INVARIANT MEANS ON TOPOLOGICAL GROUPS

by GURNAM KAUR *and* J. N. MANOCHA, *Department of Mathematics, Panjab University, Chandigarh*

(Communicated by H. Gupta, F.N.I.)

(*Received April 26, 1966*)

Let G be an Abelian topological group. In general, it is conjectured that there is only one invariant mean if and only if G is compact. The if-part is proved by Luthar (1962). In this paper we have proved that G has many invariant means if G has property P. We say that G has property P if G or a factor group of G has property P'. G is said to have property P' if there exists an infinite group H of G and symmetric neighbourhoods V, W of O in G such that

(1) $(V+V) \cap H = \{0\}$,

(2) V is maximal among the symmetric neighbourhoods of O which satisfy (1),

(3) $|(V+V+W) \cap H| < |H|$,

where for any set A, $|A|$ denotes its cardinality. This gives an improvement on a result proved by Luthar (1962).

Let G be an Abelian topological group, i.e. an Abelian group, which is provided with a Hausdorff topology under which the mapping $(g, h) \to g-h$ of $G \times G$ into G is continuous. Let $C(G)$ denote the space of continuous, bounded, real-valued functions x on G with $\|x\| = \sup_{g \in G} |x(g)|$. An element μ in $C(G)^*$, the conjugate space of $C(G)$, is called a mean if $\|\mu\| = 1 = \mu(e)$, where $e \in C(G)$ is the function which takes the value 1 at every $g \in G$. A mean μ is called an invariant mean if

$$\mu(T_g x) = \mu(x) \qquad g \in G,\ x \in C(G).$$

Here T_g is the operator from $C(G)$ to $C(G)$ defined by

$$(T_g x)(g') = x(g+g') \quad g' \in G,\ x \in C(G).$$

It is known that such a μ always exists (G is Abelian) (Day 1960). If G is discrete, then μ is unique (i.e. there is exactly one invariant mean) if and only if G is finite (Day 1957). It is conjectured that, in general, there is only one invariant mean if and only if G is compact. The if-part has been proved by Luthar (1962). The problem, therefore, is to prove the remaining part, i.e. to prove the existence of many invariant means when G is not compact. The following result in this direction is proved in Luthar (1962).

(*) If G satisfies condition P, then G has many invariant means. We say that G has property P if G or a factor group of G has property P'. G is said to have property P' if there exists a countably infinite subgroup H of G and symmetric neighbourhoods V and W of O in G such that

(1) $(V+V)\cap H=\{0\}$,

(2) V is maximal among the symmetric neighbourhoods of O which satisfy (1),

(3) $(V+V+W)\cap H$ is finite.

The object of this note is to improve the result (*). More specifically we prove the following theorem:

THEOREM: If G has property P_1, then G has many invariant means. We say that G has property P_1 if G or a factor group of G has property P_1'. G is said to have property P_1' if there exists an infinite subgroup H of G and symmetric neighbourhoods V and W of O in G such that

(1) $(V+V)\cap H=\{0\}$,

(2) V is maximal among the symmetric neighbourhoods of O which satisfy (1),

(3) $|(V+V+W)\cap H|<|H|$,

where for any set A, $|A|$ denotes its cardinality. It is obvious that property P_1 is weaker than property P and so our result is stronger than the result (*).

PROOF: By virtue of the fact that the existence of many invariant means on a factor group implies the existence of many invariant means on the group itself, we may assume that G has P_1'. We now note that the only place in the proof of (*) where condition (3) is used is in the construction of two subsets A and B of H such that

(α) $(A+V+W)\cap(B+V)=\phi$.

(β) Given $h_1,\ldots,h_t\in H$, there exist h and $h'\in H$ such that

$$h_i+h\in A,\ h_i+h'\in B,\quad 1\leqslant i\leqslant t.$$

Our object will be achieved if we can prove the existence of such two sets A and B. If H is countable, this is done in (3).

So suppose H is not countable. Let A_1 be the subgroup of H generated by $(V+V+W)\cap H$. Then $|A_1|=|(V+V+W)\cap H|<|H|$. Suppose we have defined $A_1, A_2,\ldots,A_p$ such that $H_p=\bigcup_{j=1}^{p}A_j$ is a subgroup of H of cardinality less than that of H. Let $h\in H$ and $h\notin H_p$ (such an h exists) and let A_{p+1} be the complement of H_p in the subgroup generated by H_p and h. Then $H_{p+1}=\bigcup_{j=1}^{p+1}A_j$ is a subgroup of H of cardinality less than that of H.

