TORQUE DUE TO FRICTION ON ROUGH ROTATING DISCS

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The logarithmic law of wall for rough pipes has been assumed to hold in the boundary layer on both the rotating disc and the stationary walls of the symmetrical casing in which it rotates, but with numerical coefficients different from those for the case of the pipe. Momentum integral equations in radial and tangential directions are formulated with these assumed velocity distributions and the torque coefficient for the rotating disc is calculated by equating the torque on the disc to the torque on the casing. The differential equations which result on substituting the assumed velocity distributions in the momentum integral equations are solved by an approximate method. The values of the numerical coefficients A and B in the logarithmic law of velocity distribution necessary to give good agreement with experimental results are found to be respectively 3·00 and 2·60, as compared with 2·50 and 8·48 for a rough pipe, and 1·97 and 0·03 obtained by Goldstein for a smooth free disc.

The estimation of the torque due to skin friction on rotating discs is important because of its application in the calculation of losses in hydraulic machinery. Th. von Kármán (1921) was perhaps the first to derive a theoretical expression for the torque on a smooth disc rotating in an infinite fluid when the flow is turbulent. Goldstein solved the same problem by a different method. An extensive theoretical treatment of the problem of a smooth disc rotating in a closed cylindrical enclosure was published by Daily and Nece (1958) along with experimental verification.

This paper indicates a procedure to estimate the frictional torque on rough discs rotating in a closed cylindrical chamber symmetrical about the axis of the disc. The procedure is intended to apply to the case in which (i) both the disc and walls of the chamber are rough, (ii) the disc is rough and the chamber walls are smooth, (iii) the disc is smooth and the chamber walls are rough. The case in which both the disc and the chamber walls are smooth has been treated theoretically as well as experimentally by Daily and Nece (1958), who have also presented experimental results on rough discs.

INTRODUCTION

Fig. 1 shows the disc and the enclosure. The boundary layer thickness varies from zero at the centre of the disc to its maximum value at the edge.

When the disc is rough, the boundary layer may assume different characteristics depending on the Reynolds number. At sufficiently low Reynolds numbers, it will be laminar on the whole disc. At higher Reynolds numbers, laminar flow in the boundary layer may prevail at smaller values of the radius, transition to turbulence occurring at greater radii. Beyond the transition region, the boundary layer is turbulent, with a laminar sublayer. The thickness of the sublayer is smallest at the beginning of the turbulent region, and the boundary layer is rough turbulent if the disc surface is rough enough. The rough turbulent region may extend to the edge of the disc, or if the boundary layer grows thick enough towards the edge so that the laminar sublayer submerges the roughness projections, the boundary layer will become smooth turbulent in the outer region of the disc. This is similar to the nature of the boundary layer on a rough flat plate where, too, the boundary layer is laminar first at the origin of the plate, undergoes transition to turbulence and becomes rough turbulent, finally assuming smooth turbulent character.

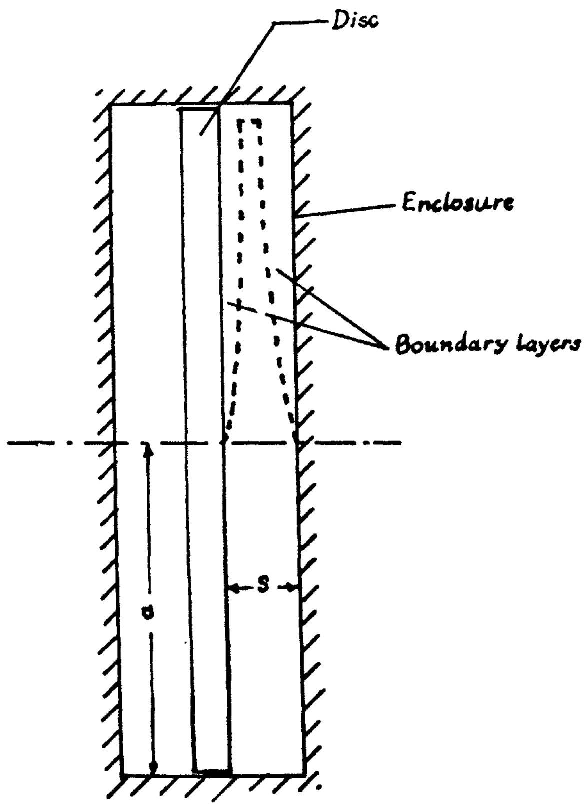

FIG. 1. Disc in a casing.

