CIRCULAR INCLUSION IN AN ELASTIC HALF PLANE

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This paper deals with the problem of a circular inclusion undergoing spontaneous dimensional changes in an elastic half plane. Explicit expressions for complex potentials, stresses and displacements are given.

INTRODUCTION

Previous papers dealing with 'Inclusion Problems in Two Dimensions' confine themselves to infinite elastic media. In this paper we extend the analysis to a homogeneous and isotropic elastic half plane: a circular region (inclusion) of the half plane undergoes non-elastic deformations which in the absence of the surrounding material (matrix) would be homogeneous in nature. Owing to the elastic constraints of the matrix a system of locked-up accommodation stresses is set up in both the inclusion and the matrix. The purpose of this paper is to construct complex potential functions for the problem satisfying appropriate boundary conditions. Eshelby's (1957) hypothetical point-force layer method has been employed for the purpose. This, in brief, hypothesizes the equivalence of the effect of a deforming inclusion and the effect of a continuous layer of point-forces along the boundary of the inclusion. The validity and the success of this hypothesis have been demonstrated in previous papers (Eshelby 1957; Jaswon and Bhargava 1961).

Let the material occupy the half of the complex plane where $y \geqslant 0$ ($z = x+iy$). If l is the distance of the centre of the circular inclusion from the leading edge and c its radius, then $(z-il)(\bar{z}+il) < c^2$ can be taken to represent the inclusion, y-axis passing through the centre as shown in Fig. 1. The remaining region is the matrix. All quantities like complex potentials, stresses and displacements pertaining to the inclusion will be marked with subscript 'i', whereas those for the matrix will have the subscript 'm'.

STATEMENT OF THE PROBLEM

The inclusion in the absence of the surrounding material will undergo a displacement characterized by

$$u = \delta_1 x + \delta_3(y-l),$$

$$v = \delta_2(y-l) + \delta_3 x$$

whence the homogeneous strains are

$$e_{xx} = \delta_1, \; e_{yy} = \delta_2 \text{ and } e_{xy} = \delta_3,$$

where δ_1, δ_2 and δ_3 are small constants of the order of magnitude admissible in the linear theory. This state of the inclusion is referred to as the 'free state'. The presence of the matrix restricts this deformation as far as possible. Thus the whole material is self-stressed. The problem is to find the displacement and the accompanying stress field both in the inclusion and in the matrix.

Boundary Conditions

The following are the boundary conditions to be satisfied by the final solution to this problem:

(*a*) The normal and the tangential stresses at the leading edge $y = 0$ must vanish.

(*b*) Assuming a perfect bond between the inclusion and the matrix, the normal and the tangential stresses must assume the same value on $(\zeta - il)(\bar{\zeta} + il) = c^2$, where $z = \zeta$ on the boundary of the inclusion.

(*c*) The displacements on $(\zeta - il)(\bar{\zeta} + il) = c^2$ must be identical for both the regions.

Complex Potentials, Isolated Force

It is well known that any two-dimensional problem can be reduced to that of finding two complex variable potential functions $\phi(z)$ and $\psi(z)$ satisfying the appropriate boundary conditions. These are related to the stresses and displacements by (Muskhelishvili 1953) the following relations:

$$\left.\begin{aligned} \sigma_x + \sigma_y &= 4Rl[\phi'(z)] \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2[\bar{z}\phi''(z) + \psi'(z)] \end{aligned}\right\} \qquad (1)$$

and

$$2\mu(u + iv) = \alpha\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \qquad (2)$$

where $\alpha = (3-\nu)/(1+\nu)$ for this problem of generalized plane stress, ν being Poisson's ratio. The remaining symbols have their usual meaning.

Complex potentials for a concentrated force $P = P_1 + iP_2$ acting at any point ζ of the half plane have been derived already by others, for example they have been given in an indirect form (Green and Zerna 1964). Their explicit expressions are

$$\phi'(z) = \frac{-P}{2\pi(\alpha+1)}\left(\frac{1}{z-\zeta} + \frac{\alpha}{z-\bar{\zeta}}\right) + \frac{\bar{P}}{2\pi(\alpha+1)}\left(\frac{1}{z-\bar{\zeta}} - \frac{z-\zeta}{(z-\bar{\zeta})^2}\right),$$

$$\psi'(z) = \frac{\bar{P}}{2\pi(\alpha+1)}\left(\frac{\alpha}{z-\zeta} + \frac{3z-\zeta}{(z-\bar{\zeta})^2} - \frac{2z(z-\zeta)}{(z-\bar{\zeta})^3}\right) + \frac{P}{2\pi(\alpha+1)}\left(\frac{-\bar{\zeta}}{(z-\zeta)^2} + \frac{\alpha}{z-\bar{\zeta}} - \frac{\alpha z}{(z-\bar{\zeta})^2}\right),$$

