ON POINCARÉ'S RECURRENCE THEOREM

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Much work has been done on the recurrence theorem of Poincaré but it is mostly based on the measure-preserving property of transformations. But actual dynamical transformations of a conservative system are also topologically invariant. It appears that the significance of this property has not yet been explored. Here attempts have been made to study the consequences of the topological invariance of measure-preserving transformations and thus to characterize topologically the set of non-recurrent points. A new notion of 'almost recurrent' has been introduced and some results regarding 'recurrence' and 'almost recurrence' have been obtained.

1. Introduction

Let T_t be a measure-preserving transformation of a set S into itself, where t is a real parameter.

Definition.—A point, $x \in E \subset S$, is said to be recurrent with respect to E, if $T_t x \in E$ for some value of t, known as the recurrent time.

The recurrence theorem of Poincaré (1890, 1899) has been discussed by a number of authors (Birkhoff 1927; Halmos 1956; Kac 1959; Kürth 1960). It can be conveniently stated as follows:

'If T_t be a measure-preserving transformation of a set S of finite measure and E be a subset of S of positive measure then, firstly, all points n except those of measure zero are recurrent and, secondly, they are recurrent for infinitely many values of the recurrent time.'

The first part of the theorem is known as 'weak recurrence theorem' and the second part as the 'strong recurrence theorem'.

In general mechanics, for Hamilton's canonical motion, the well-known Liouville's theorem implies the measure-preserving nature of transformations representing dynamical motion. Starting from this measure-preserving property, Poincaré proved simply and elegantly the recurrence theorem, which is considered as a beautiful theorem of qualitative dynamics. He (1912) applied this theorem in the general discussion of problems of stability in dynamics. He pointed out that in the phase space, the hypersurfaces, associated with a particular value of a constant of motion-like energy, might

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be taken as a set of finite measure. This theorem is also helpful in the proper understanding of the basic ideas of classical statistical mechanics. The notion of recurrence, first introduced in the theory of probability by Poincaré (1912), is also useful in discussions of asymptotic questions regarding chains in this theory.

Many investigators, being impressed by the simplicity and elegance of this theorem as a direct consequence of the measure-preserving transformations, extended and generalized it for general measure-preserving transformations in abstract measure-space (Halmos 1956). But it appears that not enough attempts have been made to investigate what more information could be obtained about the recurrence if the discussions are restricted to actually dynamically conservative systems.

For a very general motion, defined by a set of differential equations,

$$\dot{x}_i = X_i(x_1, x_2, \ldots, x_n, t), \quad i = 1, 2, \ldots, n,$$

in an n-dimensional vector space, if X_i's satisfy the Lipschitz condition, the solutions at any time t are unique and continuous functions of time t and also of the initial coordinates (cf. Birkhoff 1927; Kürth 1960). For Hamiltonian motion, Liouville's theorem is valid, i.e. dynamical transformations are measure-preserving. If the dynamical system is also conservative, the principle of reversibility (i.e. of invariance under reflection of time) is satisfied. Thus, for the Hamiltonian motion of a conservative system, the solution is bi-uniform and bi-continuous, that is, dynamical transformations are measure-preserving and topological. The aim of the present discussion is to find out what more information about the recurrent property is obtained when the measure-preserving transformations are also topological. We have defined 'almost recurrence' and tried to characterize topologically the sets of recurrent points, of almost recurrent points and of non-recurrent points.

2. Recurrence of Topological Set at a Recurrent Time

Theorem 1: If T_t be a measure-preserving topological transformation of a topological space of finite measure, and if G be an open set in S, then the set of recurrent points of G with respect to G at a recurrent time τ is open.

Remark.—If G be a null set, the theorem is trivially true. So in the present discussion by an open set, we mean a non-null open set unless otherwise explicitly stated.

Proof.—According to the recurrence theorem stated above, there exist some recurrent points. Let x be a recurrent point and let a corresponding recurrent time be τ. Let R_τ be the set of all points of G which are recurrent at the recurrent time τ.

Let us denote

$$G_\tau = G \cap T_\tau(G).$$

From the fundamental property of the topological space, G_τ is open. Let us write

$$G_{1\tau} = T_\tau^{-1}(G \cap T_\tau(G)).$$

As T_τ is a topological transformation, so $G_{1\tau}$ is open.
As $G_\tau \subset T_\tau(G)$, so $G_{1\tau} \subset G$ and $T_\tau(G_{1\tau}) = G_{1\tau} \subset G$, i.e. $G_{1\tau} \subset R_\tau$. Let y be any point so that $y \in R_\tau \subset G$, i.e. $T_\tau y \in G$.
Now $\quad T_\tau y \in T_\tau(G)$, i.e. $T_\tau y \in G_\tau$, i.e. $y \in G_{1\tau}$, i.e. $R_\tau \subset G_{1\tau}$.
So $R_\tau = G_{1\tau}$, i.e. R_τ is open.

