ON ABSTRACT STRUCTURE OF THEORIES OF INTEGRAL TRANSFORMS

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(Communicated by S. N. Bose, F.N.I.)

(*Received February 28, 1966*)

From the time of Laplace up to the present time, different theories of integral transforms have been proposed by the help of integrals with different kernels and ranges of integrations, chosen suitably. These integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. The most useful significance of integral transforms lies in the fact that they transform a class of differential equations into a class of algebraic equations, so that solutions of those differential equations can be obtained easily by algebraic methods and by use of results of integral transforms.

Here the starting points are linear operators, having their inverses, transforming on Hilbert space H to a Hilbert space $\mathfrak{H}$ and reducing some operator-equations in the Hilbert space H to algebraic equations in $\mathfrak{H}$. Then by the application of the theory of linear operators in Hilbert space, investigations have been made about the following points: (i) How are these operators represented? (ii) What are the analytic forms of these operators? (iii) When are the transformed spaces formed of functions and when of sequences? (iv) Is there any possibility of extension? The present discussion not only clearly shows the real significances of the theories of transforms but also yields some suggestions about possible extension.

1. INTRODUCTION

When the integral

$$f(s) \equiv T_s\{F(x)\} = \int_a^b w(x)K(x, s)F(x)\,dx \quad \ldots \quad \ldots \quad (1.1)$$

taken over a range (a, b) (finite, semi-infinite or doubly infinite) exists for a set of functions $F(x)$, generally referred to as object-space (after introducing its topology suitably), with a function $K(x, s)$ of definite form, generally referred to as kernel and with a weight-function $w(x)$ and when $F(x)$ is uniquely known almost everywhere from the (real-valued or complex-valued) functions $f(s)$, then $f(s)$ is known as the integral transform of $F(x)$. The weight-function $w(x)$ is generally chosen suitably keeping the convergence of the integral in view, particularly when the range of integration is infinite. The weight-function $w(x)$ generally characterizes the class of functions $F(x)$,

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whereas the kernel function $K(x, s)$ determines the nature of the transform. The study of the classes of functions $F(x)$ and $f(s)$, and their interrelations when the form of $K(x, s)$ is prescribed suitably are the main aims of a theory of integral transforms. The class of functions $f(s)$ will be referred to as the image-space after Doetsch (1943).

With the prescription of the different forms of $K(x, s)$ and of the different ranges of transforms, different theories of integral transforms have been developed, viz. transforms of Laplace (complete or truncated), of Fourier, Mellin (Doetsch 1943), Gauss, Hankel (Sneddon 1951), Whittaker (Verma 1949), Hilbert (Tricomi 1951), Legendre (Tranter 1950; Churchill 1954), Gegenbauer (Lakshmanarao 1954; Conte 1955), Laguerre (Debnath 1960, 1964*a*; McCully 1960), Jacobi (Debnath 1963, 1964*a*) and Hermite (Debnath 1964*a*, *b*). For the study of the abstract structure here, it appears to be of much convenience to group the theories of integral transforms mentioned above into two classes; one in which the image-space is a function-space named as the class of continuous image and denoted by I_f and the other in which it is a sequence space named as the class of sequential images and denoted by I_s. The transforms of Laplace, Fourier, Gauss, Mellin, Whittaker and Hilbert and also transforms similar to them are in the class I_f; while transforms of Legendre, Gegenbauer, Jacobi, Laguerre and Hermite and similar transforms are in the class I_s. Strictly speaking (Doetsch 1943), the object-space is the space of all functions which after being multiplied by $w(x)K(x, s)$ are integrable. In many developments like those of Wiener (1933) and others, the object-space is taken as a L_2-space (or L_2^w-space). The image-space is L_2- (or L_2^w-) space in the class I_f and a l_2-space in the class I_s. Now it may be noted that the theories of integral transforms are in essence practically the same as those of generalized Fourier resolutions of abstract analysis. From this point of view the major part of the theories of integral transforms is the study of a one-parametric or two-parametric family of bounded linear operators, having the inverse and transforming a Hilbert space to another Hilbert space.