where $\bar{P}$ is the complex conjugate of P. Now the cumulative effect of a continuous distribution of point-forces acting along any finite arc in the finite part of the half plane could be described by the following integrals:

$$\phi'(z) = \frac{1}{2\pi(\alpha+1)}\left[-\int_\gamma \frac{P\,ds}{z-\zeta} - \int_\gamma \frac{\alpha P\,ds}{z-\bar{\zeta}} + \int_\gamma \frac{\bar{P}\,ds}{z-\bar{\zeta}} - \int_\gamma \frac{\bar{P}(z-\zeta)\,ds}{(z-\bar{\zeta})^2}\right], \qquad (3)$$

$$\psi'(z) = \frac{1}{2\pi(\alpha+1)}\left[\int_\gamma \frac{\alpha\bar{P}\,ds}{z-\zeta} + \int_\gamma \frac{\bar{P}(3z-\zeta)\,ds}{(z-\bar{\zeta})^2} - \int_\gamma \frac{2\bar{P}z(z-\zeta)\,ds}{(z-\bar{\zeta})^3}\right.$$

$$\left. - \int_\gamma \frac{P\bar{\zeta}\,ds}{(z-\zeta)^2} + \int_\gamma \frac{\alpha P\,ds}{(z-\bar{\zeta})} - \int_\gamma \frac{\alpha P z\,ds}{(\bar{\zeta}-z)^2}\right], \qquad (4)$$

where 'γ' is the arc, ds the differential arc length and ζ any point on it.

The Inclusion Problem

According to Eshelby's point-force concept the inclusion problem can be solved by assuming an appropriate layer of point-forces along the inclusion boundary. The expression for such a layer as would come into play in the case of this problem can be found from the reversed surface tractions:

$$\left.\begin{aligned} \sigma_x^0 &= \lambda(\delta_1+\delta_2)+2\mu\delta_1 \\ \sigma_y^0 &= \lambda(\delta_1+\delta_2)+2\mu\delta_2 \\ \tau_{xy}^0 &= 2\mu\delta_3 \end{aligned}\right\}, \qquad (5)$$

and the formulae (Jaswon and Bhargava 1961)

$$\left.\begin{aligned} P\,ds &= -\frac{i}{2}\left[(\sigma_x^0+\sigma_y^0)\,d\zeta-(\sigma_x^0-\sigma_y^0)\,d\bar{\zeta}\right]+\tau_{xy}^0\,d\bar{\zeta} \\ \bar{P}\,ds &= -\frac{i}{2}\left[(\sigma_x^0-\sigma_y^0)\,d\zeta-(\sigma_x^0+\sigma_y^0)\,d\bar{\zeta}\right]+\tau_{xy}^0\,d\zeta \end{aligned}\right\}. \qquad (6)$$

Case A

Principal Strains.—It is convenient here to break the problem in two parts and consider first the principal strains only ($\delta_3 = 0$).
Thus

$$\left.\begin{aligned} P\,ds &= -ik_1\,d\zeta+ik_2\,d\bar{\zeta} \\ \bar{P}\,ds &= -ik_2\,d\zeta+ik_1\,d\bar{\zeta} \end{aligned}\right\}, \qquad (7)$$

where $k_1 = (\lambda+\mu)(\delta_1+\delta_2)$, $k_2 = \mu(\delta_1-\delta_2)$ and λ and μ are Lamé's constants.

We substitute for $P\,ds$ and $\bar{P}\,ds$ from the above eqns. (3) and (4) and evaluate the contour integrals. It may be noted that now γ is the boundary of the inclusion $(\zeta-il)(\bar{\zeta}+il) = c^2$. Also from this relation it follows that $\bar{\zeta} = \dfrac{c^2}{\zeta-il}-il$, from which we have $d\bar{\zeta} = \dfrac{-c^2}{(\zeta-il)^2}\,d\zeta$. These relations are given here because they have to be used in integration. The expressions become particularly simple by the substitutions:

$$\left.\begin{aligned} z_1 &= z+il \\ &= r_1 e^{i\theta_1} \end{aligned}\right\} \text{ and } z_2 = z-il = r_2 e^{i\theta_2}.$$