Corollary 1.—If all conditions of the theorem are satisfied and, moreover, if G is connected, then the set of all recurrent points in G is open and connected.

Proof.—As connectedness is a topological property (Dutta *et al.* 1964), so $T_\tau(G)$ is connected. Then $G_{1\tau}$ and so R_τ are connected and open.

Corollary 2.—If all conditions of the theorem are satisfied and, moreover, if G is compact, then R_τ is compact and open.

Proof.—As compactness is a topological property (Dutta *et al.* 1964), so the assertion of this corollary follows easily as in Corollary 1.

Definition.—A compact connected topological space is known as the continuum (cf. Kuratowski 1961).

Corollary 3.—If all conditions of the theorem are satisfied and, moreover, if G is a continuum, then R_τ is a continuum and open.

Proof.—It follows directly from Corollaries 1 and 2.

Corollary 4.—The set N_τ of non-recurrent points of an open set at a recurrent time τ cannot be dense in G.

Proof.—As R_τ is epen, so a point of R_τ cannot be in $\bar{N}_\tau$, i.e. the closure of N_τ.

Corollary 5.—The set N_τ of non-recurrent points of any set E of non-null interior at a recurrent time τ cannot be dense in E.

Proof.—As the set of recurrent points of the interior ${}_iE$, with respect to ${}_iE$ at a recurrent time τ, is open, it cannot be contained in $\bar{N}_\tau$, where N_τ is the set of all non-recurrent points of E.

Corollary 6.—The set of recurrent points of a set E of non-null interior at a recurrent time τ is of positive measure.

Proof.—The assertion follows from the fact that the set of recurrent points of the interior ${}_iE$ is open and so of positive measure.

THEOREM 2: If T_t be a measure-preserving topological transformation and if F be a closed set of non-null interior, then the set R_τ of all recurrent points of F at a recurrent time τ is a closed set of non-null interior.

Proof.—Proceeding as before we can easily see that

$$R_\tau = T_\tau^{-1}(F \cap T_\tau(F)).$$

As F is closed and T_τ is topological, so R_τ is closed. Now if we denote

$$G = {}_iF$$

then

$$G_{1\tau} = T_\tau^{-1}(G \cap T_\tau(G)) \subset R_\tau.$$

As $G_{1\tau}$ is open, so R_τ is of non-null interior.

3. Almost Recurrent Set at a Recurrent Time

It appears that it will be of some advantage if we introduce a notion of almost recurrence in the following way:

Definition.—A point $x \in E \subset S$ is said to be almost recurrent with respect to E, if $T_\tau x \in \bar{E}$ for some time τ, to be referred to as an almost recurrent time.

Remark.—As $E \subset \bar{E}$, so every recurrent point is also an almost recurrent point and a recurrent time an almost recurrent time.

Theorem 3: If T_τ be a measure-preserving transformation in a topological space S of finite measure and if E be any set of non-null interior, then the set A_τ of all almost recurrent points at an almost recurrent time τ is closed in E and is of non-null interior.

Proof.—Proceeding as in Theorem 1, we can see easily that

$$A_\tau = T_\tau^{-1}(\bar{E}_\tau \cap T_\tau(E)) = T_\tau^{-1}(\bar{E}_\tau) \cap E\,.$$

As $\bar{E}_\tau$ is closed and T_τ is topological, so $T_\tau^{-1}(\bar{E}_\tau)$ is closed, then A is closed in E by definition (Sierpinski 1952). Now if we take $G = {}_iE$
then

$$G_{1\tau} = T_\tau^{-1}(G \cap T_\tau(G)) = T_\tau^{-1}(G) \cap G$$

$$\subset T_\tau^{-1}(\bar{E}) \cap E = A_\tau\,,$$

since

$$G = {}_iE \subset E \subset \bar{E}.$$

As $G_{1\tau}$ is open, so A_τ has a non-null interior.

Note 1.—A set of non-null interior is of positive measure.

Note 2.—It is easy to see that at a recurrent time the set of almost recurrent points of any set E, which are not recurrent, is contained in the inverse image of the frontier set of E and so is of measure zero.

4. Recurrent and Non-recurrent Sets

Theorem 4: The set R of all recurrent points of an open set is an open set.

Proof.—Now $R = \cup_\tau R_\tau$, where the summation is taken over all recurrent times. From the fundamental property of topological space, R is open as each R_τ is open.

Theorem 5: The set N of all non-recurrent points of any set E is a border set.

Proof.—If possible, let x be an interior point of N. Then there exists an open set G of non-recurrent points containing x where $m(G)$ is positive. But

$G \subset N$, so $m(G) \leqslant m(N) = 0$ by the theorem stated in tbe Introduction. Thus we get a contradiction.

Then the interior of N is void. That is, N is a border set.

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