ON BESSEL FUNCTION OF THE FIRST KIND OF SEVERAL VARIABLES AND OF INTEGRAL ORDER

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The generating function for the Bessel function of the first kind, in n variables and of integral order m, has been taken to give an integral representation to it. After giving some interesting deductions, the uses of the Bessel function of n variables in the evaluation of reversible solution and in the change of variables have been shown.

INTRODUCTION AND DEFINITION

The Bessel function $J_m(x)$ has a great and growing importance in almost every branch of Mathematical Physics. Its extensions are going on almost every day. The principal object of these extensions is to provide for these functions convenient forms necessary for practical applications. When we include in some practical or theoretical problems more generalized conditions we may need more functions. At the same time it is also possible that we may get wider scope for the application of these functions.

The object of this paper is to extend $J_m(x)$, the Bessel function of the first kind of integral order m and of one variable to n variables and to derive its integral representations together with some applications.

Schlömilch (1857) has given $J_m(x)$ as the coefficient of u^m in the following expansion:

$$\exp \tfrac{1}{2}x\left(u-\frac{1}{u}\right) = \sum_{m=-\infty}^{\infty} u^m J_m(x).$$

We have followed a similar method and taken $J_m(x_1, \ldots, x_n)$, the Bessel function of the first kind of integral order m and of the n variables $x_1, \ldots, x_n$, as the coefficient of u^m in the expansion

$$\exp \left\{ \tfrac{1}{2}x_1\left(u-\frac{1}{u}\right)+ \ldots + \tfrac{1}{2}x_n\left(u^n-\frac{1}{u^n}\right) \right\} = \sum_{m=-\infty}^{\infty} u^m J_m(x_1, \ldots, x_n). \tag{1}$$

We observe that each of $\exp \frac{1}{2}x_1 u, \ldots, \exp \frac{1}{2}x_n u^n$ can be expanded in absolutely convergent series of ascending powers of u; and that for all values, except $u = 0$, $\exp -\frac{1}{2}x_1 u, \ldots, \exp -\frac{1}{2}x_n u^n$ can also be expanded in an absolutely convergent series of descending powers of u. When these series are

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multiplied together, their product can be arranged in powers of u as given in (1) for all finite values of $x_1, \ldots, x_n$ provided $u \neq 0$.

Integral Representations of $J_m(x_1, \ldots, x^n)$

From (1) it is easy to see that by Laurent's theorem

$$J_m(x_1, \ldots, x_n) = \frac{1}{2\pi i}\int_c^{(0+)} u^{-m-1} \exp\left\{\tfrac{1}{2}x_1\left(u-\frac{1}{u}\right)+\ldots+\tfrac{1}{2}x_n\left(u^n-\frac{1}{u^n}\right)\right\} du. \quad (2)$$

The contour c is one which encircles the origin once in the anti-clockwise direction.

Another form for $J_m(x_1, \ldots, x_n)$, similar to the one given by Appell (1915), is derived below.

On putting $u = e^{i\phi}$ in (1) and equating the real and the imaginary parts on both sides, we get

$$\cos(x_1 \sin\phi+\ldots+x_n \sin n\phi) = \sum_{m=-\infty}^{\infty} \cos m\phi \, J_m(x_1, \ldots, x_n), \quad \ldots \quad (3)$$

$$\sin(x_1 \sin\phi+\ldots+x_n \sin n\phi) = \sum_{m=-\infty}^{\infty} \sin m\phi \, J_m(x_1, \ldots, x_n). \quad \ldots \quad (4)$$

Multiplying (3) by $\cos m\phi$ and integrating with respect to ϕ between the limits 0 to π, we get

$$\int_0^\pi \cos m\phi \cos(x_1 \sin\phi+\ldots+x_n \sin n\phi)\, d\phi$$

$$= \int_0^\pi \{J_m(x_1, \ldots, x_n)+J_{-m}(x_1, \ldots, x_n)\} \cos^2 m\phi \, d\phi,$$

i.e.

