JOINT DISTRIBUTION OF INTERSECTIONS, (±) WAVES AND (±) STEPS—I

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(Communicated by R. S. Varma, F.N.I.)

(*Received August 17, 1965; after revision July 27, 1966*)

In connection with a statistical test for the equality of means of two populations, Csaki and Vincze (1961) obtained by path methods the joint distribution of the numbers of positive steps and of crosses in the case of a particle performing a Bernoullian symmetric random walk returning to the starting position O on the last step. In this paper these results are generalized by introducing variables, namely how often the particle reaches O from the positive side and also from the negative side. Generating functions, path methods and, in addition, the combinatorial technique are employed to deduce the number of paths in a variety of cases.

1. Notations

Symbol	*Definition*
Positive step	the ith step is said to be positive if $S_i+S_{i-1} > 0$.
Negative step	the ith step is said to be negative if $S_i+S_{i-1} < 0$.
	It may be noted that a positive step is not necessarily of the size $+1$. A positive step in the path (i, S_i) always occurs on the positive side of the axis and a negative step occurs on the negative side.
+ve wave of length i ..	a path for which $S_0 = 0, S_1 > 0, S_2 > 0, \ldots, S_{i-1} > 0, S_i = 0$.
−ve wave of length i ..	a path for which $S_0 = 0, S_1 < 0, S_2 < 0, \ldots, S_{i-1} < 0, S_i = 0$.
O_i ..	a point of the path, where $S_i = 0$, $i \geqslant 0$.
T_i ..	a point of the path, where $S_{i-1} . S_{i+1} = -1$, so that $S_i = 0$.
E_{2n} ..	a path from $(0, 0)$ to $(2n, 0)$.
$E_{2n, l}$ ..	an E_{2n} path containing exactly $(l+1)$ O_i- points, i.e. an E_{2n} path with the lth return to origin at the $2n$th step.
E_{2n, l, l_1} ..	an $E_{2n, l}$ path having l_1 positive waves and $l-l_1$ negative waves, i.e. an E_{2n} path touching the axis l_1 times from the positive side and $l-l_1$ times from the negative side.
E^{2g}_{2n, l, l_1} ..	an E_{2n, l, l_1} path having $2g$ steps above the axis.

Symbol		*Definition*
E^{2g+}_{2n,l,l_1}	..	an E^{2g}_{2n,l,l_1} path with the first $2g$ steps comprising the l_1 positive waves.
$E^{2g,k}_{2n,l,l_1}$	..	an E^{2g}_{2n,l,l_1} path containing exactly k/T_i-points.
H^r_m	..	a path starting at the origin and reaching for the first time the position r at the mth step.
L^t_m	..	a path such that $S_0 = 0, S_1 \geqslant 0, \ldots, S_m = 0$ with t positive waves, or a path such that $S_0 = 0, S_1 \leqslant 0, \ldots, S_m = 0$ with t negative waves.
$(\ldots)$	..	number of paths of the type ...

These definitions can, of course, be made more precise in certain cases, e.g. in a positive wave $S_1 = 1$ and $S_{i-1} = 1$, and in an L^t_m path either $S_1 = 1$ and $S_{m-1} = 1$ or $S_1 = -1$ and $S_{m-1} = -1$.

2. Introduction

Let us denote by ν_i $(i = 1, 2, \ldots)$ the independent variables, taking the values $+1$ and -1 with probabilities $\frac{1}{2}$ each and by S_i the partial sum

$$S_i = \nu_1 + \nu_2 + \ldots + \nu_i, \quad S_0 \equiv 0.$$

If the points (i, S_i) are joined one to the next, we obtain a path of the particle performing a Bernoullian symmetric random walk (BSRW); $1/\binom{2n}{n}$ is the probability of each path in the space of paths starting from and returning to the origin on the $2n$th step. The study of certain sets of paths has proved useful in testing the homogeneity of the distributions from which two samples are available.

Consider two independent samples of n measurements $\xi_1, \xi_2, \ldots, \xi_n$ and $\eta_1, \eta_2, \ldots, \eta_n$ drawn from continuous populations with unknown distribution functions. Let $\mu_1 < \mu_2 \ldots < \mu_{2n}$ denote their arrangement in ascending order. Making the variable ν_i take $+1$ if $\mu_i = \xi_j$ and -1 if $\mu_i = \eta_k$ determines a path terminating on the axis.