Thus

$$\phi_i'(z) = \frac{k_1}{\alpha+1} - k_1\frac{\alpha-1}{\alpha+1}\frac{c^2}{z_1^2} + \frac{k_2}{\alpha+1}\left(\frac{-c^2}{z_1^2} + \frac{4ilc^2}{z_1^3} + \frac{3c^4}{z_1^4}\right), \quad \ldots \quad (8)$$

$$\psi_i'(z) = k_1\frac{\alpha-1}{\alpha+1}\left(\frac{-c^2}{z_1^2} + \frac{2ilc^2}{z_1^3}\right) \frac{-k_2}{\alpha+1}\left(\alpha - \frac{10ilc^2}{z_1^3} - \frac{12l^2c^2}{z_1^4} - \frac{9c^4}{z_1^4} + \frac{12ilc^4}{z_1^5}\right), \quad (9)$$

$$\phi_m'(z) = -k_1\frac{(\alpha-1)}{(\alpha+1)}\frac{c^2}{z_1^2} + \frac{k_2}{\alpha+1}\left(\frac{-c^2}{z_1^2} + \frac{4ilc^2}{z_1^3} + \frac{3c^4}{z_1^4} - \frac{c^2}{z_2^2}\right), \quad \ldots \quad (10)$$

$$\psi_m'(z) = k_1\frac{\alpha-1}{\alpha+1}\left(\frac{c^2}{z_2^2} - \frac{c^2}{z_1^2} + \frac{2ilc^2}{z_1^3}\right) - \frac{k_2}{\alpha+1}\left(\frac{3c^4}{z_2^4} - \frac{2ilc^2}{z_2^3} - \frac{10ilc^2}{z_1^3} - \frac{12l^2c^2}{z_1^4} - \frac{9c^4}{z_1^4} + \frac{12ilc^4}{z_1^5}\right). \quad \ldots \quad (11)$$

Stress Fields.—By means of relation (1) we can find the stress fields from the above potentials. But it must be emphasized here that the inclusion had an initial stress field $(-\sigma_x^0, -\sigma_y^0, 0)$ and this must be added to the one got from the complex potentials $\phi_i'(z)$ and $\psi_i'(z)$. Moreover, at this stage it can be verified that the boundary conditions on the stresses on the edge $y = 0$ and at the inclusion boundary $(\zeta - il)(\zeta + il) = c^2$ are satisfied.

Cartesian Form of Stresses.—Stresses in Cartesian form are given below, where we put

$$y_1 = y+l,\ y_2 = y-l,\ r_1^2 = x^2+y_1^2,\ r_2^2 = x^2+y_2^2 \text{ and } r^2 = x^2+y^2.$$

$$\begin{aligned}(\sigma_x)_i = &-k_1\frac{\alpha-1}{\alpha+1}\left[1 + \frac{c^2(x^2+y_1^2)}{r_1^4} + \frac{4lc^2(3y_1x^2-y_1^3)}{r_1^6} + \frac{2c^2(x^4-6x^2y_1^2+y_1^4)}{r_1^6}\right] \\ &-\frac{k_2}{\alpha+1}\left[1 + \frac{2c^2(x^2-y_1^2)}{r_1^4} + \frac{4lc^2(3y_1x^2-y_1^3)}{r_1^6}\right. \\ &+\frac{(2c^2r_1^2+3c^4+24l^2c^2)(x^4-6x^2y_1^2+y_1^4)}{r_1^8} \\ &-\frac{12lc^2(r_1^2+2c^2)(5y_1x^4-10x^2y_1^3+y_1^5)}{r_1^{10}} \\ &\left.-\frac{12c^4(x^6-15y_1^2x^4+15y_1^4x^2-y_1^6)}{r_1^{10}}\right],\end{aligned}$$

$$(\sigma_y)_i = -k_1 \frac{\alpha-1}{\alpha+1}\left[1+\frac{3c^2(x^2-y_1^2)}{r_1^4}-\frac{4lc^2(3y_1x^2-y_1^3)}{r_1^6}-\frac{2c^2(x^4-6x^2y_1^2+y_1^4)}{r_1^6}\right]$$

$$+\frac{k_2}{\alpha+1}\left[1-\frac{2c^2(x^2-y_1^2)}{r_1^4}+\frac{20lc^2(3y_1x^2-y_1^3)}{r_1^6}+\frac{c^2(2r_1^2+15c^2+24l^2)(x^4-6x^2y_1^2+y_1^4)}{r_1^8}\right.$$

$$\left.-\frac{12lc^2(r_1^2+2c^2)(5y_1x^4-10x^2y_1^3+y_1^5)}{r_1^{10}}-\frac{12c^4(x^6-15y_1^2x^4+15y_1^4x^2-y_1^6)}{r_1^{10}}\right],$$

$$(\tau_{xy})_i = k_1\frac{\alpha-1}{\alpha+1}xc^2\left[\frac{2y_1}{r_1^4}+\frac{4l(x^2-3y_1^2)}{r_1^6}+\frac{8(y_1^3-x^2y_1)}{r_1^6}\right]$$

$$+\frac{k_2xc^2}{\alpha+1}\left[\frac{4(2r_1^2+12l^2+9c^2)(y_1^3-x^2y_1)}{r_1^8}-\frac{12l(r_1^2+2c^2)(x^4-10x^2y_1^2+5y_1^4)}{r_1^{10}}\right.$$

$$\left.+\frac{12l(x^2-3y_1^2)}{r_1^6}+\frac{24c^2(3y_1x^4-10y_1^3x^2+3y_1^5)}{r_1^{10}}\right].$$

