EXTREME QUADRATIC FORMS FOR COVERINGS IN FOUR VARIABLES

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In this paper we prove that the polynomial $\phi_1 = \sum x_i^2 + \sum x_i x_j$, $i, j, = 1, 2, \ldots, 4$ which is extreme in the packing case is not extreme in the covering case.

The object of this paper is to prove that the polynomial

$$\phi_1 = \Sigma x_i^2 + x_i x_j \quad i, j = 1, 2, \ldots, 4$$

which is extreme in the packing case is not extreme in the covering case.

We start with the following definitions.

Let X denote the n-tuple (x_i), $i = 1, 2, \ldots, n$. Let $f(x)$ be a positive definite quadratic form of determinant D_f. The inhomogeneous minimum m_f of f is defined as

$$m_f = \max_{\alpha \in R_n} \min_{X \in \Lambda_0} f(\alpha + X)$$

where R_n denotes the n-dimensional Euclidean space and Λ_0 is the n-dimensional fundamental lattice. Let θ_n be the density of the most economical lattice covering of R_n by n-dimensional spheres. Then we know that

$$\left(\frac{\theta_n}{J_n}\right)^{2/n} = \min_f \frac{m_f}{D_f^{1/n}}$$

where J_n is the volume of the unit sphere $x_1^2 + x_2^2 + \ldots + x_n^2 < 1$.

A positive definite quadratic form $f(X)$ is said to be extreme if the ratio $\frac{m_f}{D_f^{1/n}}$ is a local minimum, i.e. if it does not decrease by any sufficiently small variation of the coefficients.

If f is extreme, so clearly is cf for all real $c > 0$. Also forms equivalent under unimodular transformations of f are also extreme. Identifying all such forms the question arises, 'How many extreme forms are there for a given n?'. It has been shown by Bleicher (1962) that for large n there are at least two such forms. For $n = 2, 3$, Barnes (1956) has shown that there is only one extreme form. It is not known how many extreme forms there are for $n = 4$. It is known that

$$f_1 = 4\,\Sigma x_i^2 - 2\,\Sigma x_i x_j,\ i, j = 1, 2, \ldots, 4$$

is extreme for $n = 4$, while

$$\phi_2 = \Sigma\,(x_i^2 + x_i x_j) - x_1 x_2,\ i, j = 1, \ldots, 4$$

and its adjoint are not extreme. If we could prove that there is exactly one extreme form, it will follow that the best lattice covering of spheres in R_4 corresponds to f_1 and has density $\theta_4 = (2\pi^2/5\sqrt{5})$ as conjectured by Bambah (1964). In the corresponding packing case, it was shown by Korkine and Zolotareff (1877) that ϕ_1 and ϕ_2 are the only extreme forms. Now, f_1 is adjoint to ϕ_1 whereas ϕ_2 is self-adjoint and is known to be not extreme. From now on, $i, j = 1, 2, \ldots, 4$.

We shall prove now that ϕ_1 is not extreme. In order to show this we note that it is equivalent to showing that $f(x) = 2\phi_1$ is not extreme. Now

$$f(x) = 2\,\Sigma\,(x_i^2 + x_i x_j)$$
$$= \Sigma\,x_i^2 + (x_1 + x_2 + x_3 + x_4)^2.$$

We replace $f(x)$ by an equivalent form by applying the unimodular transformation $x_1 \to x_1 - x_2$, $x_2 \to x_2$, $x_3 \to -x_3 + x_4$, $x_4 \to -x_4$ and get

$$f(x) = x_2^2 + x_4^2 + (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_3 - x_4)^2.$$

This is a Voronoi form (1909) of type I.

Let P be the Voronoi parallelohedron consisting of points X such that $f(X) \leqslant f(X - A)$ for all $A \in \Lambda_0$. By the method of Voronoi (1908) we know that P is the intersection of half-strips $f(X) \leqslant f(X \pm A)$ where $A \in \Sigma$, a subset of Λ_0 defined as follows.