Sir Francis Galton was the first to arrange in an ascending order data comprising two sets of measurements. The measurements related to heights of plants, 15 each of treated subjects and of control subjects, referred to him by Charles Darwin in 1876. Galton observed that the number g of times a treated subject exceeded a control subject of the same rank was 13. This was taken by him as an evidence for the treatment having increased the measurements. In the corresponding BSRW the number of positive steps is $2g$. Recently Chung and Feller (1949) while deriving the arc sine law showed that the distribution of g is a uniform one. On the basis of the probability formula $P(G \geqslant g) = (n-g+1)/(n+1)$, a significance level of $\frac{3}{16}$ can be attached to Galton's observation.

Csaki and Vincze (1961) made the Galton statistic more precise by considering the joint distribution of the number of positive steps and intersections with the axis; this can be improved further by incorporating the four variables mentioned below.

While Chung and Feller (1949) used a double generating function and Csaki and Vincze (1961) the path method, we employ here both these and, in addition, the combinatorial technique to investigate more completely the joint distribution of the numbers of (i) positive steps, (ii) positive returns to the axis, (iii) negative ones and (iv) intersections with the axis.

3. Two Theorems

Theorem I.

$$(E^{2g}_{2n,\,l,\,l_1}) = \binom{l}{l_1} \frac{l_1}{2g-l_1} \binom{2g-l_1}{g} \frac{l-l_1}{2n-2g+l_1-l} \binom{2n-2g+l_1-l}{n-g};$$

$$1 \leqslant l_1 \leqslant g, \quad 2 \leqslant l \leqslant n.$$

For $l_1 = l$ or 0,

$$(E_{2n,\,l}) = \frac{l}{2n-l} \binom{2n-l}{n}; \quad 0 \leqslant l \leqslant n.$$

Proof. Consider a path $E^{2g}_{2n,\,l,\,l_1}$ (Fig. 1) and its corresponding path of the type $E^{2g+}_{2n,\,l,\,l_1}$ (Fig. 2) obtained by keeping the structure of each wave unchanged but rearranging them in such a way that the relative positions of

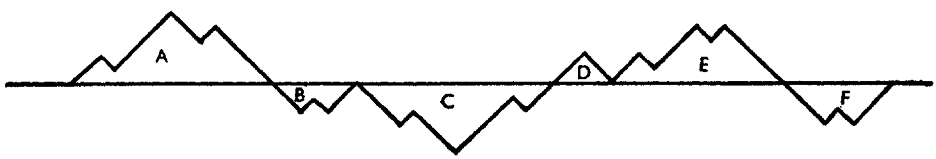

Fig. 1. An $E^{2g}_{2n,\,l,\,l_1}$ path.

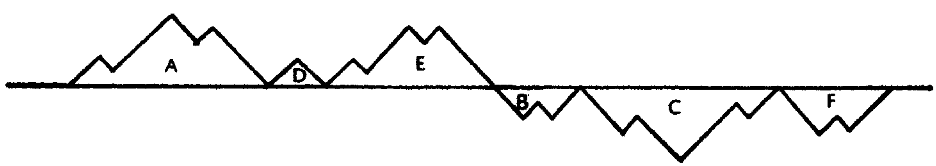

Fig. 2. An $E^{2g+}_{2n,\,l,\,l_1}$ path.

the +ve waves amongst themselves do not change nor those of the −ve waves.

The number of E^{2g}_{2n,l,l_1} paths corresponding to the E^{2g+}_{2n,l,l_1} path shown in Figs. 1 and 2 is equal to the number of ways in which l_1 positive signs and $l-l_1$ negative signs can be arranged in a row and is $\binom{l}{l_1}$. Hence

$$(E^{2g}_{2n,l,l_1}) = \binom{l}{l_1}(E^{2g+}_{2n,l,l_1})$$
$$= \binom{l}{l_1}(L^{l_1}_{2g})(L^{l-l_1}_{2n-2g}). \qquad \qquad (1)$$

To find $(L^{l_1}_{2g})$, we omit the first step of each one of the l_1 positive waves comprising $2g$ steps and translate the truncated waves, without disturbing their order, so that the gaps cover up; we then get a $H^{-l_1}_{2g-l_1}$ path. Thus (Feller 1957, p. 76)

$$(L^{l_1}_{2g}) = (H^{-l_1}_{2g-l_1}) = \frac{l_1}{2g-l_1}\binom{2g-l_1}{g}. \qquad \qquad (2)$$

Similarly the value of $(L^{l-l_1}_{2n-2g})$ can be written.