$$\int_0^\pi \cos m\phi \cos(x_1 \sin\phi+\ldots+x_n \sin n\phi)\, d\phi = \tfrac{1}{2}\pi\{J_m(x_1, \ldots, x_n)+J_{-m}(x_1, \ldots, x_n)\}. \quad \ldots \quad (5)$$

Similarly we have

$$\int_0^\pi \sin m\phi \sin(x_1 \sin\phi+\ldots+x_n \sin n\phi)\, d\phi = \tfrac{1}{2}\pi\{J_m(x_1, \ldots, x_n)-J_{-m}(x_1, \ldots, x_n)\}. \quad \ldots \quad (6)$$

Adding (5) and (6) we get

$$J_m(x_1, \ldots, x_n) = \frac{1}{\pi}\int_0^\pi \cos(m\phi-x_1 \sin\phi-\ldots-x_n \sin n\phi)\, d\phi,$$

and on subtracting we get

$$J_{-m}(x_1, \ldots, x_n) = \frac{1}{\pi}\int_0^\pi \cos(-m\phi-x_1 \sin\phi-\ldots-x_n \sin n\phi)\, d\phi.$$

Thus combining these two, we get

$$J_m(x_1, \ldots, x_n) = \frac{1}{\pi}\int_0^\pi \cos(m\phi - x_1 \sin\phi - \ldots - x_n \sin n\phi)\, d\phi \quad .. \quad (7)$$

for positive and negative integral values of m and $m = 0$. The above integral for $J_m(x_1, \ldots, x_n)$ is similar to Bessel's Integral (1824) for $J_m(x)$. We can also deduce this integral representation directly from (2).

Deductions

In (3) and (4) on putting $\phi = \frac{\pi}{2}$ and 0 respectively we get the following interesting results:

$$(a)\quad \cos(x_1 - x_3 + x_5 - \ldots) = \sum_{m=-\infty}^{\infty} (-1)^m J_{2m}(x_1, \ldots, x_n),$$

$$(b)\quad \sin(x_1 - x_3 + x_5 - \ldots) = \sum_{m=-\infty}^{\infty} (-1)^m J_{2m+1}(x_1, \ldots, x_n),$$

$$(c)\quad 1 = \sum_{m=-\infty}^{\infty} J_m(x_1, \ldots, x_n).$$

From (7) we can also obtain

$$(d)\quad \int_0^\pi \cos r\phi \cos(m\phi - x_1 \sin\phi - \ldots - x_n \sin n\phi)\, d\phi$$

$$= \frac{\pi}{2}\{J_{m-r}(x_1, \ldots, x_n) + J_{m+r}(x_1, \ldots, x_n)\}$$

and

$$(e)\quad \int_0^\pi \sin r\phi \sin(m\phi - x_1 \sin\phi - \ldots - x_n \sin n\phi)\, d\phi$$

$$= \frac{\pi}{2}\{J_{m-r}(x_1, \ldots, x_n) - J_{m+r}(x_1, \ldots, x_n)\}.$$

Again, if we put $u = e^{i\phi}$ in (1) and multiply both sides by $e^{i\phi}$, then, on equating the real and imaginary parts of both sides, we get

$$(f)\quad \cos(\theta + x_1 \sin\phi + \ldots + x_n \sin n\phi) = \sum_{m=-\infty}^{\infty} J_m(x_1, \ldots, x_n) \cos(\theta + m\phi)$$

and

$$(g)\quad \sin(\theta + x_1 \sin\phi + \ldots + x_n \sin n\phi) = \sum_{m=-\infty}^{\infty} J_m(x_1, \ldots, x_n) \sin(\theta + m\phi).$$

On writing $-r\theta$ for θ and $z \sin \theta$ for ϕ in (f) and integrating with respect to θ between the limits 0 to 2π, we get, on dividing both the sides by 2π,