The stresses in the matrix are given directly by the potentials $\phi_m'(z)$ and $\psi_m'(z)$ given in the relations (10) and (11).

$$(\sigma_x)_m = -k_1\frac{\alpha-1}{\alpha+1}c^2\left[\frac{(x^2-y_1^2)}{r_1^4}+\frac{(x^2-y_2^2)}{r_2^4}+\frac{4l(3y_1x^2-y_1^3)}{r_1^6}+\frac{2(x^4-6x^2y_1^2+y_1^4)}{r_1^6}\right]$$

$$-k_2\frac{c^2}{\alpha+1}\left[\frac{2(x^2-y_2^2)}{r_2^4}+\frac{2(x^2-y_1^2)}{r_1^4}+\frac{4l(3y_1x^2-y_1^3)}{r_1^6}\right.$$

$$+\frac{(2r_1^2+3c^2+24l^2)(x^4-6x^2y_1^2+y_1^4)}{r_1^8}+\frac{(2r_2^2-3c^2)(x^4-6x^2y_2^2+y_2^4)}{r_2^8}$$

$$\left.-\frac{12l(r_1^2+2c^2)(5y_1x^4-10x^2y_1^3+y_1^5)}{r_1^{10}}-\frac{12c^2(x^6-15y_1^2x^4+15y_1^4x^2-y_1^6)}{r_1^{10}}\right],$$

$$(\sigma_y)_m = k_1\frac{\alpha-1}{\alpha+1}c^2\left\{\frac{(x^2-y_2^2)}{r_2^4}-\frac{3(x^2-y_1^2)}{r_1^4}+\frac{4l(3y_1x^2-y_1^3)}{r_1^6}+\frac{2(x^4-6x^2y_1^2+y_1^4)}{r_1^6}\right\}$$

$$+k_2\frac{c^2}{\alpha+1}\left\{-\frac{2(x^2-y_1^2)}{r_1^4}-\frac{2(x^2-y_2^2)}{r_2^4}+\frac{20l(3y_1x^2-y_1^3)}{r_1^6}\right.$$

$$+\frac{(2r_1^2+15c^2+24l^2)(x^4-6x^2y_1^2+y_1^4)}{r_1^8}+\frac{(2r_2^2-3c^2)(x^4-6x^2y_2^2+y_2^4)}{r_2^8}$$

$$\left.-\frac{12l(r_1^2+2c^2)(5y_1x^4-10x^2y_1^3+y_1^5)}{r_1^{10}}-\frac{12c^2(x^6-15y_1^2x^2+15x^2y_1^4-y_1^6)}{r_1^{10}}\right\},$$

$$(\tau_{xy})_m = k_1 \frac{\alpha-1}{\alpha+1} xc^2 \left\{ \frac{2y_1}{r_1^4} - \frac{2y_2}{r_2^4} + \frac{4l(x^2-3y_1^2)}{r_1^6} + \frac{8(y_1^3-x^2y_1)}{r_1^6} \right\}$$

$$+ \frac{k_2}{\alpha+1} xc^2 \left\{ \frac{(8r_2^2-12c^2)(y_2^3-x^2y_2)}{r_2^8} - \frac{12l(r_1^2+2c^2)(x^4-10x^2y_1^2+5y_1^4)}{r_1^{10}} \right.$$

$$\left. + \frac{4(2r_1^2+24l^2+9c^2)(y_1^3-x^2y_1)}{r_1^8} + \frac{12l(x^2-3y_1^2)}{r_1^6} + \frac{24c^2(3y_1x^4-10x^2y_1^3+3y_1^5)}{r_1^{10}} \right\}.$$

Stresses on the Edge.—The non-vanishing component of stress on the edge can be found from the expression for $(\sigma_x)_m$ given above by putting $y = 0$,

$$(\sigma_x)_{y=0} = -4k_1 \frac{\alpha-1}{\alpha+1} c^2 \frac{(x^2-l^2)}{r_1^4} + \frac{4k_2}{\alpha+1} c^2 \left\{ - \frac{2(x^2-l^2)}{r_1^4} \right.$$

$$\left. + \frac{4l^2(3x^2-l^2)}{r_1^6} + \frac{3c^2(x^4-6x^2l^2+l^4)}{r_1^8} \right\}.$$