A use of (1) and (2) leads us to the first result.

To prove the second result we note that for $l_1 = l$ or 0, the number of paths is

$$(E^{2n+}_{2n,l,l}) = (E^{0}_{2n,l,0}) = (L^{l}_{2n}) = (H^{l}_{2n-l})$$
$$= \frac{l}{2n-l}\binom{2n-l}{n}.$$

THEOREM II.

For $1 \leqslant k \leqslant l-l_1$ and $2l_1 \geqslant l$,

$$(E^{2g,\,2k-1}_{2n,l,l_1}) = 2\binom{l_1-1}{k-1}\binom{l-l_1-1}{k-1}\frac{l_1}{2g-l_1}\binom{2g-l_1}{g}\frac{l-l_1}{2n-2g-l+l_1}\binom{2n-2g-l+l_1}{n-g}; \qquad (3a)$$

$$(E^{2g,\,2k}_{2n,l,l_1}) = \frac{l-2k}{k}\binom{l_1-1}{k-1}\binom{l-l_1-1}{k-1}\frac{l_1}{2g-l_1}\binom{2g-l_1}{g}\frac{l-l_1}{2n-2g-l+l_1}\binom{2n-2g-l+l_1}{n-g}. \qquad (3b)$$

For $k = 0$, either $n-g = 0 = l-l_1$ or $g = 0 = l_1$; then

$$(E^{2n,\,0}_{2n,l,l}) = (E^{0,\,0}_{2n,l,0}) = (L^{l}_{2n})$$
$$= (H^{l}_{2n-l}) = \frac{l}{2n-l}\binom{2n-l}{n}.$$

Proof. We first find those E^{2g}_{2n,l,l_1} paths which involve a given number of intersections and correspond to a given E^{2g+}_{2n,l,l_1} path. Consider the segment of the given path comprised between and including the first and the last +ve waves. All the intersections lie within or at one of the ends of this segment.

Let us identify the exponent of y with the number of intersections and that of t with the number of negative waves.

By interposing one or more negative waves in between two consecutive +ve waves we shall get only two intersections and, of course, no interaction if we don't interpose any −ve wave. This is represented by the series

$$y^0t^0+y^2(t+t^2+t^3+\ldots).$$

This expression raised to the power l_1-1 will give the number of intersections by all possible placings of −ve waves in between the l_1 +ve waves. By placing one or more −ve waves to the right of the +ve wave on the right extreme of the segment we shall get only one intersection and, of course, no intersection if we don't place any. This is indicated by the series

$$y^0t^0+y(t+t^2+t^3+\ldots).$$

This also indicates the situation when −ve waves are placed to the left of the +ve wave on the left extreme of the segment.

Hence the number of E^{2g}_{2n,l,l_1} paths with i intersections, with the relative positions amongst themselves of +ve waves unchanged as also those of the negative waves, is given by the coefficient of $t^{l-l_1}y^i$ in the expression

$$[1+y^2(t+t^2+\ldots)]^{l_1-1}[1+y(t+t^2+\ldots)]^2$$
$$=\left(1+\frac{y^2t}{1-t}\right)^{l_1-1}\left(1+\frac{yt}{1-t}\right)^2. \quad \ldots \quad (4)$$

The coefficient of y^{2k} in (4) is

$$\binom{l_1-1}{k}\left(\frac{t}{1-t}\right)^k+\binom{l_1-1}{k-1}\left(\frac{t}{1-t}\right)^{k+1},$$

and the coefficient of t^{l-l_1} in its expansion is

$$\binom{l_1-1}{k}\binom{-k}{l-l_1-k}(-)^{l-l_1-k}+\binom{l_1-1}{k-1}\binom{-k-1}{l-l_1-k-1}(-)^{l-l_1-k-1}$$
$$=\binom{l_1-1}{k}\binom{l-l_1-1}{k-1}+\binom{l_1-1}{k-1}\binom{l-l_1-1}{k}=\frac{l-2k}{k}\binom{l_1-1}{k-1}\binom{l-l_1-1}{k-1}. \quad (5)$$

Again, the coefficient of y^{2k-1} in (4) is

$$2\binom{l_1-1}{k-1}\left(\frac{t}{1-t}\right)^k,$$

and the coefficient of t^{l-l_1} in its expansion is

$$2\binom{l_1-1}{k-1}\binom{-k}{l-l_1-k}(-)^{l-l_1-k}=2\binom{l_1-1}{k-1}\binom{l-l_1-1}{k-1}. \quad \ldots \quad (6)$$

Multiplying respectively by (5) and (6) the value of (E^{2g+}_{2n,l,l_1}) as obtained in (1), we prove theorem II.