Let

$$A' = \bigcup_{j=1}^{\infty} A_{2j},\ B' = \bigcup_{j=0}^{\infty} A_{2j+1} \text{ and } H' = \bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} H_j.$$

We claim

(α') $(A'+V+W) \cap (B'+V) = \phi.$

(β') Given $h'_1, \ldots, h'_t \in H'$ there exist h' and h'' such that

$$h'_i + h' \in A',\ h'_i + h'' \in B', \quad 1 < i < t.$$

To prove (α'), suppose

$$d \in (A'+V+W) \cap (B'+V);$$

so there exist $a' \in A'$, $b' \in B'$, $w \in W$ and v, $v' \in V$ such that

$$d = a'+v+w = b'+v'.$$

Let

$$a' = a_{2j} \in A_{2j},\ b' = b_{2k+1} \in A_{2k+1},$$

then

$$a_{2j} - b_{2k+1} = v - v' - w \in V+V+W \quad [V \text{ and } W \text{ are symmetric}]$$

$$a_{2j} - b_{2k+1} \in (V+V+W) \cap H$$

and so

$$a_{2j} - b_{2k+1} \in A_1 \subset H_{2j},\ H_{2k+1}.$$

Therefore,

$$a_{2j} - b_{2k+1} \in H_{2j} \cap H_{2k+1} = H_{\min}(2j,\ 2k+1)$$

which is impossible.

To prove (β'), let

$$h'_i \in H_{j_i} \quad \left(h'_i \in H' = \bigcup_{p=1}^{\infty} H_p\right).$$

Let $\max_{1<i<t} \{j_i\} = m$. Take $2n \geqslant m+1$, then there exists h' such that h' belongs to H_{2n} but not to H_{2n-1}.

Now $h'_i + h' \in H_{2n}$ (because $h'_i \in H_{j_i} \subset H_{2n}$), and $h'_i + h'$ does not belong to H_{2n-1} (for $h'_i \in H_{j_i} \subset H_{2n-1}$).

Consequently $h'_i + h' \in A_{2n} \subset A'$.

Similarly by taking $2n+1 \geqslant m+2$, we can prove the existence of h'' belonging to H_{2n+1} such that $h'_i + h'' \in A_{2n+1} \subset B'$.

This proves (β'). Now we decompose H into the cosets of H' in H.

Let

$$H = H' \cup (x+H') \cup (y+H') \ldots$$

Let now

$$A = A' \cup (x+A') \cup (y+A') \ldots$$

$$B = B' \cup (x+B') \cup (y+B') \ \ldots$$

We claim A and B satisfy the conditions (α) and (β).
To prove (α), let

$$d \in (A+V+W) \cap (B+V).$$

Then there exist x, $y \in H$; $a \in A'$, $b \in B'$; v, $v' \in V$ and $w \in W$ such that

$$x+a'+v+w = y+b'+v',$$

$$\text{i.e. } x+a'-b'-y \in V+V+W,$$

$$\text{i.e. } x+a'-b'-y \in (V+V+W) \cap H \subset H'.$$

So $x-y \in H'$, which gives $x = y$.
Therefore, $a'+v+w = b'+v'$ which is impossible by (α').
To prove (β), let

$$h_1, \ldots, h_t \in H, \text{ then } h_i = x_i+h_i' \quad (h_i' \in H').$$

By (β') there exist h' and h'' such that $h_i'+h' \in A'$, $h_i'+h'' \in B'$.
Therefore,

$$h_i+h' = x_i+h_i'+h' \in x_i+A' \subset A,$$

$$h_i+h'' = x_i+h_i'+h'' \in x_i+B' \subset B.$$

This proves (β), which completes the proof of our theorem.

References

Day, M. M. (1957). Amenable semi-groups. *Illinois J. Math.*, **1**, 509–544.
——— (1960). Means for the bounded functions and Ergodicity of the bounded representations of semigroups. *Trans. Am. math. Soc.*, **69**, No. 2, 276–291.
Luthar, I. S. (1962). Uniqueness of the invariant mean on Abelian topological semi-groups. *Trans. Am. math. Soc.*, **104**, 403–411.