2}\cdots g_{\dot{1}n} & g_{\dot{1}\dot{1}}\cdots g_{\dot{1}\dot{n}} \\ \cdot & \cdot\ \cdots\ \cdot & \cdot\ \cdots\ \cdot \\ \cdot & \cdot\ \cdots\ \cdot & \cdot\ \cdots\ \cdot \\ \cdot & \cdot\ \cdots\ \cdot & \cdot\ \cdots\ \cdot \\ g_{\dot{n}1} & g_{\dot{n}2}\cdots g_{\dot{n}n} & g_{\dot{n}\dot{1}}\cdots g_{\dot{n}\dot{n}} \\ g_{11} & g_{12}\cdots g_{1n} & g_{1\dot{1}}\cdots g_{1\dot{n}} \\ \cdot & \cdot\ \cdots\ \cdot & \cdot\ \cdots\ \cdot \\ \cdot & \cdot\ \cdots\ \cdot & \cdot\ \cdots\ \cdot \\ \cdot & \cdot\ \cdots\ \cdot & \cdot\ \cdots\ \cdot \\ g_{n1} & g_{n2}\cdots g_{nn} & g_{n\dot{1}}\cdots g_{n\dot{n}} \end{vmatrix} \qquad \ldots \quad (3.2)$$

one can define the contravariant metric spinor $g^{\mu\nu}$ satisfying the conditions

$$\left.\begin{aligned} g_{\dot\mu\sigma}g^{\dot\nu\sigma} &= 1, \text{ if } \dot\mu = \dot\nu \\ &= 0, \text{ if } \dot\mu \neq \dot\nu \\ g_{\dot\mu\sigma}g^{\nu\sigma} &= 0 \end{aligned}\right\} . \quad (3.3)$$

Raising and lowering of indices in a spinor can be performed by the metric spinors according to the following scheme:

$$\left.\begin{aligned} \psi^{\mu} &= g^{\mu\nu}\psi_{\nu} \\ \psi_{\mu} &= g_{\mu\nu}\psi^{\nu} \end{aligned}\right\} . \quad (3.4)$$

In the auxiliary real space R_{2n} there will be a symmetric tensor t_{kl} which corresponds to $g_{\mu\nu}$ in S_n. The components of t_{kl} are given by (2.7) as

$$T_{kl} = g_{\mu\nu}S^{\mu}_{k}S^{\nu}_{l}. \quad (3.5)$$

Evaluating the above we obtain for values of α, $\beta = 1, 2, \ldots, n$

$$\left.\begin{aligned} t_{\alpha-1,\, \beta-1} &= g_{\alpha\beta}+g_{\dot\alpha\dot\beta}+g_{\dot\alpha\beta}+g_{\alpha\dot\beta} \\ t_{\alpha-1,\, 2n-\beta} &= ig_{\alpha\beta}-ig_{\dot\alpha\dot\beta}+ig_{\dot\alpha\beta}-ig_{\alpha\dot\beta} \\ t_{2n-\alpha,\, \beta-1} &= ig_{\alpha\beta}-ig_{\dot\alpha\dot\beta}-ig_{\dot\alpha\beta}+ig_{\alpha\dot\beta} \\ t_{2n-\alpha,\, 2n-\beta} &= -g_{\alpha\beta}-g_{\dot\alpha\dot\beta}+g_{\dot\alpha\beta}+g_{\alpha\dot\beta} \end{aligned}\right\} . \quad (3.6)$$

Metrical affinities $\Gamma^{\sigma}_{\mu\nu}$ in S_n satisfying Hermitian symmetry with respect to the indices μ, ν will be defined by

$$\Gamma^{\sigma}_{\mu\nu} = \tfrac{1}{2}g^{\sigma\lambda}(\xi_{\mu}g_{\lambda\nu}+\xi_{\nu}g_{\mu\lambda}-\xi_{\lambda}g_{\mu\nu}), \quad (3.7)$$

μ, ν, σ ranging over the values $1, 2, \ldots n$; $\dot 1, \dot 2, \ldots \dot n$. There will be four distinct types of such Γ's. It may be shown (Ghosh 1950) that the law of transformation satisfied by $\Gamma^{\sigma}_{\mu\nu}$ is given by

$$\Gamma'^{\sigma}_{\mu\nu} = (\xi'_{\mu}\xi'_{\nu}X^{\rho})(\xi_{\rho}X'^{\sigma})+(\xi'_{\mu}X^{\alpha})(\xi'_{\nu}X^{\beta})(\xi_{\rho}X'^{\sigma})\Gamma^{\rho}_{\alpha\beta}. \quad (3.8)$$

The analogue of $\Gamma^{\sigma}_{\mu\nu}$ in R_{2n} is

$$\Gamma^{l}_{mn} = \Gamma^{\sigma}_{\mu\nu}S^{\mu}_{m}S^{\nu}_{n}T^{l}_{\sigma}. \quad (3.9)$$

The formulae for covariant differentiation of spinors yielding the definition of curvature spinor can all be written down with an obvious similarity with their analogues in R_{2n}. The connection formulae discussed in Sec. 2 will hold good with respect to similar entities in S_n and R_{2n}, their forms being preserved for any change of coordinates.