4. Generating Function of $(E^{2g,\,i}_{2n,\,l,\,l_1})$

In determining the generating function (GF) we make use of the well-known relation (*see also* Erdelyi 1953, p. 101) for l a positive integer or zero:

$$\sum_{n=0}^{\infty} \binom{2n+l}{n} \omega^n = \frac{1}{\sqrt{(1-4\omega)}} \left[\frac{2}{1+\sqrt{(1-4\omega)}}\right]^l, \qquad \ldots \quad \ldots \quad (7)$$

and its variant

$$\sum_{n=1}^{\infty} \frac{l}{n} \binom{2n}{n-l} \omega^n = \omega^l \left(\frac{1-\sqrt{(1-4\omega)}}{2\omega}\right)^{2l}. \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

Formula (3*a*) can be broken up into two parts, one of which is

$$L(k, l_1, g) \equiv \binom{l_1-1}{k-1} \frac{l_1}{2g-l_1} \binom{2g-l_1}{g},$$

and the other is $2L(k, l_2, h)$, where $l_2 = l-l_1 =$ the number of $-$ve waves and $h = n-g =$ half of the number of $-$ve steps.

We now find the generating function of $L(k, l_1, g)$.

Since l_1 and g can vary between the limits inserted in the GF,

$$A_{k-1}(\mu, \nu) \equiv \sum_{g=k}^{\infty} \sum_{l_1=k}^{g} L(k, l_1, g) \mu^{l_1} \nu^g$$

$$= \sum_{l_1=k}^{\infty} \sum_{g=l_1}^{\infty}, \text{ same summand as above}$$

$$= \sum_{l_1=k}^{\infty} \binom{l_1-1}{k-1} l_1 \mu^{l_1} \int_0^{\nu} \sum_{m=0}^{\infty} \binom{2m+l_1-1}{m} t^{m+l_1-1}\, dt,$$

on putting $g-l_1 = m$.

The integral on using (7) is

$$\int_0^{\nu} \frac{t^{l_1-1}}{\sqrt{(1-4t)}} \left(\frac{2}{1+\sqrt{(1-4t)}}\right)^{l_1-1} dt = \int_0^{\nu} \frac{[1-\sqrt{(1-4t)}]^{l_1-1}}{2^{l_1}} \frac{2}{\sqrt{(1-4t)}}\, dt$$

$$= [1-\sqrt{(1-4\nu)}]^{l_1}/2^{l_1} l_1.$$

Hence writing

$$\lambda = \tfrac{1}{2}\mu[1-\sqrt{(1-4\nu)}], \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

$$A_{k-1}(\mu, \nu) = \sum_{l_1=k}^{\infty} \binom{l_1-1}{k-1} \lambda^{l_1} = \sum_{i=0}^{\infty} \binom{i+k-1}{k-1} \lambda^{k+i}$$

$$= [\lambda/(1-\lambda)]^k. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Similarly,

$$B_{k-1}(\phi, \psi) \equiv \sum_{h=k}^{\infty} \sum_{l_2=k}^{h} L(k, l_2, h) \phi^{l_2} \psi^h$$

$$= [\delta/(1-\delta)]^k, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where

$$\delta = \tfrac{1}{2}\phi[1-\sqrt{(1-4\psi)}]. \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

(i) Hence the quadrivariate GF for an odd number $2k-1$ of intersections with parameters μ, ϕ, ν, ψ respectively for the number of +ve waves, −ve waves, +ve steps and −ve steps is

$$G_{2k-1}(\mu, \phi, \nu, \psi) \equiv \sum_{h=k}^{\infty} \sum_{g=k}^{\infty} \sum_{l_2=k}^{h} \sum_{l_1=k}^{g} (E^{2g,\, 2k-1}_{2g+2h,\, l,\, l_1})\mu^{l_1}\phi^{l_2}\nu^{g}\psi^{h}$$

$$= 2\,[\lambda/(1-\lambda)]^k[\delta/(1-\delta)]^k, \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

on using (10) and (11) and noting that l_2 and h vary independently of l_1 and g.