Let S be a set consisting of the following points:

(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (1,1,1,0), (1,1,0,0), (1,0,1,1), (0,1,1,1), (1,1,1,1). Then Σ consists of those points of S which satisfy the following conditions:

$$A \in \Sigma$$

if and only if the minimum of $f(X)$ for all $X \equiv A$ (mod 2) is attained only at $\pm A$.

The faces of P are then $f(X) = f(X \pm A)$, $A \in \Sigma$. The vertices of P are the intersections of faces lying in P.

Applying this criterion we find that the faces of the parallelohedron P corresponding to f arise from all the points of S except for the points (1,0,0,1), (0,1,1,0), (1,1,0,1), (0,1,0,1), (0,1,1,1). The faces of P can be easily got by the formula $f(X) = f(X \pm A)$ where A runs over all the points of S except the above five points. We get twenty faces. By the method of Voronoi (1909) we get distinct vertices of types I, II, VI and X only:

$V_1 = (\frac{6}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5})$ represents a vertex of type I.

$V_2 = (\frac{2}{5}, \frac{1}{5}, -\frac{2}{5}, -\frac{1}{5})$ represents a vertex of type II.

$V_3 = (\frac{4}{5}, \frac{2}{5}, \frac{1}{5}, -\frac{1}{5})$ represents a vertex of type VI.

$V_4 = (\frac{1}{5}, \frac{3}{5}, -\frac{1}{5}, -\frac{3}{5})$ represents a vertex of type X.

The reflexions of some of these vertices coincide with one another. Vertices of all other types coincide with these vertices.

$f(V)$ is the same for all vertices of the same type.
Then the inhomogeneous minimum

$$m_f = \max_{\alpha \in R_4} \min_{X \in \Lambda_0} f(X+\alpha) = \max_{V \in \{V_1, V_2, V_3, V_4\}} f(V) = \tfrac{30}{25}.$$

Moreover, the determinant D_f of f is 5. Therefore

$$\phi_4(f) = \frac{m_f^2}{D_f^{1/2}} = \frac{36}{25\sqrt{5}}.$$

Now consider the neighbouring forms

$$f'(x) = (1-\epsilon)x_2^2 + (1-\epsilon)x_4^2 + (x_1-x_2)^2 + (x_1-x_3)^2 + \epsilon(x_2-x_4)^2 + (x_3-x_4)^2,$$

where $\epsilon > 0$ is small.
This is also a Voronoi form of type I.
As before, the faces of the Voronoi parallelohedron P' correspond to all those points to which the faces of P correspond.
We get 20 faces by the formula $f'(X) = f'(X \pm A)$. By the method of Voronoi (1909) we get distinct vertices of types I, II, VI and VIII only.

$V_1' = \left(\frac{6+4\epsilon}{d}, \frac{3+3\epsilon}{d}, \frac{4+2\epsilon}{d}, \frac{2}{d}\right)$ represents a vertex of type I,

$V_2' = \left(\frac{2+\epsilon}{d}, \frac{1}{d}, -\frac{2+\epsilon}{d}, -\frac{1}{d}\right)$ represents a vertex of type II,

$V_3' = \left(\frac{4+2\epsilon}{d}, \frac{2}{d}, \frac{1+\epsilon}{d}, -\frac{2}{d}\right)$ represents a vertex of type VI, and

$V_4' = \left(\frac{1}{d}, \frac{3}{d}, \frac{4+3\epsilon}{d}, \frac{2+3\epsilon}{d}\right)$ represents a vertex of type VIII,

where $d = 5+3\epsilon$.
The vertices of all other types coincide with these vertices.

By examination of these vertices we get

$$m_{f'} = \frac{30+28\epsilon+6\epsilon^2}{(5+3\epsilon)^2}.$$

Also

$$D_{f'} = (1-\epsilon)(5+3\epsilon).$$

Therefore

$$\phi_4(f') = \frac{36}{25\sqrt{5}} \frac{\left(1+\frac{28}{15}\epsilon+\dots\right)}{\left(1+\frac{11}{5}\epsilon+\dots\right)}$$

$$< \frac{36}{25\sqrt{5}} = \phi_4(f)$$

if $\epsilon > 0$ is sufficiently small.

Hence f is not extreme.

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