AN EXACT SOLUTION FOR THE PROBLEM OF UNSTEADY TEMPERATURE DISTRIBUTION IN A VISCOUS FLOW

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An exact solution for the problem of unsteady temperature distribution in the case of a fully developed laminar flow through a pipe has been obtained. The heat due to friction has been taken into account but the process of heat transfer considered here is of forced convection.

INTRODUCTION

The exact solutions of steady temperature distribution for the cases of Couette flow and Poiseuille flow through a channel with flat walls have been discussed by Schlichting (1951). In both these cases, he had assumed the temperature to be constant in the direction of flow and to vary only in a direction perpendicular to the direction of flow. The heat due to friction has been taken into account by him. The process of heat transfer considered by him is of forced convection. In the present paper, the author investigates the axisymmetric unsteady temperature distribution for the case of a fully developed laminar flow inside a circular pipe under the same conditions. The situation can be realized if initially the temperature of the wall of the pipe is the same as that of the liquid and then the temperature of the wall is suddenly raised or lowered and kept constant. The case of steady temperature distribution has been obtained by taking the time since the start of the process of heat transfer to be infinite.

ENERGY EQUATION

Consider a fully developed laminar flow through a circular pipe with heat transfer taking place. Assume that the temperature is constant along the wall; and let the initial and boundary conditions be

$$\left.\begin{array}{ll} t=0: T=T_0 & \text{for } 0 < r < a \\ t>0: T=T_1 & \text{for } r=a \end{array}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where a is the radius of the pipe, T is the temperature of the fluid at any time t at any point in the pipe whose distance from the axis is r.

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With the conditions (1), it is clear that the temperature will vary only across the cross-section of the pipe. Hence, the equation for temperature distribution in this case will take the form

$$\rho gc\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)+\frac{\mu 16U^2r^2}{a^4}. \qquad (2)$$

Here ρ is the density of the liquid, g is the acceleration due to gravity, c is the specific heat, k is the thermal conductivity, μ is the coefficient of dynamic viscosity and U is the average velocity over the cross-section of the pipe. All these have been taken to be constant for the temperature changes.

Solution

Writing $\theta = \dfrac{T-T_1}{T_0-T_1}$ and $r = au$, the eqn. (2) after simplification reduces to

$$\rho gc\frac{\partial \theta}{\partial t} = \frac{k}{a^2}\left(\frac{\partial^2 \theta}{\partial u^2}+\frac{1}{u}\frac{\partial \theta}{\partial u}\right)+\frac{16\mu U^2u^2}{(T_0-T_1)a^2}. \qquad (3)$$

And the initial and boundary conditions (1) become

$$\left.\begin{array}{ll} t=0:\theta=1 & \text{for } 0\leqslant u\leqslant 1 \\ t>0:\theta=0 & \text{for } u=1 \end{array}\right\}. \qquad (4)$$

By applying the finite Hankel transform, eqn. (3) can be reduced to

$$\rho gc\int_0^1\frac{\partial \theta}{\partial t}uJ_0(pu)\,du = \frac{k}{a^2}\int_0^1\left(\frac{\partial^2 \theta}{\partial u^2}+\frac{1}{u}\frac{\partial \theta}{\partial u}\right)uJ_0(pu)\,du$$
$$+\frac{16\mu U^2}{a^2(T_0-T_1)}\int_0^1 u^3J_0(pu)\,du, \qquad (5)$$

where p is chosen as a positive root of the equation $J_0(p) = 0$.

Now

$$\int_0^1\left(\frac{\partial^2 \theta}{\partial u^2}+\frac{1}{u}\frac{\partial \theta}{\partial u}\right)uJ_0(pu)\,du$$
$$= \int_0^1\frac{\partial}{\partial u}\left(u\frac{\partial \theta}{\partial u}\right)J_0(pu)\,du$$
$$= \left[u\frac{\partial \theta}{\partial u}J_0(pu)\right]_0^1 - p\int_0^1 u\frac{\partial \theta}{\partial u}J_0'(pu)\,du$$
$$= -p\int_0^1 u\frac{\partial \theta}{\partial u}J_0'(pu)\,du, \quad \text{as } \left[u\frac{\partial \theta}{\partial u}J_0(pu)\right]_0^1 = 0,$$
$$= -p\left[\theta uJ_0'(pu)\right]_0^1 + p\int_0^1 \theta\left[J_0'(pu)+puJ_0''(pu)\right]du$$
$$= -p^2\int_0^1 \theta uJ_0(pu)\,du, \quad \text{as } \left[u\theta J_0'(pu)\right]_0^1 = 0 \text{ and } J_0(pu)$$