(ii) To find the corresponding GF for an even number $2k$ of intersections we make use of the identity (5) in (3*b*) and use the symbol $P^{g,\, l_1}_{h,\, l_2}(2k)$ in place of $E^{2g,\, 2k}_{2n,\, l,\, l_1}$, where $n = g+h$ and $l = l_1+l_2$. Then (5) can be written

$$(P^{g,\, l_1}_{h,\, l_2}(2k)) = L(k+1, l_1, g)L(k, l_2, h)+L(k, l_1, g)L(k+1, l_2, h),$$

so that

$$\sum_{h} \sum_{g} \sum_{l_2} \sum_{l_1} P^{g,\, l_1}_{h,\, l_2}(2k)\mu^{l_1}\phi^{l_2}\nu^{g}\psi^{h} = \left(\frac{\lambda}{1-\lambda}\right)^{k+1}\left(\frac{\delta}{1-\delta}\right)^{k}+\left(\frac{\lambda}{1-\lambda}\right)^{k}\left(\frac{\delta}{1-\delta}\right)^{k+1}. \quad (14)$$

(iii) Finally making use of (13) and (14) we find the quintic variate GF with parameters μ, ϕ, ν, ψ as in (i) and ω for the number k of intersections

$$H(\mu, \phi, \nu, \psi, \omega) \equiv \sum_{i=0}^{\infty} G_i(\mu, \phi, \nu, \psi)\omega^i$$

$$= \sum_{k=0}^{\infty} \left[\frac{\lambda\delta}{(1-\lambda)(1-\delta)}\right]^k \left[\frac{\lambda}{1-\lambda}+\frac{\delta}{1-\delta}\right]\omega^{2k}+2\sum_{k=1}^{\infty}\left[\frac{\lambda\delta}{(1-\lambda)(1-\delta)}\right]^k\omega^{2k-1}$$

$$= \frac{\lambda+\delta-2\lambda\delta(1-\omega)}{1-\delta-\lambda+\delta\lambda(1-\omega^2)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

5. Odd Number of Intersections

(*a*) To get $(E^{*\,2k-1}_{2n})$, the number of paths with $2k-1$ intersections in $2n$ steps and $S_{2n} = 0$, we put $\psi = \nu$, $\phi = \mu = 1$ in (13) and collect in the expansion thereof the coefficient of ν^n. With the above substitutions (13) becomes

$$2\left[\frac{1-\sqrt{(1-4\nu)}}{1+\sqrt{(1-4\nu)}}\right]^{2k} = 2\nu^{2k}\left[\frac{1-\sqrt{(1-4\nu)}}{2\nu}\right]^{4k}.$$

The coefficient of ν^n in its expansion by (8) is

$$2\,.\,\frac{2k}{n}\binom{2k}{n-2k} = 2(H^{4k}_{2n}), \quad \text{by (2)}.$$

Hence

$$(E_{2n}^{*\,2k-1}) = 2(H_{2n}^{1k}),$$

a result obtained by Csaki and Vincze (1961).

(*b*) To find $(E_{2n}^{2g,\,2k-1})$, the number of paths with (i) $2k-1$ intersections, (ii) $2g$ steps above the axis and (iii) $S_{2n} = 0$, we put $\phi \equiv \mu = 1$ in (13) and collecting as above the coefficient of $\nu^g \psi^{n-g}$ in its expansion, we verify the following result due to Csaki and Vincze (1961):

$$(E_{2n}^{2g,\,2k-1}) = 2\frac{k}{g}\binom{2g}{g-k}\frac{k}{n-g}\binom{2n-2g}{n-g-k} = 2(H_{2g}^{2k})(H_{2n-2g}^{2k}).$$

(*c*) To get $(E_{2n,\,*l_1}^{*\,2k-1})$, the number of paths with l_1 +ve waves and $2k-1$ intersections in $2n$ steps, we put $\psi = \nu$ and $\phi = 1$ in (13) and collect in the expansion thereof the coefficient of $\mu^{l_1}\nu^n$. With these substitutions (13) becomes

$$2\left\{\frac{\frac{1}{2}\mu[1-\surd(1-4\nu)]}{1-\frac{1}{2}\mu[1-\surd(1-4\nu)]}\right\}^k\left[\frac{1-\surd(1-4\nu)}{1+\surd(1-4\nu)}\right]^k.$$