The proximity of the plane wall of the enclosure determines whether the boundary layers on the disc and the plane wall merge or remain separate.

METHOD OF SOLUTION

The procedure of determining the torque consists of solving the momentum integral equations by an approximate method. For the present problem, two integral equations each can be written for the disc and the plane wall, apart from the continuity equation. Daily and Nece (1958) have shown that though the continuity equation is not satisfied by the assumed velocity distributions, the results can still be satisfactory for a smooth disc. Accordingly we do not seek to satisfy the continuity equation, in order to simplify the calculations. The momentum integral equations are

Disc:

$$\rho\int_0^{\delta} u^2\,dz - r\tau_r - \frac{dp}{dr}\,\delta\,.\,r = \frac{d}{dr}\left[\rho r\int_0^{\delta} v^2\,dz\right], \qquad (1)$$

$$r^2\tau_t = \frac{d}{dr}\left[\rho\int_0^{\delta} r^2uv\,dz\right]; \qquad (2)$$

Plane wall of enclosure:

$$\rho\int_0^{\delta'} u^2\,dz + r\tau_r - \frac{dp}{dr}\,\delta'\,.\,r = \frac{d}{dr}\left[\rho r\int_0^{\delta'} v^2\,dz\right], \qquad (3)$$

$$r^2\tau_t = \frac{d}{dr}\left[\rho\int_0^{\delta'} r^2uv\,dz\right], \qquad (4)$$

where ρ = density of the fluid,
u = velocity of the fluid in the tangential direction,
v = velocity of the fluid in the radial direction,
r = distance from the disc axis,
τ_r = radial component of the shear stress on the surface,
τ_t = tangential component of the shear stress on the surface,
p = pressure at r,
z = distance normal to the surface towards the fluid,
δ = boundary layer thickness on the disc,
and δ' = boundary layer thickness on the plane wall.

The above equations are derived on the assumption that the pressure does not vary across the space between the disc and the plane wall.

We shall proceed with the analysis by assuming that the boundary layers on the disc and plane wall are separate, and that the space between the boundary layers is occupied by fluid rotating with an angular velocity $K\omega$, where ω is the angular velocity of the disc and K (< 1) is independent of r.

The velocity distributions to be assumed should satisfy the following boundary conditions:

Disc $\quad u = r\omega$ at $z = 0$

$u = Kr\omega$ at $z = \delta$

$v = 0$ at $z = 0$ and $z = \delta$.

Plane wall $u = 0$ at $z = 0$

$u = Kr\omega$ at $z = \delta'$

$v = 0$ at $z = 0$ and $z = \delta'$.

Disc

The velocity distribution in the boundary layer on a rough flat plate follows a logarithmic law. The boundary layer on a disc is a skewed one, but it will be assumed that the distribution of the resultant velocity relative to the disc follows the same law.

Let α be the angle which the resultant relative velocity near the rotating disc makes with the radius vector. We shall assume that it is also the angle between the resultant shear stress and the radius vector. The resultant relative velocity near the disc is $\dfrac{r\omega - u}{\sin\alpha}$ and the tangential velocity distribution is given by

$$\frac{r\omega - u}{u_\tau \sin\alpha} = A \log \frac{z}{k_s} + B, \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

where A and B are constants. It follows that

$$v = (r\omega - u)\cot\alpha = u_\tau \cos\alpha \left(A \log \frac{z}{k_s} + B\right). \qquad \ldots \quad \ldots \quad (6)$$

If u satisfies the condition $u = Kr\omega$ at $z = \delta$, then from eqns. (5) and (6), v does not become zero at $z = \delta$. This objection is overcome by making v follow a law different from (6) when z exceeds a certain value z_1 within the boundary layer. Hence v is given by

$$\left.\begin{aligned} v &= (r\omega - u)\cot\alpha = u_\tau \cos\alpha \left(A \log \frac{z}{k_s} + B\right), \qquad 0 < z \leqslant z_1 \\ v &= (u - Kr\omega)\cot\alpha \\ &= \cot\alpha \left[r\omega(1-K) - u_\tau \sin\alpha \left(A \log \frac{z}{k_s} + B\right)\right], \quad z_1 \leqslant z \leqslant \delta \end{aligned}\right\} \quad \ldots \quad \ldots \quad (7)$$