Stresses at the Boundary of the Inclusion.—The normal and the tangential stresses continuously transmitted by the bond are given by

$$(\sigma_n)_{z=\zeta} = -k_1 \frac{\alpha-1}{\alpha+1} \left[1 + \frac{2c^2}{r_1^2} \cos 2\theta_1 + \frac{2rc^2}{r_1^3} \cos (2\theta_2-3\theta_1-\theta) - \frac{c^2}{r_1^2} \cos (2\theta_2-2\theta_1) \right.$$

$$\left. - \frac{2lc^2}{r_1^3} \sin (2\theta_2-3\theta_1) \right] - \frac{k_2}{\alpha+1} \left[\cos 2\theta_2 + \frac{2c^2}{r_1^2} \cos 2\theta_1 \right.$$

$$- \frac{8lc^2}{r_1^3} \sin 3\theta_1 - \frac{6c^4}{r_1^4} \cos 4\theta_1 + \frac{2rc^2}{r_1^3} \cos (2\theta_2-3\theta_1-\theta)$$

$$+ \frac{12lc^2r}{r_1^4} \sin (2\theta_2-4\theta_1-\theta) - \frac{12rc^4}{r_1^5} \cos (2\theta_2-5\theta_1-\theta)$$

$$- \frac{10lc^2}{r_1^3} \sin (2\theta_2-3\theta_1) + \frac{3c^2(3c^2+4l^2)}{r_1^4} \cos (2\theta_2-4\theta_1)$$

$$\left. + \frac{12lc^4}{r_1^5} \sin (2\theta_2-5\theta_1) \right],$$

$$(\tau_{ns})_{z=\zeta} = k_1 \frac{\alpha-1}{\alpha+1} c^2 \left[\frac{2r}{r_1^3} \sin (2\theta_2-3\theta_1-\theta) - \frac{1}{r_1^2} \sin (2\theta_2-2\theta_1) \right.$$

$$\left. + \frac{2l}{r_1^3} \cos (2\theta_2-3\theta_1) \right] + \frac{k_2}{\alpha+1} \left[\sin 2\theta_2 + \frac{2rc^2}{r_1^3} \sin (2\theta_2-3\theta_1-\theta) \right.$$

$$- \frac{12lc^2r}{r_1^4} \cos (2\theta_2-5\theta_1-\theta) - \frac{12c^4r}{r_1^5} \sin (2\theta_2-5\theta_1-\theta)$$

$$+ \frac{10lc^2}{r_1^3} \cos (2\theta_2-3\theta_1) + \frac{3c^2(3c^2+4l^2)}{r_1^4} \sin (2\theta_2-4\theta_1)$$

$$\left. - \frac{12lc^4}{r_1^5} \cos (2\theta_2-5\theta_1) \right].$$

The hoop stress is discontinuous across the boundary:

$$[(\sigma_s)_{z=\zeta}]_i = -k_1\frac{\alpha-1}{\alpha+1}\Big[1+\frac{2c^2}{r_1^2}\cos 2\theta_1+\frac{c^2}{r_1^2}\cos(2\theta_2-2\theta_1)$$

$$-\frac{2rc^2}{r_1^3}\cos(2\theta_2-3\theta_1-\theta)+\frac{2lc^2}{r_1^3}\sin(2\theta_2-3\theta_1)\Big]$$

$$+\frac{k_2}{\alpha+1}\Big[\cos 2\theta_2-\frac{2c^2}{r_1^2}\cos 2\theta_1+\frac{8lc^2}{r_1^3}\sin 3\theta_1$$

$$+\frac{6c^4}{r_1^4}\cos 4\theta_1+\frac{2rc^2}{r_1^3}\cos(2\theta_2-3\theta_1-\theta)$$

$$+\frac{12lrc^2}{r_1^4}\sin(2\theta_2-4\theta_1-\theta)-\frac{12rc^4}{r_1^5}\cos(2\theta_2-5\theta_1-\theta)$$

$$-\frac{10lc^2}{r_1^3}\sin(2\theta_2-3\theta_1)+\frac{3c^2}{r_1^4}(3c^2+4l^2)\cos(2\theta_2-4\theta_1)$$

$$+\frac{12lc^4}{r_1^5}\sin(2\theta_2-5\theta_1)\Big].$$

The jump in the hoop stress across the inclusion boundary is

$$[(\sigma_s)_{z=\zeta}]_i-[(\sigma_s)_{z=\zeta}]_m = -2k_1\frac{\alpha-1}{\alpha+1}+\frac{4k_2}{\alpha+1}\cos 2\theta_2.$$