The genesis of the theories of integral transforms is in Laplace's introduction of moment-generating function in the theory of probability. But the chief importance of these theories lies in solving differential equations by transforming them into algebraic equations which are solved by algebraic methods and then by writing the solution of original equations from the solutions of algebraic equations by application of inverse transforms. It can be easily noticed that a certain theory of integral transforms is most convenient in solving certain types of equations. Recently (Dutta and Debnath 1965, 1966) a general method was sketched to solve certain classes of differential equations by developing a suitable theory of integral transforms.

For our discussion here, keeping the utilitarian point of view in mind, we shall summarize the common basic features of almost all theories of integral transforms in a bit more abstract language in the following way:

(i) There exists a Hilbert space, H, defined over a complex (or real) field Φ.

(ii) There exists a certain type of operator-equation involving certain elementary operator, L to define over a domain D of H and to transform D into H.

(iii) Investigations are made of one-parametric (or two-parametric) family of linear continuous operator, T_s of H into complex (real) space Φ with its natural topology, so that

$$T_s\{L(F)\} = \lambda_s T_s\{F\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

and

$$T_s\{g(L)(F)\} = g(\lambda_s) T_s\{F\}, \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

where $F \in H$, and $g(z)$ is a polynomial (or integral) function of z.

The object of this paper is to investigate the following points for a mathematical system with the above basic features:

(i) What is the analytic form (or representation) of the linear transform T_s?

(ii) How is the kernel determined completely from the knowledge of L on D in H?

(iii) Under what conditions is a theory of integral transforms in the class I_f or in the class I_s?

(iv) Whether there is any possibility of existence of any other class of integral transforms or not?

(v) Is there any advantage of the present study of the abstract structure of theories of integral transforms?

In the present discussion, we shall consider a one-parametric family of transformations on H to the complex field Φ. The extension of the present discussion to a two-parametric family of transformations can be made in a direct manner.

2. Representation of T_s

From the well-known theorem (Dunford and Schwartz 1958) for linear continuous transformation on a Hilbert space H into the complex field Φ, we have

$$T_s\{F\} = (F, u_s), \text{ for all } F, \quad \ldots \quad \ldots \quad \ldots \quad (2.01)$$

where u_s is an unique element of H.

Now if the original space is a L_2-space (a particular realization of H), so that the scalar product is defined as usual as

$$(F, u) = \int_a^b w(x) F(x) \bar{u}(x)\, dx, \quad \ldots \quad \ldots \quad \ldots \quad (2.02)$$

where $\bar{u}(x)$ is conjugate to $u(x)$, then from (2.01) we get

$$f(s) \equiv T_s\{F\} = \int_a^b w(x)F(x)\bar{u}_s(x)\,dx. \qquad \text{(2.03)}$$

Usually in the theories of integral transforms, the definition of transform is introduced *ad hoc* by integrals. The above discussion shows that for the representation theorem, this becomes possible.

Also it is easy and interesting to note some general properties from well-known theorems for Hilbert space. It is well known (Dunford and Schwartz 1958) that all linear continuous transformations of a Hilbert space H into the complex field Φ form a Hilbert space $\mathfrak{H}$. Thus the set of all $f(s)$ is a Hilbert space.

It is also easy to see that $\mathfrak{H}$ is isomorphic and isometric to H. So the set of all $f(s)$ is either a L_2-space or a l_2-space.

3. Determination of the Kernel

From the relation (2.01), we have, for all $F \in D$,

$$T_s\{L(F)\} = (LF, u_s) = (F, L^* u_s), \qquad \text{(3.1)}$$

where L^* is the adjoint of L. Again by (1.2)

$$T_s\{L(F)\} = \lambda_s T_s\{F\} = (F, \bar{\lambda}_s u_s), \qquad \text{(3.2)}$$

where $\bar{\lambda}$ is the conjugate to the complex number.

