RANDOM WALK IN THE PRESENCE OF ABSORBING BARRIERS

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(Communicated by R. S. Varma, F.N.I.)

(*Received August 18, 1965*)

The paper deals with one-dimensional random walk by discrete movements in the presence of two absorbing barriers in the two cases: (1) the steps are Bernoullian, with probability p of a step being to the right and (2) the consecutive steps are correlated, with probability p of continuation of direction and q $(= 1-p)$ that of reversal, the steps being all of unit length. The analysis carried out in case (2) is equally applicable to the more general two-state Markov chain process, but the expressions in the results would be still more complicated. The probability generating functions (PGF) for the ith passage of a particle to a specified point, as also that of the rth return to the starting position, conditional on the particle not having been absorbed earlier, are obtained. The expressions for probabilities become much simplified when the starting position is the middle point between the barriers. It has been shown that return to the starting position is a transient event and that in the Bernoullian walk the probabilities of absorption at the barriers a and $-b$, conditional on no return to 0, are respectively $p^a(q-p) \div (q^a-p^a)$ and $q^b(q-p)/(q^b-p^b)$. Probabilities of the particle returning at least r times to the origin before being absorbed at an absorbing barrier have been explicitly given. The treatment of the correlated case has been made possible by the introduction of suitable conditional probabilities. The problem of the number of passages through an arbitrary point before absorption has been solved by the authors, using similar methods; but has not been given here in view of the heavy work involved.

1. Introduction

Suppose that a particle is performing a discrete random walk along a straight line when there are absorbing barriers at the points $-b$ and a. At each step the particle moves a unit distance either to the right or to the left of its original position, with probabilities depending on the type of walk the particle is performing. For the non-correlated walk these probabilities are p and q; and for the correlated one they are governed by the transition probability matrix.†

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† The more general transition probability matrix with diagonal elements p_1 and p_2 could be treated by the very same method, but the expressions would be cumbersome.

$$\begin{array}{lcc} \text{From/To} & \text{Right} & \text{Left} \\ \text{Right} & \multicolumn{2}{c}{\left(\begin{array}{cc} p & q \\ q & p \end{array}\right);} \\ \text{Left} & & \end{array} \quad p+q = 1.$$

Whenever the particle reaches either barrier, the walk terminates. Feller (1957) and Mohan (1955) considered the analogous problem of the gambler's ruin and the expected duration of the game. Kac (1947) determined by the use of matrix algebra the probability that the length of life of the particle is greater than a given number of instants. Weesakul (1961) considered the problem when there are only one absorbing and one reflecting barrier; while Seth and Gupta (1964) discussed the random walk in the presence of two reflecting barriers. In another paper Seth (1963) investigated the correlated unrestricted random walk. We obtain here the probability generating functions (PGF) for the particle to arrive at a particular point for the lth time and also for its lth return to the original position. These can be utilized to give the various statistics of the distribution.

While in § 2 we summarize known results, in other sections we give results which are believed to be new. The formulation of variables and difference equations, involving the lth conditional passage, along with the method of solution (e.g. in § 6) will, it is hoped, present points of interest.

(I) Non-correlated Walk

2. *PGF for absorption*

Denote respectively by $p_{z,n}$ and $q_{z,n}$ the probabilities of absorption on the nth step at a and $-b$ of a particle starting from the position z. Clearly $p_{z,n}$ and $q_{z,n}$ are conditional on the particle not reaching the other barrier earlier. Denoting the PGF by the corresponding capital letter, e.g.

$$P_z(s) = \sum_{n=0}^{\infty} p_{z,n} s^n,$$

and using Laplace's method of generating functions, we have (Feller, 1957)

$$P_z(s) = L_{z+b}/L_{b+a},$$

where

$$L_i = \lambda_1^i - \lambda_2^i, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

with

$$\left.\begin{aligned} \lambda_1 = \frac{1+(1-4pqs^2)^{\frac{1}{2}}}{2ps} \text{ and } \lambda_2 = \frac{1-(1-4pqs^2)^{\frac{1}{2}}}{2ps}, \\ \lambda_1\lambda_2 = q/p = \rho, \text{ say.} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

so that

It may be noted that

(i) $[P_0(s)]_{a=r}$ is the PGF for the first passage through $r(>0)$ conditional on the particle starting from 0 and not reaching $-b$ earlier;

(ii) $p_{z,n}$ can be interpreted as the first passage probability from z to a at the nth step, conditional on the particle not reaching $-b$ earlier; and

(iii) $$L_{-i} = -\rho^{-i}L_i. \qquad (3)$$

Similarly, or by suitable interchange of symbols in the above, we have

$$Q_z(s) = -\rho^{a+b}L_{z-a}/L_{a+b}. \qquad (4)$$

It is easily seen that $P_z(1)+Q_z(1) = 1$, implying that the particle is eventually absorbed at one of the barriers.

3. *PGF for the lth conditional passage to the point r (> 0)*

For $0 < r < a$, let

$f^{(l)}_{r,n} \equiv$ probability of the particle reaching r for the lth time at the nth step, conditional on the particle starting from 0 and not reaching a or $-b$ earlier.

Then

$$f^{(2)}_{r,n} = \sum_{k=1}^{n-2} f^{(1)}_{r,n-k-1}[p\pi_{r+1,r}(k)+q\pi_{r-1,r}(k)],$$

where $\pi_{\omega,r}(k)$ = conditional probability of arriving at the position r from ω for the first time at the kth step.