In order that $v(z)$ be continuous

$$u_\tau \sin\alpha \left[A \log \frac{z_1}{k_s} + B\right] = \frac{r\omega(1-K)}{2}$$

so that

$$z_1 = k_s \exp\left[\frac{r\omega(1-K)}{2Au_\tau \sin\alpha} - \frac{B}{A}\right].$$

Since $u = Kr\omega$ at $z = \delta$, we get from eqn. (5)

$$\therefore \quad \frac{r\omega(1-K)}{u_\tau \sin\alpha} = A \log \frac{\delta}{k_s} + B$$

$$\text{or} \quad \delta = k_s \exp\left[\frac{r\omega(1-K)}{Au_\tau \sin\alpha} - \frac{B}{A}\right].$$

Putting $\frac{r\omega(1-K)}{Au_\tau \sin\alpha} = X$, we have

$$\delta = k_s \exp\left[X - \frac{B}{A}\right]. \qquad (8)$$

Plane enclosure wall

For the plane wall of the enclosure, we can assume, in a similar manner, that

$$\frac{u}{u'_\tau} = \sin\beta\left[A\log\frac{z}{k'_s}+B\right],$$

where β has a meaning similar to α.

$$v = u\cot\beta = u'_\tau\cos\beta\left[A\log\frac{z}{k'_s}+B\right], \qquad 0 < z \leqslant z_2 \qquad (9)$$

$$v = (Kr\omega-u)\cot\beta = \cot\beta\left[Kr\omega-u'_\tau\sin\beta\left(A\log\frac{z}{k'_s}+B\right)\right], \quad z_2 \leqslant z \leqslant \delta' \qquad (10)$$

where

$$z_2 = \delta' e^{-Y/2},\ \delta' = k'_s \exp\left[Y-\frac{B}{A}\right] \qquad (11)$$

in which

$$Y = \frac{Kr\omega}{Au'_\tau \sin\beta}. \qquad (12)$$

The integrals which occur in eqns. (1) to (4) can now be evaluated to give for the disc:

$$\left.\begin{aligned}
\int_0^\delta u^2\,dz &= \frac{r^2\omega^2\delta}{X^2}[K^2X^2+2K(1-K)X+2(1-K)^2] \\
\int_0^\delta v^2\,dz &= \frac{2(1-K)^2r^2\omega^2\cot^2\alpha\delta}{X^2}[1-Xe^{-X/2}] \\
\int_0^\delta r^2uv\,dz &= \omega^2r^4\delta\cot\alpha\left[-\left(\frac{1-K^2}{X}+\frac{4(1-K)^2}{X^2}\right)e^{-X/2}\right. \\
&\qquad\left.+\frac{K(1-K)}{X}+\frac{2(1-K)^2}{X^2}\right]
\end{aligned}\right\}; \qquad (13)$$

for the plane wall:

$$\left.\begin{aligned}
\int_0^{\delta'} u^2\,dz &= \frac{K^2\omega^2r^2\delta'}{Y^2}[Y^2-2Y+2] \\
\int_0^{\delta'} v^2\,dz &= \frac{K^2\omega^2r^2\delta'}{Y^2}\cot^2\beta[2-2Ye^{-Y/2}] \\
\int_0^{\delta'} uv\,dz &= \frac{K^2\omega^2r^2\delta'}{Y^2}\cot\beta[Y-2-Ye^{-Y/2}+4e^{-Y/2}]
\end{aligned}\right\}. \qquad (14)$$