Displacement Fields.—Integrating the relations (8), (9), (10) and (11) with respect to z we get

$$\phi_i(z) = \frac{k_1}{\alpha+1}z_2+k_1\frac{\alpha-1}{\alpha+1}\frac{c^2}{z_1}+\frac{k_2}{\alpha+1}\left(\frac{c^2}{z_1}-\frac{2ilc^2}{z_1^2}-\frac{c^4}{z_1^3}\right), \quad \ldots \quad (12)$$

$$\psi_i(z) = k_1\frac{\alpha-1}{\alpha+1}\left(\frac{c^2}{z_1}-\frac{ilc^2}{z_1^2}\right)+\frac{ilk_1}{\alpha+1}-\frac{k_2}{\alpha+1}\left(\alpha z_2+\frac{5ilc^2}{z_1^2}+\frac{4l^2c^2}{z_1^3}+\frac{3c^4}{z_1^3}-\frac{3ilc^4}{z_1^4}\right), \quad (13)$$

$$\phi_m(z) = k_1\frac{\alpha-1}{\alpha+1}\frac{c^2}{z_1}+\frac{k_2}{\alpha+1}\left(\frac{c^2}{z_1}+\frac{c^2}{z_2}-\frac{2ilc^2}{z_1^2}-\frac{c^4}{z_1^3}\right), \quad \ldots \quad (14)$$

$$\psi_m(z) = k_1\frac{\alpha-1}{\alpha+1}\left(\frac{c^2}{z_1}-\frac{c^2}{z_2}-\frac{ilc^2}{z_1^2}\right)-\frac{k_2}{\alpha+1}\left(\frac{5ilc^2}{z_1^2}+\frac{4l^2c^2}{z_1^3}+\frac{3c^4}{z_1^3}-\frac{3ilc^4}{z_1^4}+\frac{ilc^2}{z_2^2}-\frac{c^4}{z_2^3}\right). \quad \ldots \quad (15)$$

Here we have added suitable constants of integration so as to satisfy the continuity condition on displacements across the boundary. The displacement fields both in the inclusion and in the matrix are given directly by the complex potentials and eqn. (2).

In Cartesian form the displacements are

$$(2\mu u)_i = k_1\frac{\alpha-1}{\alpha+1}x\left[1-\frac{c^2}{r_1^2}+\frac{\alpha c^2}{r_1^2}+\frac{4lc^2y_1}{r_1^4}+\frac{c^2(x^2-3y_1^2)}{r_1^4}\right]$$
$$+\frac{\alpha k_2}{\alpha+1}x\left[1+\frac{c^2}{r_1^2}-\frac{4lc^2y_1}{r_1^4}-\frac{c^4(x^2-3y_1^2)}{r_1^6}\right]$$
$$+\frac{k_2c^2x}{\alpha+1}\left[\frac{12ly_1}{r_1^4}+\frac{(r_1^2+3c^2+8l^2)(x^2-3y_1^2)}{r_1^6}\right.$$
$$\left.-\frac{8l(3c^2+2r_1^2)(y_1x^2-y_1^3)}{r_1^8}-\frac{3c^2(x^4-10x^2y_1^2+5y_1^4)}{r_1^8}\right],$$

$$(2\mu v)_i = k_1\frac{\alpha-1}{\alpha+1}\left[y_2-\frac{c^2y_1}{r_1^2}-\frac{\alpha c^2y_1}{r_1^2}-\frac{2lc^2(x^2-y_1^2)}{r_1^4}+\frac{c^2(3y_1x^2-y_1^3)}{r_1^4}\right]$$
$$-\frac{\alpha k_2}{\alpha+1}\left[y_2+\frac{c^2y_1}{r_1^2}+\frac{2lc^2(x^2-y_1^2)}{r_1^4}-\frac{c^4(3y_1x^2-y_1^3)}{r_1^6}\right]$$
$$+\frac{k_2c^2}{\alpha+1}\left[-\frac{6l(x^2-y_1^2)}{r_1^4}+\frac{(r_1^2+3c^2+8l^2)(3y_1x^2-y_1^3)}{r_1^6}\right.$$
$$\left.-\frac{8l(3c^2+2r_1^2)(y_1x^3-y_1^3x)}{r_1^8}-\frac{3c^2(5y_1x^4-10x^2y_1^3+y_1^5)}{r_1^8}\right],$$

$$(2\mu u)_m = k_1\frac{\alpha-1}{\alpha+1}xc^2\left[\frac{\alpha}{r_1^2}-\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{4ly_1}{r_1^4}+\frac{(x^2-3y_1^2)}{r_1^4}\right]+\frac{\alpha k_2}{\alpha+1}xc^2\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{4ly_1}{r_1^4}\right.$$
$$\left.-\frac{c^2(x^2-3y_1^2)}{r_1^6}\right]+\frac{k_2}{\alpha+1}xc^2\left[\frac{12ly_1}{r_1^4}+\frac{(r_1^2+3c^2+8l^2)(x^2-3y_1^2)}{r_1^6}\right.$$
$$\left.+\frac{(r_2^2-c^2)(x^2-3y_2^2)}{r_2^6}-\frac{8l(3c^2+2r_1^2)(y_1x^2-y_1^3)}{r_1^8}-\frac{3c^2(x^4-10x^2y_1^2+5y_1^4)}{r_1^8}\right],$$