satisfies Bessel's equation $u^2\dfrac{d^2y}{du^2}+u\dfrac{dy}{du}+p^2u^2y=0$,

$$= -p^2\bar{\theta}.$$

Next consider

$$\int_0^1 u^3J_0(pu)\,du.$$

Let $pu = v$. Then we have

$$\int_0^1 u^3J_0(pu)\,du$$

$$= \int_0^p \frac{v^3}{p^4}J_0(v)\,dv$$

$$= \frac{1}{p^4}\left[\{v^2\,.\,vJ_1(v)\}_0^p - \int_0^p 2v\,.\,vJ_1(v)\,dv\right]$$

$$= \frac{1}{p^4}\left[p^3J_1(p)-2\{v^2J_2(v)\}_0^p\right]$$

$$= \frac{1}{p^4}\left[p^3J_1(p)-2p^2J_2(p)\right].$$

Substituting these values, we see that eqn. (5) becomes

$$\rho gc\frac{d\bar{\theta}}{dt} = \frac{-kp^2\bar{\theta}}{a^2}+\frac{16\mu U^2}{a^2(T_0-T_1)p^4}[p^3J_1(p)-2p^2J_2(p)]$$

or

$$\frac{d\bar{\theta}}{dt}+\frac{kp^2\bar{\theta}}{\rho gca^2} = \frac{16\mu U^2}{\rho gca^2(T_0-T_1)p^2}[pJ_1(p)-2J_2(p)].$$

Integrating, we get

$$\bar{\theta} = Ce^{-kp^2t/\rho gca^2}+\frac{16\mu U^2}{k(T_0-T_1)p^4}[pJ_1(p)-2J_2(p)], \quad \ldots \quad \ldots \quad (6)$$

where C is an arbitrary constant.

The first of the conditions (4) becomes

$$\bar{\theta} = \frac{J_1(p)}{p} \quad \text{at } t=0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Applying the condition (7) in eqn. (6), we get

$$\frac{J_1(p)}{p} = C+\frac{16\mu U^2}{k(T_0-T_1)p^4}[pJ_1(p)-2J_2(p)]. \quad \ldots \quad \ldots \quad (8)$$

Hence, substituting the value of C from eqn. (8) in eqn. (6), we get

$$\bar{\theta} = \frac{J_1(p)}{p}e^{-kp^2t/\rho gca^2}+\frac{16\mu U^2}{k(T_0-T_1)p^4}[pJ_1(p)-2J_2(p)](1-e^{-kp^2t/\rho gca^2}). \quad (9)$$

Hence, by the inversion formula (1951), we have

$$\theta = \sum_{p} \left\{ \frac{2J_1(p)}{p} e^{-kp^2t/\rho gca^2} + \frac{32\mu U^2}{k(T_0-T_1)p^4} [pJ_1(p) - 2J_2(p)](1-e^{-kp^2t/\rho gca^2}) \right\} \frac{J_0(pu)}{J_1^2(p)}, \quad \ldots \quad (10)$$

the summation being over the positive roots of the equation

$$J_0(p) = 0.$$

Substituting $\theta = \frac{T-T_1}{T_0-T_1}$ and $u = r/a$ in (10) and rearranging, we have

$$\frac{T-T_0}{T_1-T_0} = \left[1 - \sum_{p} \frac{2J_0\left(\frac{pr}{a}\right) e^{-kp^2t/\rho gca^2}}{pJ_1(p)} \right] + \left[\frac{32\mu U^2}{k(T_1-T_0)} \sum_{p} \left\{ \frac{1}{p^3 J_1(p)} - \frac{2J_2(p)}{p^4 J_1^2(p)} \right\} (1-e^{-kp^2t/\rho gca^2}) J_0\left(\frac{pr}{a}\right) \right]. \quad \ldots \quad (11)$$