Substituting for these integrals in eqns. (1) to (4), we get, after multiplying the resulting equations for the disc and the plane wall by $\frac{dr}{dX}$ and $\frac{dr}{dY}$ respectively, for the disc:

$$2Kk_s e^{-B/A}\frac{r^2e^X}{X}\frac{dr}{dX}+2(1-K)k_s e^{-B/A}\frac{r^2e^X}{X^2}\frac{dr}{dX}-\frac{1-K}{A^2}\frac{r^3\theta\sqrt{1+\theta^2}}{X^2}\frac{dr}{dX}$$
$$=\frac{d}{dX}\left[2(1-K)k_s e^{-B/A}\frac{r^3\theta^2e^X}{X^2}(1-Xe^{-X/2})\right], \quad \ldots \quad \ldots \quad (15)$$

$$\frac{r^4(1-K)\sqrt{1+\theta^2}}{A^2X^2}\frac{dr}{dX}=\frac{d}{dX}\left[k_s e^{-B/A}r^4\theta e^X\left\{-\left(\frac{1+K}{X}+\frac{4(1-K)}{X^2}\right)e^{-X/2}\right.\right.$$
$$\left.\left.+\frac{K}{X}+\frac{2(1-K)}{X^2}\right\}\right]; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

for the plane wall:

$$2k_s' e^{-B/A}\frac{r^2}{Y^2}e^Y\frac{dr}{dY}-2k_s' e^{-B/A}\frac{r^2}{Y}e^Y\cdot\frac{dr}{dY}+\frac{r^3\gamma\sqrt{1+\gamma^2}}{A^2Y^2}\frac{dr}{dY}$$
$$=\frac{d}{dY}\left[k_s' e^{-B/A}\gamma^2\frac{r^3e^Y}{Y^2}(2-2Ye^{-Y/2})\right], \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$\frac{\sqrt{1+\gamma^2}}{A^2}\frac{r^4}{Y^2}\frac{dr}{dY}=\frac{d}{dY}\left[k_s' e^{-B/A}\gamma\frac{r^4e^Y}{Y^2}(Y-2-Ye^{-Y/2}+4e^{-Y/2})\right], \quad \ldots \quad (18)$$

where $\theta = \cot\alpha$ and $\gamma = \cot\beta$.

Basing on experimental data, Prandtl and Schlichting (1934) had classified the flow in the turbulent boundary layer on a rough flat plate into four regimes. Each regime is characterized by the manner in which the value of B in the logarithmic law of velocity distribution varies with the parameter $\frac{k_s u_\tau}{\nu}$:

$$\left.\begin{array}{lll} \text{Rough turbulent:} & B = 8{\cdot}48; & 70{\cdot}8 \leqslant \frac{k_s u_\tau}{\nu} \\ \text{Transition I:} & B = 11{\cdot}50-0{\cdot}704\log\frac{k_s u_\tau}{\nu}; & 14{\cdot}10 < \frac{k_s u_\tau}{\nu} \leqslant 70{\cdot}8 \\ \text{Transition II:} & B = 9{\cdot}58; & 7{\cdot}08 < \frac{k_s u_\tau}{\nu} \leqslant 14{\cdot}10 \\ \text{Smooth turbulent:} & B = 5{\cdot}5+2{\cdot}5\log\frac{k_s u_\tau}{\nu}; & 0 < \frac{k_s u_\tau}{\nu} \leqslant 7{\cdot}08 \end{array}\right\}. \quad \ldots \quad (19)$$

The transition regimes are to be understood as transition from smooth turbulent to rough turbulent flow, and not from laminar to turbulent flow. We cannot, however, borrow this classification with the values of the parameter indicated above for the rough disc boundary layer. For, in the case of a flat plate, the value of u_τ decreases with distance along the plate, since the free stream velocity remains the same while the boundary layer thickness increases. The lowest values of $\frac{k_s u_\tau}{\nu}$ occur farthest from the leading edge of the plate, and the highest values near it. But in the case of a rough rotating disc (for constant roughness over the whole disc), the shear stress on the disc increases with radius. The highest values of $\frac{k_s u_\tau}{\nu}$ occur at the outer radii and the least values in the central region of the disc. Thus, whereas the classification (19) is consistent with the change in the character of the boundary layer along the length of a flat plate, it contradicts that along the radius of the rough disc. It is probable that the classification is qualitatively correct for the rough disc, but new limits for $\frac{k_s u_\tau}{\nu}$ have to be fixed. However, since these are not known yet, and in order to simplify the calculations, we shall assume that the boundary layer is rough turbulent on the disc and the plane wall and ignore the effect of the presence of a laminar and transition region. Thus, B will be assumed to have a constant value on both the disc and plane wall.