The coefficient of μ^{l_1} in its expansion in ascending powers of μ is seen to be

$$2\binom{l_1-1}{k-1}\left[\frac{1-\surd(1-4\nu)}{2}\right]^{l_1+2k}\left(\frac{1}{\nu}\right)^k, \qquad \ldots \quad (16)$$

where we have used the formula

$$\binom{-k}{l_1-k}(-)^{l_1-k} = \binom{l_1-1}{k-1}; \qquad \ldots \quad (17)$$

and the coefficient of ν^n in the expansion of (16) in powers of ν by means of (8) shows that

$$(E_{2n,\,*l_1}^{*\,2k-1}) = 2\binom{l_1-1}{k-1}\binom{2n-l_1}{n+k}\left(\frac{l_1+2k}{2n-l_1}\right) = 2\binom{l_1-1}{k-1}(H_{2n-l_1}^{2k+l_1}).$$

(*d*) To get $(E_{2n,\,*l_1}^{2g,\,2k-1})$, the number of paths with (i) l_1 +ve waves comprising $2g$ steps, (ii) $2k-1$ intersections and (iii) $S_{2n} = 0$, we put $\phi = 1$ in (13) and collect the coefficient of $\mu^{l_1}\nu^g\psi^{n-g}$ in its expansion.

We are thus finally led to the result

$$(E_{2n,\,*l_1}^{2g,\,2k-1}) = 2\binom{l_1-1}{k-1}\binom{2g-l_1}{g}\frac{l_1}{2g-l_1}\binom{2n-2g}{n-g-k}\frac{k}{n-g}$$

$$= 2\binom{l_1-1}{k-1}(H_{2g-l_1}^{l_1})(H_{2n-2g}^{2k}).$$

(*e*) To get $(E_{2n,\,l}^{*\,2k-1})$, the number of paths with $2k-1$ intersections and lth return to the origin at the $2n$th step, we put $\phi = \mu$ and $\psi = \nu$ in (13) and collect the coefficient of $\mu^l\nu^n$ in its expansion, getting finally

$$(E_{2n,\,l}^{*\,2k-1}) = 2\binom{l-1}{2k-1}\binom{2n-l}{n}\frac{l}{2n-l} = 2\binom{l-1}{2k-1}(H_{2n-l}^{l}).$$

(*f*) To get $(E^{2g,\,2k-1}_{2n,\,l})$, the number of paths with $2k-1$ intersections and $2g$ steps above the axis and lth return to the origin at the $2n$th step, we substitute $\phi=\mu$ in (13) and collect in the expansion thereof the coefficient of $\mu^l\nu^g\psi^{n-g}$. Now (13) becomes

$$2\left\{\frac{\frac{1}{2}\mu[1-\sqrt{(1-4\nu)}]}{1-\frac{1}{2}\mu[1-\sqrt{(1-4\nu)}]}\right\}^k\left\{\frac{\frac{1}{2}\mu[1-\sqrt{(1-4\psi)}]}{1-\frac{1}{2}\mu[1-\sqrt{(1-4\psi)}]}\right\}^k.$$

Expanding each factor as a power series in μ, the coefficient of μ^l is seen on using a variant of (17) to be

$$2\sum_{i=0}^{l-2k}\binom{i+k-1}{k-1}\binom{l-i-k-1}{k-1}\left[\frac{1-\sqrt{(1-4\nu)}}{2}\right]^{i+k}\left[\frac{1-\sqrt{(1-4\psi)}}{2}\right]^{l-i-k}$$

and finally its expansion in powers of ν and ψ shows that

$$(E^{2g,\,2k-1}_{2n,\,l}) = 2\sum_{i=0}^{l-2k}\binom{i+k-1}{k-1}\binom{l-i-k-1}{k-1}\binom{2g-i-k}{g}\frac{i+k}{2g-i-k}$$
$$\times\binom{2n-2g-l+i+k}{n-g}\frac{l-i-k}{2n-2g-l+i+k}$$
$$=2\sum_{i=0}^{l-2k}\binom{i+k-1}{k-1}\binom{l-i-k-1}{k-1}(H^{i+k}_{2g-i-k})(H^{l-i-k}_{2n-2g-l+i+k}).$$

6. Even Number of Intersections

The results for an even number of intersections corresponding to those of the preceding section are the following. Of these (*a*) and (*b*) are due to Csaki and Vincze (1961).