$$(2\mu v)_m = k_1\frac{\alpha-1}{\alpha+1}c^2\left\{-\frac{\alpha y_1}{r_1^2}-\frac{2l(x^2-y_1^2)}{r_1^4}-\frac{y_1}{r_1^2}+\frac{y_2}{r_2^2}+\frac{(3y_1x^2-y_1^3)}{r_1^4}\right\}$$
$$+\frac{\alpha k_2}{\alpha+1}c^2\left\{-\frac{y_1}{r_1^2}-\frac{y_2}{r_2^2}-\frac{2l(x^2-y_1^2)}{r_1^4}+\frac{c^2(3y_1x^2-y_1^3)}{r_1^6}\right\}$$
$$+\frac{k_2c^2}{\alpha+1}\left\{-\frac{6l(x^2-y_1^2)}{r_1^4}+\frac{(r_1^2+3c^2+8l^2)(3y_1x^2-y_1^3)}{r_1^6}+\frac{(r_2^2-c^2)(3y_2x^2-y_2^3)}{r_2^6}\right.$$
$$\left.-\frac{8l(3c^2+2r_1^2)(y_1x^3-y_1^3x)}{r_1^8}-\frac{3c^2(5y_1x^4-10x^2y_1^3+y_1^5)}{r_1^8}\right\}.$$

CASE B

Pure Shear.—The case of pure shear ($\delta_1 = \delta_2 = 0$, $\delta_3 \neq 0$) can be proceeded with in a similar way. Thus from (6)

$$P\,ds = \tau_{xy}^0\, d\bar{\zeta},$$

$$\bar{P}\,ds = \tau_{xy}^0\, d\zeta.$$

With this distribution of point-forces along the inclusion boundary, the cumulative effect can be found by evaluating the integrals in (3) and (4). Thus the potentials are

$$\phi_i'(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(\frac{ic^2}{z_1^2}+\frac{4lc^2}{z_1^3}-\frac{3ic^4}{z_1^4}\right),$$

$$\psi_i'(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(i\alpha+\frac{10lc^2}{z_1^3}-\frac{12il^2c^2}{z_1^4}-\frac{9ic^4}{z_1^4}-\frac{12lc^4}{z_1^5}\right),$$

$$\phi_m'(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(\frac{ic^2}{z_1^2}-\frac{ic^2}{z_2^2}+\frac{4lc^2}{z_1^3}-\frac{3ic^4}{z_1^4}\right),$$

$$\psi_m'(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(\frac{-3ic^4}{z_2^4}-\frac{2lc^2}{z_2^3}+\frac{10lc^2}{z_1^3}-\frac{12il^2c^2}{z_1^4}-\frac{9ic^4}{z_1^4}-\frac{12lc^4}{z_1^5}\right).$$

By integration with respect to z we find from above

$$\phi_i(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(\frac{-ic^2}{z_1}-\frac{2lc^2}{z_1^2}+\frac{ic^4}{z_1^3}\right),$$

$$\psi_i(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(i\alpha z_2-\frac{5lc^2}{z_1^2}+\frac{4il^2c^2}{z_1^3}+\frac{3ic^4}{z_1^3}+\frac{3lc^4}{z_1^4}\right),$$

$$\phi_m(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(\frac{-ic^2}{z_1}+\frac{ic^2}{z_2}-\frac{2lc^2}{z_1^2}+\frac{ic^2}{z_1^3}\right)$$

$$\psi_m(z) = \frac{\tau_{xy}^0}{\alpha+1}\left(\frac{-5lc^2}{z_1^2}+\frac{4il^2c^2}{z_1^3}+\frac{3ic^4}{z_1^3}+\frac{3lc^4}{z_1^4}+\frac{lc^2}{z_2^2}+\frac{ic^4}{z_2^3}\right),$$

where again we have added a suitable constant of integration to $\psi_i(z)$, so as to satisfy the continuity of displacements across the inclusion boundary.

Actual verification of the boundary conditions on stresses and displacements provides an excellent check on the analysis. The stresses and the displacements can now be found from the potential functions and eqns. (1) and (2).