The eqns. (15) to (18) can be solved approximately by a method adopted by Goldstein (1935) in solving a similar set of equations. We expand r and θ for the disc and r and γ for the plane wall in asymptotic series in X and Y, thus:

Disc:

$$r = a_0 k_s X e^X\left(1+\frac{A_1}{X}+\frac{A_2}{X^2}+\ldots\right),$$

$$\theta = \theta_0\left(1+\frac{\theta_1}{X}+\frac{\theta_2}{X^2}+\ldots\right).$$

Plane wall:

$$r = b_0 k_s' Y e^Y\left(1+\frac{B_1}{Y}+\frac{B_2}{Y^2}+\ldots\right),$$

$$\gamma = \gamma_0\left(1+\frac{\gamma_1}{Y}+\frac{\gamma_2}{Y^2}+\ldots\right).$$

The series are valid for large values of X and Y, for which we reduce them to their first terms. Substituting into the respective equations, we find that

$$r = 5\sqrt{\tfrac{2}{7}}\,\frac{KA^2e^{-B/A}}{1-K}\,k_s X e^X,$$

$\theta = \sqrt{\tfrac{2}{5}}$ for the disc and that

$$r = 5\sqrt{\tfrac{2}{7}}A^2e^{-B/A}k_s' Y e^Y,$$

$\gamma = \sqrt{\tfrac{2}{5}}$ for the plane wall.

The total torque on both sides of the disc is equal to

$$T_d = 4\pi \int_0^a r^2 \tau_t \, dr,$$

where a is the radius of the disc.

From eqns. (2) and (13), after substituting for δ and r,

$$\therefore \quad T_d = 4\sqrt{\tfrac{2}{5}}\pi e^{-B/A} \rho \omega^2 a^5 \frac{1-K}{X_1} \frac{k_s}{a} e^{X_1} \left[-\left(1+K+\frac{4(1-K)}{X_1}\right) e^{-X_1/2} + K + \frac{2(1-K)}{X_1} \right].$$

Hence the torque coefficient is

$$\left. \begin{aligned} C_{md} &= \frac{T_d}{\frac{1}{2}\rho\omega^2 a^5} \\ &= 8\sqrt{\tfrac{2}{5}}\pi e^{-B/A} \frac{1-K}{X_1} \frac{k_s}{a} e^{X_1} \left[-\left(1+K+\frac{4(1-K)}{X_1}\right) e^{-X_1/2} + K + \frac{2(1-K)}{X_1} \right] \end{aligned} \right\} \quad (20)$$

Similarly the torque coefficient C_{mp} on the two plane walls of enclosure is

$$C_{mp} = 8\sqrt{\tfrac{2}{5}}\pi e^{-B/A} \frac{k_s'}{a} \frac{K^2}{Y_1^2} e^{Y_1} [Y_1 - 2 - (Y_1 - 4) e^{-Y_1/2}]. \quad \ldots \quad (21)$$

X_1 and Y_1 are given by the equations

$$a = 5\sqrt{\tfrac{2}{7}} \frac{K A^2 e^{-B/A}}{1-K} k_s X_1 e^{X_1} \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

and

$$a = 5\sqrt{\tfrac{2}{7}} A^2 e^{-B/A} k_s' Y_1 e^{Y_1}. \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

If either the disc or the plane wall is smooth, we can use the proper expression for C_{md} or C_{mp} derived by Daily and Nece (1958) using the one-seventh power law of velocity distribution, with possibly a modification of the numerical coefficient.

The expressions for C_{md} and C_{mp} above, however, still contain the unknown quantity K. To find this out, we use the equation

$$C_{md} = C_{mp} + C_{mc}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where C_{mc} is the torque coefficient of the cylindrical wall (on both sides of the disc) of enclosure.

We can calculate C_{mc} by assuming that the boundary layer thickness on the cylindrical wall is equal to the boundary layer thickness at the outer radius of the rotating disc, δ_a. If the cylindrical wall is smooth, we can assume the velocity distribution near it to follow the one-seventh power law and write the shear stress on it as

$$\tau_c = 0{\cdot}0225 \, \rho U^{7/4} \left(\frac{\nu}{\delta_a}\right)^{1/4}.$$

Putting $U = Ka\omega$ and $\delta = k_s \exp\,(X_1 - B/A)$, we get the expression for the torque coefficient on the cylindrical wall as

$$C_{mc} = 1{\cdot}32 K^{7/4} \frac{s}{a} \left(\frac{k_s}{a}\right)^{-1/4} R^{-1/4} e^{-X_1/4}, \qquad \ldots \quad \ldots \quad (25)$$

where s is the clear distance between the disc and the plane wall and R is the Reynolds number, $\dfrac{\rho\omega a^2}{\nu}$.