Writing $k/\rho gc = \alpha$ (the thermal diffusivity), $\frac{\mu gc}{k} = P$ (the Prandtl number) and $\frac{U^2}{gc(T_1-T_0)} = E$ (the Eckert number), we see that eqn. (11) becomes

$$\frac{T-T_0}{T_1-T_0} = \left[1 - \sum_{p} \frac{2J_0\left(\frac{pr}{a}\right) e^{-\alpha tp^2/a^2}}{pJ_1(p)} \right] + \left[32P\cdot E \sum_{p} \left\{ \frac{J_0\left(\frac{pr}{a}\right)}{p^3 J_1(p)} - \frac{2J_2(p) J_0\left(\frac{pr}{a}\right)}{p^4 J_1^2(p)} \right\} (1-e^{-\alpha tp^2/a^2}) \right]. \quad \ldots \quad (12)$$

Steady State Solution

The steady state temperature distribution is obtained from the solution (11) by taking t tending to ∞. This steady temperature distribution is given by

$$T = T_1 + \frac{32\mu U^2}{k} \sum_{p} \left\{ \frac{1}{p^3 J_1(p)} - \frac{2J_2(p)}{p^4 J_1^2(p)} \right\} J_0\left(\frac{pr}{a}\right). \quad \ldots \quad \ldots \quad (13)$$

In order to obtain this solution (13) from eqn. (3), we proceed as follows: For steady state, eqn. (3) reduces to

$$\frac{d^2\theta}{du^2} + \frac{1}{u}\frac{d\theta}{du} + \frac{16\mu U^2 u^2}{k(T_0-T_1)} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

The boundary condition is

$$\theta = 0 \quad \text{for } u = 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Replacing θ by $\dfrac{T-T_1}{T_0-T_1}$ in eqns. (14) and (15), we have

$$\frac{d^2T}{du^2}+\frac{1}{u}\frac{dT}{du}+\frac{16\mu U^2u^2}{k}=0 \qquad \ldots \quad \ldots \quad \ldots \quad (16)$$

and

$$T=T_1 \quad \text{for } u=1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

The solution of eqn. (16) subject to the condition (17) is given by

$$T=T_1+\frac{\mu U^2}{k}(1-u^4). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

The Fourier-Bessel expansion for $(1-u^4)$ over the range 0 to 1, as given by Relton (1946), is

$$1-u^4=\sum_p A_pJ_0(pu), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

where the summation is over the positive roots of the equation $J_0(p)=0$ and

$$A_p=\frac{2}{J_0'^2(p)}\int_0^1(1-u^4)uJ_0(pu)\,du. \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

As $J_0'(p)=-J_1(p)$, we have from (20) that

$$A_p=\frac{2}{J_1^2(p)}\int_0^1(1-u^4)uJ_0(pu)\,du.$$

Putting $pu=v$, we have

$$A_p=\frac{2}{J_1^2(p)}\int_0^p\left(1-\frac{v^4}{p^4}\right)\frac{v}{p^2}J_0(v)\,dv$$

$$=\frac{2}{J_1^2(p)}\left[\left\{\left(1-\frac{v^4}{p^4}\right)\frac{v}{p^2}J_1(v)\right\}_0^p+\int_0^p\frac{4v^3}{p^6}\,.\,vJ_1(v)\,dv\right]$$

$$=\frac{2}{J_1^2(p)}\left[\int_0^p\frac{4v^2}{p^6}\,.\,v^2J_1(v)\,dv\right]$$

$$=\frac{2}{J_1^2(p)}\left[\left\{\frac{4v^2}{p^6}\,.\,v^2J_2(v)\right\}_0^p-\int_0^p\frac{8v}{p^6}\,.\,v^2J_2(v)\,dv\right]$$

$$=\frac{2}{J_1^2(p)}\left[\frac{4J_2(p)}{p^2}-\int_0^p\frac{8v^3J_2(v)}{p^6}\,dv\right]$$

$$=\frac{2}{J_1^2(p)}\left[\frac{4J_2(p)}{p^2}-\left\{\frac{8v^3J_3(v)}{p^6}\right\}_0^p\right]$$

$$=\frac{2}{J_1^2(p)}\left[\frac{4J_2(p)}{p^2}-\frac{8J_3(p)}{p^3}\right].$$

Since $2J_1(p) = p\{J_0(p)+J_2(p)\}$ and $4J_2(p) = p\{J_1(p)+J_3(p)\}$, we have

$$A_p = \frac{2}{J_1^2(p)}\left[\frac{4}{p^2}\left\{\frac{2J_1(p)}{p}-J_0(p)\right\}-\frac{8}{p^3}\left\{\frac{4J_2(p)}{p}-J_1(p)\right\}\right]$$

$$= \frac{2}{J_1^2(p)}\left[\frac{16J_1(p)}{p^3}-\frac{32J_2(p)}{p^4}\right], \quad \text{as } J_0(p) = 0$$

$$= 32\left[\frac{1}{p^3J_1(p)}-\frac{2J_2(p)}{p^4J_1^2(p)}\right].$$

Substituting this value of A_p in (19), we have

$$1-u^4 = \sum_p 32\left[\frac{1}{p^3J_1(p)}-\frac{2J_2(p)}{p^4J_1^2(p)}\right]J_0(pu). \quad \dots \quad \dots \quad (21)$$