Some Results

It has been suggested by H. L. Cox that the stresses at the points A and B (Fig. 1) will be of greater interest from a practical point of view. The stress

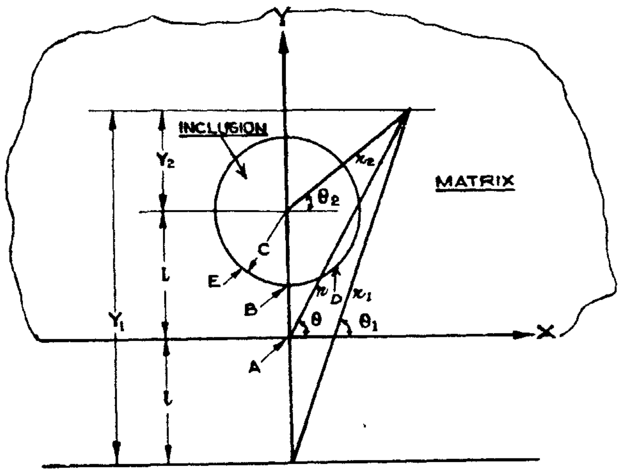

Fig. 1. Coordinate system.

field near the inclusion and the leading edge was evaluated. Some figures showing variation of stresses with the variation of the distance of the inclusion from the leading edge have been drawn (Figs. 2 and 3). Poisson's ratio ν has been taken to be 1/3. It is observed that when the distance l of the centre of the inclusion is about five times its radius c, the stresses in the matrix would differ from those in a similar region of an infinite plate (with an inclusion embedded in it) by about 5 per cent. And, therefore, for practical

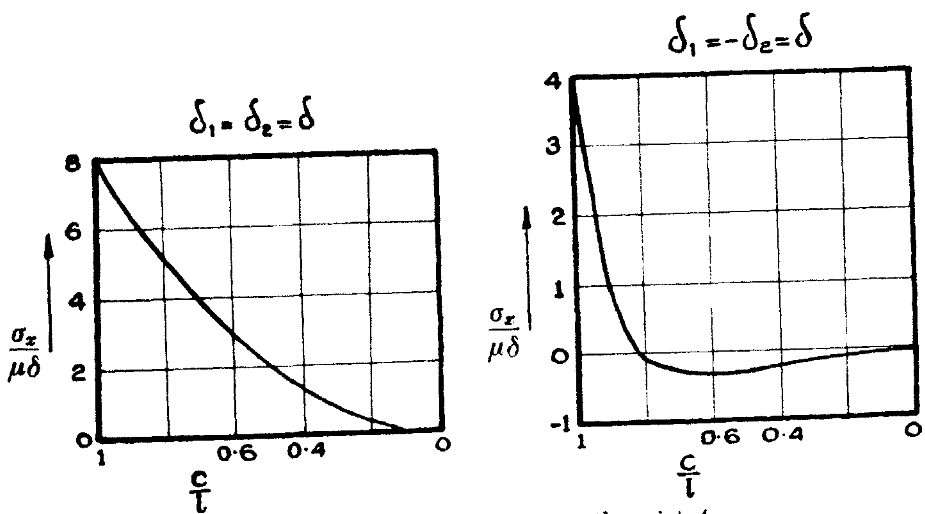

Fig. 2. Variation of hoop stress at the point A.

purposes, when $l/c = 5$, the region can be considered infinite. It is also observed that when the inclusion is almost touching the edge, the tangential stress on the inclusion boundary is maximum at points like D and F (Fig. 1) where θ_2 almost equals $-65°$ and $245°$. Moreover the hoop stress σ_θ at the point B is greater in the case when $\delta_1 = \delta$, $\delta_2 = 0$ than that in the case when $\delta_1 = 0$, $\delta_2 = \delta$.

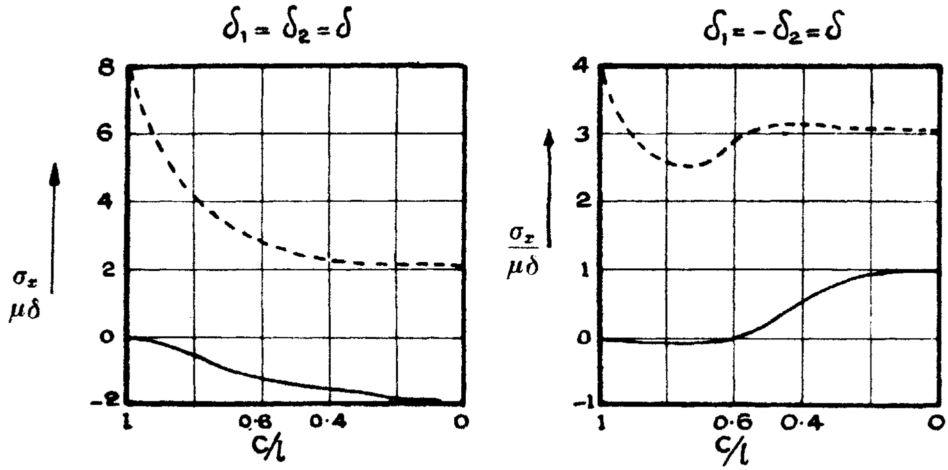

FIG. 3. Variation of hoop stress at the point B of matrix is shown dotted and that of normal stress in continuous lines.

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