Numerical Example

As regards the values of A and B in the preceding equations, Goldstein (1935) has shown that best results for smooth free discs are obtained by putting $A = 1{\cdot}97$ and $B = 0{\cdot}03$. For the case under discussion, best results are

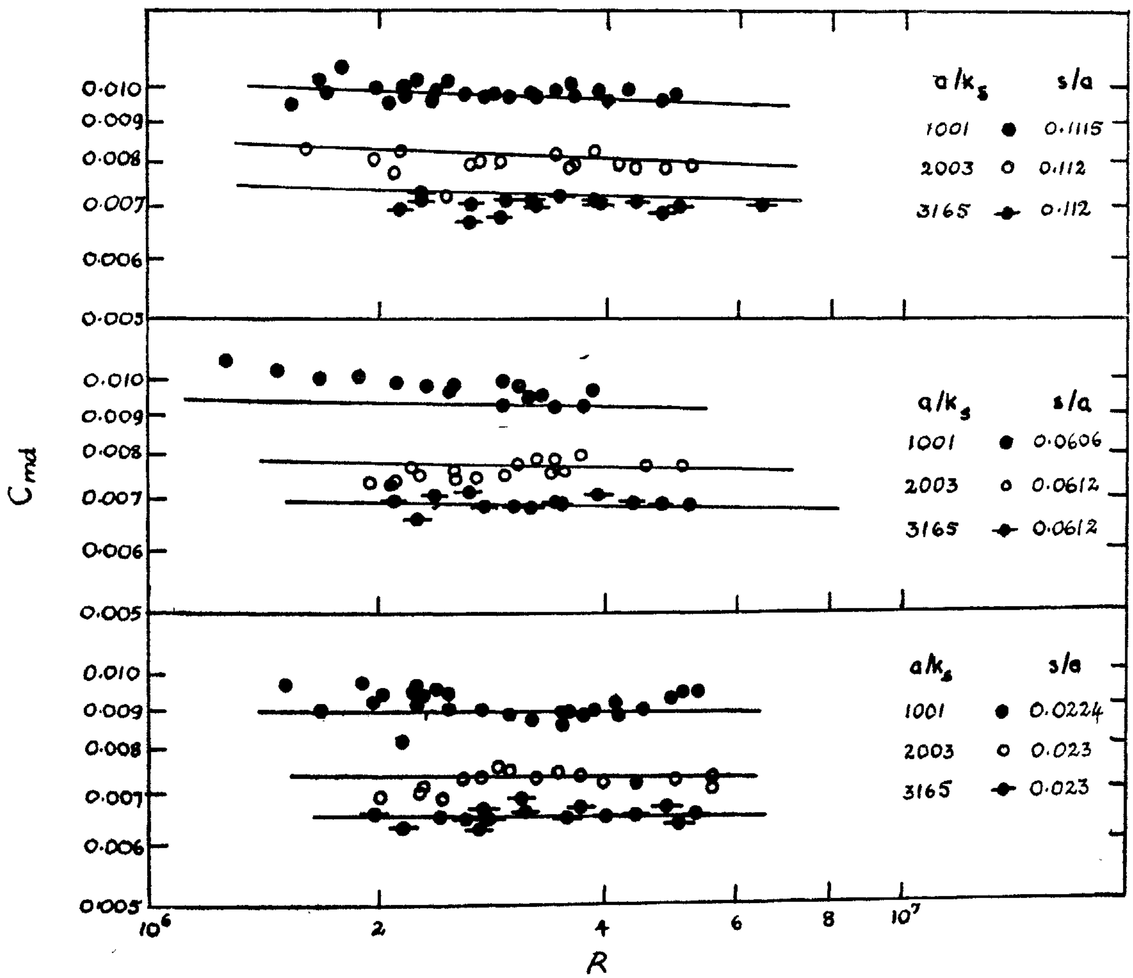

Fig. 2. Torque coefficient for equal roughnesses on the disc and casing wall.