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\{r\theta - x_1 \sin(z \sin\theta) - \ldots - x_1 \sin(nz\sin\theta)\}\, d\theta$$

$$= \sum_{m=-\infty}^{\infty} J_m(x_1, \ldots, x_n) J_r(mz). \qquad (8)$$

From (7) we find that

$$J_m(x_1, \ldots, x_n) = J_{-m}(-x_1, \ldots, -x_n). \qquad (9)$$

Now, on multiplying the expansions for

$$\exp\left[\tfrac{1}{2}x_1\left(u - \frac{1}{u}\right) + \ldots + \tfrac{1}{2}x_n\left(u^n - \frac{1}{u^n}\right)\right] \text{ and}$$

$$\exp\left[-\tfrac{1}{2}x_1\left(u - \frac{1}{u}\right) - \ldots - \tfrac{1}{2}x_n\left(u^n - \frac{1}{u^n}\right)\right] \text{ and}$$

considering the terms independent of u, we get, on using (9),

$$\sum_{m=-\infty}^{\infty} \{J_m(x_1, \ldots, x_n)\}^2 = 1. \qquad (10)$$

Evidently,

$$|J_m(x_1, \ldots, x_n)| < 1. \qquad (11)$$

Applications

(A) If $M = E - x_1 \sin E - \ldots - x_n \sin nE$, then we can find the reversible solution for E in terms of M (Watson 1958) with the help of the Bessel function of n variables.

Here we observe that when $E = \pi$, M is also equal to π and if $E = 0$, then $M = 0$. Also if M and E vary, while $x_1, \ldots, x_n$ remain constants, $E - M$ is a periodic function of M, which vanishes when M is a multiple of π. We may, therefore, assume that

$$E - M = \sum_{r=1}^{\infty} A_r \sin rM,$$

where the coefficients A_r's are functions of $x_1, \ldots, x_n$, which are to be determined. Differentiating the above expression with respect to M we get

$$\sum^{\infty} rA_r \cos rM = \frac{dE}{dM} - 1.$$

Now, multiplying both sides by cos rM and integrating with respect to M between the limits 0 and π, we get

$$\tfrac{1}{2}\pi r\, A_r = \int_0^\pi \left(\frac{dE}{dM} - 1\right) \cos rM\, dM = \int_0^\pi \frac{dE}{dM} \cos rM\, dM.$$

On changing the independent variable from M to E, since the limits remain the same, we get

$$A_r = \frac{2}{r\pi}\int_0^\pi \cos r(E - x_1 \sin E - \ldots - x_n \sin nE)\, dE$$

$$= \frac{2}{r} J_r(rx_1, \ldots, rx_n).$$

Therefore,

$$E = M + 2\sum_{r=1}^{\infty} J_r(rx_1, \ldots, rx_n) \frac{\sin rM}{r}.$$

(B) If $\frac{e}{\sqrt{1+e^2}}\eta = a_1 \sin \xi + \ldots + a_n \sin n\xi$ be the equation of a curve referred to oblique axes, whose inclination is $\cot^{-1} e$, then with the help of the Bessel function of n variables we can find the equation of the curve referred to rectangular axes with x-axis coinciding with the ξ-axis (Gray *et al.* 1952).

Here the relations between the coordinates (x, y) of rectangular system and (ξ, η) of oblique system is given by

$$x = \xi + \frac{e}{\sqrt{1+e^2}}\eta \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

and

$$y = \frac{1}{\sqrt{1+e^2}}\eta. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Therefore, from (12), we get

$$x = \xi + a_1 \sin \xi + \ldots + a_n \sin n\xi,$$

following (A) we may, therefore, assume

$$\xi - x = \sum_{r=1}^{\infty} Ar \sin rx, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where A_r can be determined, as before, in the following form:

$$A_r = \frac{2}{r} J_r(-ra_1, \ldots, -ra_n). \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

From the above relations (12)–(14), we get

$$ey = x - \xi = -\sum_{r=1}^{\infty} A_r \sin rx.$$

Putting the value of A_r we get

$$y = \frac{-2}{e} \sum_{r=1}^{\infty} J_r(-ra_1, \ldots, -ra_n) \frac{\sin rx}{r},$$

which is the required equation of the curve.

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