(*a*) $$(E^{*\,2k}_{2n}) = 2\,.\,\frac{2k+1}{n}\binom{2n}{n+2k+1} = 2(H^{4k+2}_{2n}).$$

(*b*) $$(E^{2g,\,2k}_{2n}) = \frac{k+1}{g}\binom{2g}{g-k-1}\frac{k}{n-g}\binom{2n-2g}{n-g-k}+\frac{k}{g}\binom{2g}{g-k}\frac{k+1}{n-g}\binom{2n-2g}{n-g-k-1}$$
$$=(H^{2k+2}_{2g})(H^{2k}_{2n-2g})+(H^{2k}_{2g})(H^{2k+2}_{2n-2g}).$$

(*c*) $$(E^{*\,2k}_{2n,\,*l_1}) = \binom{l_1-1}{k}\binom{2n-l_1}{n+k}\frac{l_1+2k}{2n-l_1}+\binom{l_1-1}{k-1}\frac{2k+l_1+2}{2n-l_1}\binom{2n-l_1}{n+k+1}$$
$$=\binom{l_1-1}{k}(H^{2k+l_1}_{2n-l_1})+\binom{l_1-1}{k-1}(H^{2k+l_1+2}_{2n-l_1}).$$

(*d*) $$(E^{2g,\,2k}_{2n,\,*l_1}) = (H^{l_1}_{2g-l_1})\left[\binom{l_1-1}{k}(H^{2k}_{2n-2g})+\binom{l_1-1}{k-1}(H^{2k+2}_{2n-2g})\right].$$

(*e*) $$(E^{*\,2k}_{2n,\,l}) = 2\binom{l-1}{2k}\binom{2n-l}{n}\frac{l}{2n-l} = 2\binom{l-1}{2k}(H^{l}_{2n-l}).$$

$$(f)\ (E^{2g,\,2k}_{2n,\,l}) = \sum_{i=0}^{l-2k-1} \binom{k+i-1}{k-1}\binom{l-i-k-1}{k}\Big\{(H^{i+k}_{2g-i-k})(H^{l-i-k}_{2n-2g+i+k-l})$$

$$+(H^{i+k}_{2n-2g-i-k})(H^{l-i-k}_{2g+i+k-l})\Big\}.$$

7. Paths with Intersections Unspecified

We now deduce the number of paths when the number of intersections is indifferent, i.e. unspecified. On putting $\omega = 1$, (15) becomes

$$(\lambda+\delta)/(1-\lambda-\delta). \qquad \ldots \qquad (18)$$

(*a*) To get $(E_{2n,\,*l_1})$, the number of paths with l_1 +ve waves in $2n$ steps and with $S_{2n} = 0$, we put $\psi = \nu$, $\phi = 1$ in (18) and collect in the expansion thereof the coefficient of $\nu_n \mu^{l_1}$. Now the GF (18) becomes

$$\frac{[1-\surd(1-4\nu)]+\mu[1-\surd(1-4\nu)]}{2-[1-\surd(1-4\nu)]-\mu[1-\surd(1-4\nu)]} = \frac{[1-\surd(1-4\nu)]}{1+\surd(1-4\nu)}(1+\mu)\left[1-\frac{\mu[1-\surd(1-4\nu)]}{1+\surd(1-4\nu)}\right]^{-1}.$$

The coefficient of μ^{l_1} in its expansion in an ascending power series in μ is

$$\left(\frac{1-\surd(1-4\nu)}{1+\surd(1-4\nu)}\right)^{l_1+1}+\left(\frac{1-\surd(1-4\nu)}{1+\surd(1-4\nu)}\right)^{l_1}.$$

Finally the coefficient of ν^n in its expansion, by using (8), leads to the result

$$(E_{2n,\,*l_1}) = \frac{l_1+1}{n}\binom{2n}{n-l_1-1}+\frac{l_1}{n}\binom{2n}{n-l_1} = (H^{2l_1+2}_{2n})+(H^{2l_1}_{2n})$$
$$= (H^{2l_1+1}_{2n+1}).$$

(*b*) To find $(E^{2g}_{2n,\,*l_1})$, the number of paths with l_1 +ve waves comprising $2g$ steps and with $S_{2n} = 0$, we put $\phi = 1$ in (18) and collect the coefficient of $\mu^{l_1}\nu^g\psi^{n-g}$ in its expansion. Proceeding as above the final result obtained is

$$(E^{2g}_{2n,\,*l_1}) = (H^{l_1}_{2g-l_1})[(H^{l_1+2}_{2n-2g+l_1})+(H^{l_1}_{2n-2g+l_1})]$$
$$= (H^{l_1}_{2g-l_1})(H^{l_1+1}_{2n-2g+l_1+1}).$$

(*c*) To get $(E_{2n,\,l,\,l_1})$, as defined in § 2, we put $\psi = \nu$ in (18) and collect the coefficient of $\mu^{l_1}\phi^{l_2}\nu^n$ in its expansion.