4. Representation of $2n$-dimensional Lorentz Transformation in S_n

In the $2n$-dimensional real space R_{2n} we consider a set of linear transformation of coordinates defined by

$$x'^k = L^k_l x^l (k, l = 0, 1, \ldots, 2n-1) \qquad \ldots \quad (4.1)$$

which leaves invariant an elementary tensor ϵ_{kl} with non-vanishing components

$$\left.\begin{aligned} \epsilon_{00} = \epsilon_{11} = \ldots = \epsilon_{t-1,\,t-1} = -1 \\ \epsilon_{tt} = \epsilon_{t+1,\,t+1} = \ldots = \epsilon_{2n-1,\,2n-1} = 1 \end{aligned}\right\} \quad t < n,$$

so that the relation

$$\epsilon_{mp} = \epsilon_{kl} L^k_m L^l_p \qquad \ldots \quad (4.2)$$

is preserved among the coefficients L^k_l. This Lorentz transformation in R_{2n} will induce a corresponding spinor transformation in the auxiliary spin-space S_n. Referring to (2.10) we obtain the contravariant spinor coefficients expressed as

$$\left.\begin{aligned} \xi_\alpha X'^\mu = \tfrac{1}{2}\left[\left(L^{\mu-1}_{\alpha-1} + L^{2n-\mu}_{2n-\alpha}\right) + i\left(L^{2n-\mu}_{\alpha-1} - L^{\mu-1}_{2n-\alpha}\right)\right] \\ \xi_\alpha X'^{\dot\mu} = \tfrac{1}{2}\left[\left(L^{\mu-1}_{\alpha-1} - L^{2n-\mu}_{2n-\alpha}\right) - i\left(L^{2n-\mu}_{\alpha-1} + L^{\mu-1}_{2n-\alpha}\right)\right] \end{aligned}\right]. \qquad \ldots \quad (4.3)$$

Inverting (4.1) we write

$$x^k = \tilde{L}^k_l x'^l, \qquad \ldots \quad (4.4)$$

where

$$L^k_l \tilde{L}^l_m = \delta^k_m.$$

The covariant spinor coefficients of transformation are then given by

$$\xi'_\mu X^\alpha = T^k_\mu S^\alpha_l \tilde{L}^l_k. \qquad \ldots \quad (4.5)$$

Corresponding to the fundamental tensor ϵ_{kl} in R_{2n} we have the invariant spinor $\rho_{\mu\nu}$ in S_n connected by the equation

$$\rho_{\mu\nu} = T^k_\mu T^l_\nu \epsilon_{kl}. \qquad \ldots \quad (4.6)$$

The non-vanishing components of $\rho_{\mu\nu}$ are then

$$\left.\begin{aligned} \rho_{11} = \rho_{22} = \ldots = \rho_{tt} = -\tfrac{1}{2} \\ \rho_{t+1(t+1)^\cdot} = \ldots = \rho_{n\dot n} = \tfrac{1}{2} \\ \rho_{\dot\alpha\dot\alpha} = \text{conj}\, \rho_{\alpha\alpha}, \quad \rho_{\dot\alpha\alpha} = \text{conj}\, \rho_{\alpha\dot\alpha} \end{aligned}\right\}. \qquad \ldots \quad (4.7)$$

The invariance of $\rho_{\mu\nu}$ under the spinor transformation yields the conditions

$$\rho_{\mu\nu} = \rho_{\alpha\beta}(\xi'_\mu X^\alpha)(\xi'_\nu X^\beta) \qquad \ldots \quad (4.8)$$

to be satisfied by the spinor coefficients of transformation. The associated contravariant spinor $\rho^{\mu\nu}$ is defined by the non-vanishing components

$$\left.\begin{array}{l}\rho^{11} = \rho^{22} = \ldots = \rho^{tt} = -2 \\ \rho^{t+1\, t+1} = \ldots = \rho^{n\dot{n}} = 2\end{array}\right\} \qquad \ldots \quad (4.9)$$

with their complex conjugates, so that

$$\rho^{\mu\nu}\rho_{\mu\lambda} = \delta^{\nu}_{\lambda}, \quad \rho^{\mu\dot{\nu}}\rho_{\mu\lambda} = 0. \qquad \ldots \quad (4.10)$$

The spinors $\rho^{\mu\nu}$, $\rho_{\mu\nu}$ may be used as metric spinors in S_n in raising and lowering of spinor indices according to the following scheme

$$\psi^{\mu} = \psi_{\nu}\rho^{\nu\mu}, \quad \psi_{\mu} = \rho_{\mu\nu}\psi^{\nu}. \qquad \ldots \quad (4.11)$$

We shall call the non-analytic spinor transformation in S_n defined as above a *Minkowskian spinor transformation.* The spinor representation as obtained here is in (1–1) correspondence with the $2n$-dimensional Lorentz transformation R_{2n}.

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