Then $$(F, L^* u_s) = (F, \bar{\lambda} u_s), \quad \text{for all } F,$$

or $$(F, (L^* - \bar{\lambda}_s)u_s) = 0, \quad \text{for all } F. \qquad \text{(3.3)}$$

So the sufficient condition* for the validity of (3.3) is that

$$L^* u_s = \bar{\lambda}_s u_s. \qquad \text{(3.4)}$$

Now from the general properties of eigen-value and eigen-functions, we have

$$\bar{g}(L^*)u_s = \bar{g}(\bar{\lambda}_s)u_s = \overline{g(\lambda_s)}u_s, \qquad \text{(3.5)}$$

where $\overline{g(z)}$ is the conjugate of $g(z)$. Then

$$\left.\begin{aligned} T_s\{g(L)F\} &= (g(L)F, u_s) \\ &= (F, \bar{g}(L^*)u_s) = (F, \overline{g(\lambda_s)}u_s) \\ &= (F, \overline{g(\lambda_s)}u_s) = g(\lambda_s)(F, u_s) \\ &= g(\lambda_s)T\{F\} \end{aligned}\right\}, \qquad \text{(3.6)}$$

i.e. the condition (1.3) is satisfied.

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* The condition will also be necessary only if D satisfies some suitable conditions.

Thus the problem of developing a theory of integral transforms most convenient for solving the class of the operator-equations

$$L(y) = 0 \qquad (3.7)$$

or

$$g(L)(y) = 0 \qquad (3.8)$$

$$y \in D \subset H,$$

where $g(x)$ is a polynomial (or an integral function of x), reduces to the problem of the existence and the evaluation of the eigen-values and eigen vector of the adjoint operator L^*. When the eigen-function has been evaluated, the kernel is known by (2.03). Now if the operator be self-adjoint, it is known (Neumark 1960) that eigen-values are real and there is a possibility of building up a theory of real integral transforms.

4. Criteria for I_f or I_s

From the above discussion, it is easy to see that the theory of integral transforms will be in I_f or I_s according as the operator-equation (3.4) has a pure continuous or a pure point spectrum. In most of the usual theories of integral transforms, the major interest is in the case when L is a differential operator of finite order. For differential operators, it is known (Neumark 1960) that the eigen-values are either discretely enumerable or continuous. So in the usual theories of integral transform we notice only two classes, I_f and I_s.

5. Possibility of Existence of other Classes

It is known that the spectrum of a linear operator consists of a point-spectrum, a continuous spectrum and a residual spectrum. We have noticed two classes, I_s corresponding to the point-spectrum and I_f corresponding to the continuous spectrum. If the residual spectrum of L is void and the spectrum of L consists of point-spectrum and continuous spectrum only, it is not difficult to see how the theories are to be extended. It appears that in the definition of the transform no modification is necessary, and only in the inversion formula we shall have to take an integral and a sum, or better to take the integral in the Lebesgue-Stieltjes sense in place of only the Lebesgue sense. But the nature of the class corresponding to the residual spectrum is not clear. This class may be useful in the case of discussions with aperiodic functions.

6. Advantage of the Abstract Study

In most of the theories of integral transforms developed yet, the kernel is associated with differential operators of the first or the second order. This point is clearly discussed in some recent papers (Dutta and Debnath 1965, 1966). From the article 3 we can see a general method for developing a theory of integral transforms in which the kernel is associated with differential operators of higher order. This is one simple advantage of the abstract study, as seen easily. Other advantages may be seen in detailed future studies in the line.

7. Concluding Remarks

Here we have studied significances and consequences of only three most basic features of the theories of integral transforms. But there are also some other important features of these theories, like convolution property, asymptotic behaviour, etc. Their significance will be the subject-matter of future studies. Also, for simplicity, the object-space is taken to be a Hilbert space. The actual abstract structure of the general object-space and the significance of the theories of integral transforms in such an abstract object-space appear to be interesting and important.

Moreover, it is also very interesting to note that the basic problem of theories of integral transforms is quite similar to that of quantum mechanics. Eigen-value problem of an operator in a Hilbert space is the kernel in both the subjects. In quantum mechanics, the operator is mostly self-adjoint. Here operators need not be necessarily self-adjoint.

Acknowledgement

The author expresses his deep gratitude to National Professor S. N. Bose, F.R.S., for his constant help and encouragement in the present investigation and also for his support by including the present investigation as a part of his research scheme.

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