obtained by putting $A = 3{\cdot}00$ and $B = 2{\cdot}60$. (The values for a rough flat plate are $A = 2{\cdot}5$ and $B = 8{\cdot}48$). Equations (20) and (21) then become

$$C_{md} = 6{\cdot}68 \frac{1-K}{X_1} \frac{k_s}{a} e^{X_1}\left[-\left(1+K+\frac{4(1-K)}{X_1}\right)e^{-X_1/2}+K+\frac{2(1-K)}{X_1}\right], \qquad (26)$$

$$C_{mp} = 6{\cdot}68 \frac{k_s'}{a} K^2 \frac{e^{Y_1}}{Y_1^2}\left[Y_1-2-(Y_1-4)e^{-Y_1/2}\right]. \qquad (27)$$

Substituting in eqn. (24) the value of K is obtained by trial and error for any given set of R, $\frac{s}{a}$, $\frac{k_s}{a}$ and $\frac{k_s'}{a}$. Numerical calculations have been carried out only in cases for which experimental data are available. The calculations are plotted along with the data of Daily and Nece in Figs. 2 and 3.

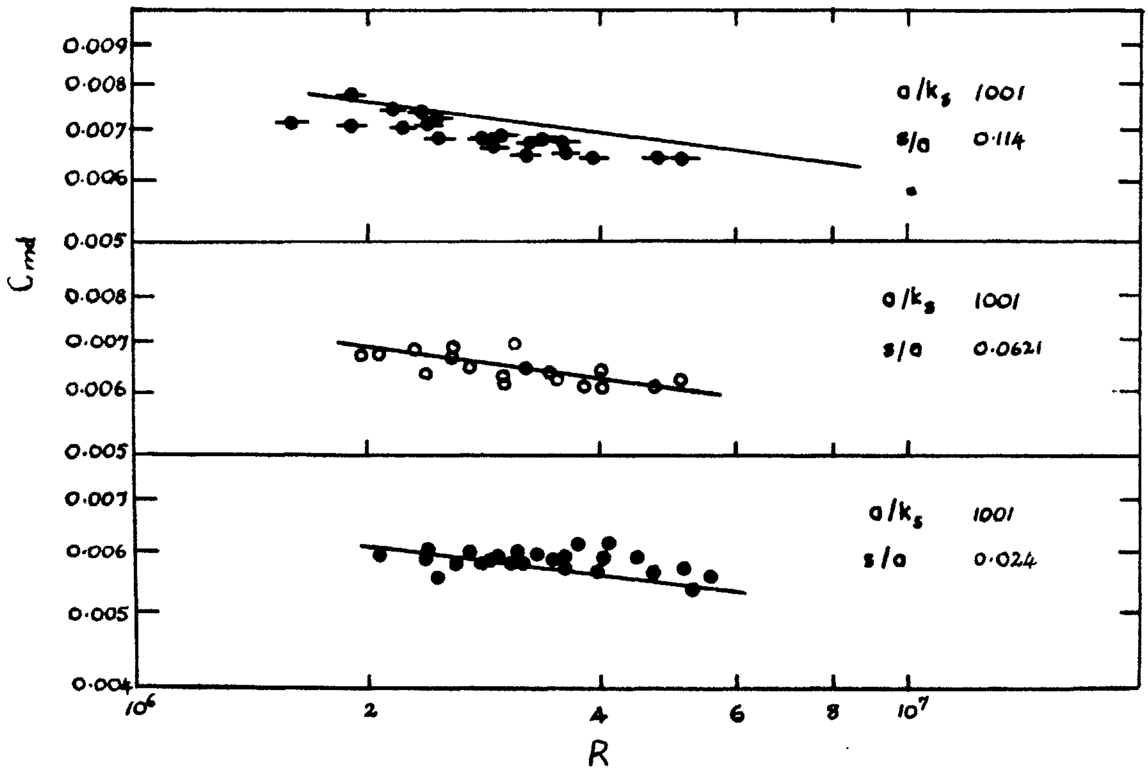

FIG. 3. Torque coefficient for rough disc and smooth casing wall.

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