And substituting the value of $1-u^4$ from (21) in (18), we have

$$T = T_1+\frac{32\mu U^2}{k}\sum_p\left[\frac{1}{p^3J_1(p)}-\frac{2J_2(p)}{p^4J_1^2(p)}\right]J_0(pu).$$

Replacing u by r/a, we get

$$T = T_1+\frac{32\mu U^2}{k}\sum_p\left[\frac{1}{p^3J_1(p)}-\frac{2J_2(p)}{p^4J_1^2(p)}\right]J_0\left(\frac{pr}{a}\right). \quad \dots \quad \dots \quad (22)$$

This result (22) is identical with the result (13).

Numerical Results and Conclusions

It is worth noting that the terms in the first square bracket on the right-hand side of eqn. (12) are due to conduction of heat, and are the same as in the case of a fluid at rest with no frictional heat generated; and the terms in the second square bracket are due to the heat generated through friction. The symbol ϕ will be used to denote the terms in the first bracket on the r.h.s. of eqn. (12) and the symbol ψ to denote those in the second bracket. Tables 1 and 2 show the values of ϕ and $\psi/P{\cdot}E$ corresponding to various values of the dimensionless parameter $\alpha t/a^2$ at $r/a = 0$ and at $r/a = 0{\cdot}9$ respectively.

From the above values of ϕ and ψ corresponding to various values of $\alpha t/a^2$, the following conclusions may be derived: The heat due to conduction and the frictional heat increase with the increase of time at all points in the pipe, a fact easily seen from eqn. (12) also. The frictional heat increases from its value zero at the wall of the pipe to its maximum value at the centre of the pipe. On the other hand, the heat due to conduction decreases from its maximum value at the pipe wall to its minimum value at the centre of the pipe but as time passes the heat due to conduction becomes more and more evenly distributed throughout the pipe and ultimately in the steady case it

TABLE I

Values of ϕ and $\psi/P{\cdot}E$ at $r/a = 0$

$\alpha t/a^2$	ϕ	$\psi/P{\cdot}E$
0·04	0·004	0·048
0·06	0·030	0·106
0·08	0·082	0·179
0·10	0·152	0·256
0·20	0·499	0·570
0·40	0·842	0·864
0·60	0·950	0·957
0·80	0·984	0·986
1·00	0·995	0·995
∞	1·000	0·998

TABLE II

Values of ϕ and $\psi/P{\cdot}E$ at $r/a = 0{\cdot}9$

$\alpha t/a^2$	ϕ	$\psi/P{\cdot}E$
0·04	0·766	0·176
0·06	0·819	0·205
0·08	0·851	0·227
0·10	0·874	0·243
0·20	0·934	0·291
0·40	0·979	0·330
0·60	0·994	0·339
0·80	0·998	0·342
1·00	0·999	0·346
∞	1·000	0·346

We note that $\phi = 1$ and $\psi/P{\cdot}E = 0$ for all values of $\alpha t/a^2$ at $r/a = 1$.

becomes uniformly distributed throughout the pipe. It may be observed that the heat due to friction will be small if the dimensionless product $P{\cdot}E$ is small. And as the product $P{\cdot}E$ is generally small (specially in the case of liquids of small viscosity flowing with small velocities), it follows that the heat due to friction is small. But it cannot be said that the heat due to friction is small as compared to the heat due to conduction even for small values of the product $P{\cdot}E$ at all points inside the pipe for all times. In fact, the heat due to friction will be greater than that due to conduction near the centre of the pipe even for small values of the product $P{\cdot}E$ up to a certain instant of time after which the heat due to friction will be less as compared to that due to conduction. Of course, near the pipe wall, the heat due to friction will be small as compared to that due to conduction for all times if the product $P \cdot E$ is small.

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