Now (18) becomes

$$\frac{\tfrac{1}{2}(\mu+\phi)[1-\surd(1-4\nu)]}{1-\tfrac{1}{2}[1-\surd(1-4\nu)](\mu+\phi)},$$

and the coefficient of $\mu^{l_1}\phi^{l_2}$ in its expansion is

$$\binom{l_1+l_2}{l_1}\left[\frac{1-\surd(1-4\nu)}{2}\right]^{l_1+l_2}.$$

Finally the coefficient of ν^n in its expansion by (8) leads to the result

$$(E_{2n,\,l,\,l_1}) = \binom{l}{l_1}\binom{2n-l}{n}\frac{l}{2n-l} = \binom{l}{l_1}(H^l_{2n-l}).$$

(*d*) To get $(E_{2n},\,l)$, as defined in § 2, we put $\phi = \mu$ and $\psi = \nu$ in (18) and collecting in the expansion thereof the coefficient of $\mu^l\nu^n$ one can verify the following well-known result (Feller 1957, p. 82):

$$(E_{2n,\,l}) = 2^l\frac{l}{2n-l}\binom{2n-l}{n} = 2^l(H^l_{2n-l}).$$

(*e*) To find $(E^{2g}_{2n,\,l})$, the number of paths with (i) l returns to the origin, (ii) $2g$ steps above the axis and (iii) $S_{2n} = 0$, we put $\phi = \mu$ in (18) and collect the coefficient of $\mu^l\nu^g\psi^{n-g}$ in its expansion. With these substitutions (18) becomes

$$\tfrac{1}{2}\left\{\frac{\mu[1-\surd(1-4\nu)]+\mu[1-\surd(1-4\psi)]}{1-\frac{1}{2}\mu[1-\surd(1-4\nu)]-\frac{1}{2}\mu[1-\surd(1-4\psi)]}\right\},$$

and the coefficient of μ^l in its expansion is

$$\{\tfrac{1}{2}[1-\surd(1-4\nu)]+\tfrac{1}{2}[1-\surd(1-4\psi)]\}^l$$
$$=\sum_{j=0}^{l}\binom{l}{j}\left[\frac{1-\surd(1-4\psi)}{2}\right]^j\left[\frac{1-\surd(1-4\nu)}{2}\right]^{l-j}.$$

The coefficient of $\nu^g\psi^{n-g}$ in its expansion shows that

$$(E^{2g}_{2n,\,l}) = \sum_{j=0}^{l}\binom{l}{j}(H^j_{2n-2g-j})(H^{l-j}_{2g-l+j}).$$

(*f*) To find (E^{2g}_{2n}), the number of paths with $2g$ steps above the axis when $S_{2n} = 0$, we put $\phi \equiv \mu \equiv 1$ in (18) and get

$$\frac{2-\surd(1-4\nu)-\surd(1-4\psi)}{\surd(1-4\nu)+\surd(1-4\psi)} = \frac{1}{2}\,\frac{\surd(1-4\nu)-\surd(1-4\psi)}{\psi-\nu}-1$$
$$=\frac{1}{2}\,\frac{[1-\surd(1-4\psi)]-[1-\surd(1-4\nu)]}{\psi-\nu}-1$$
$$=\frac{1}{\psi}\,\frac{1-\surd(1-4\psi)}{2}\sum_{j=0}^{\infty}\left(\frac{\nu}{\psi}\right)^j-\frac{1}{\psi}\,\frac{1-\surd(1-4\nu)}{2}\sum_{j=0}^{\infty}\left(\frac{\nu}{\psi}\right)^j-1.$$

The coefficient of $\nu^g\psi^{n-g}$ in it leads to the following well-known result due to Chung and Feller (1949):

$$(E^{2g}_{2n}) = \binom{2n+1}{n+1}\frac{1}{2n+1} = \binom{2n}{n}\frac{1}{n+1}.$$

Of the deductions in §§ 5, 6, 7, those which have not been quoted as verifications of known results are believed to be new.

Acknowledgements

I wish to express my deep indebtedness to Dr. H. C. Gupta for supervision and to the referee for suggesting improvements in the paper.

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