Therefore

$$F^{(2)}_r(s) \equiv \sum_{n=0}^{\infty} f^{(2)}_{r,n}s^n = sF^{(1)}_r(s)[p\Pi_{r+1,r}(s)+q\Pi_{r-1,r}(s)],$$

while

$$F^{(1)}_r(s) = [P_z(s)]_{z=0,\, a=r} = L_b/L_{b+r}.$$

Also

$$\Pi_{r+1,r}(s) = [Q_z(s)]_{b=-r,\, z=r+1} = -\rho^{a-r}L_{r-a+1}/L_{a-r}, \text{ by (4)},$$

$$= \rho L_{a-r-1}/L_{a-r}, \text{ by (3)};$$

while

$$\Pi_{r-1,r}(s) = [P_z(s)]_{a=r,\, z=r-1} = L_{b+r-1}/L_{b+r}.$$

Hence

$$Y_r(s) \equiv p\Pi_{r+1,r}(s)+q\Pi_{r-1,r}(s)$$

$$= q\left[\frac{L_{a-r-1}}{L_{a-r}}+\frac{L_{b+r-1}}{L_{b+r}}\right]. \qquad (5)$$

In general,

$$f^{(l)}_{r,n} = \sum_{k=1}^{n-2} f^{(l-1)}_{r,n-k-1}[p\pi_{r+1,r}(k)+q\pi_{r-1,r}(k)],$$

so that

$$F^{(l)}_r(s) = sY_r(s)F^{(l-1)}_r(s)$$

$$= (sY_r)^{l-1}F^{(1)}_r(s), \quad l > 1.$$

Hence the PGF for arriving at r from 0 in the presence of absorbing barriers at $-b$ and a, not necessarily for the first time, is

$$U_r(s) = \sum_{l=1}^{\infty} (sY_r)^{l-1} F_r^{(1)}(s) = L_b/[1-sY_r(s)]L_{b+r}. \quad \ldots \quad \ldots \quad (6)$$

The explicit expression can be written by using in succession the results (5), (1) and (2).

4. *Return to origin*

(i) The PGF of conditional return to the origin for the lth time is, in view of $F_0^{(1)}(s)$ being unity by definition,

$$G_0^{(l)}(s) = F_0^{(l+1)}(s) = (sY_0)^l F_0^{(1)}(s) = (qs)^l \left[\frac{L_{a-1}}{L_a} + \frac{L_{b-1}}{L_b}\right]^l.$$

For $p > q$, if $s = 1$, then $\lambda_1 = 1$ and $\lambda_2 = \rho$; and

$$G_0^{(1)}(1) = q^l \left[\frac{1-\rho^{a-1}}{1-\rho^a} + \frac{1-\rho^{b-1}}{1-\rho^b}\right]^l \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

is the probability of the particle returning at least l times to the origin before being absorbed.

$G_0^{(1)}(1)$ is the probability of ever returning to the origin; and this being less than unity, a return to the origin is not a *certain* recurrent event. The equation $G_0^{(1)}(1) = 1$, when rid of fractions, admits of the only solution $\rho = 1$, i.e. $p = q = \frac{1}{2}$; but in this case the theory of limits shows that

$$G_0^{(1)}(1) = 1 - \tfrac{1}{2}(1/a + 1/b) \neq 1. \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

In fact, $1 - G_0^{(1)}(1)$ is the probability of absorption at a or $-b$ without the particle returning to the origin. The probability of absorption at a is

$$[pP_1(1)]_{b=0} = p^a(q-p)/(q^a-p^a),$$

that of absorption at $-b$ is

$$[qQ_{-1}(1)]_{a=0} = q^b(q-p)/(q^b-p^b);$$

and it is easily verified that the sum of these two is $1 - G_0^{(1)}(1)$.

(ii) If $u_{0,n}$ denotes the probability of a conditional return to the origin at the nth step, not necessarily for the first time, then

$$u_{0,n} = \sum_{r=0}^{n} g_{0,n-r}^{(1)} u_{0,r},$$

where

$$u_{0,0} \equiv 1 \text{ and } g_{0,0}^{(1)} \equiv 0;$$

and the convolution theorem gives, in conformity with (6),

$$1/U_0(s) = 1 - G_0^{(1)}(s) = 1 - sY_0.$$

(II) Correlated Walk

5. *PGF for the absorption probability*

To deal with this case it is necessary to introduce the conditional probabilities:

$\alpha_{z,n} \equiv$ probability of absorption at a on the nth step, given that the starting position is z and the first step is to the right;

$\beta_{z,n} \equiv$ the corresponding probability when the first step is to the left.

Also define

$$\alpha_{z,0} \equiv 0,\ z \neq a;\quad \alpha_{a,0} \equiv 1$$
$$\beta_{z,0} \equiv 0,\ z \neq a;\quad \beta_{a,0} \equiv 1$$
$$\alpha_{a-1,n} \equiv \delta_{n-1};\quad \delta_i = \begin{cases} 0, & i \neq 0 \\ 1, & i = 0 \end{cases}$$
$$\beta_{-b+1,n} \equiv 0,\ n \geqslant 1. \qquad (9)$$

Then

$$\left.\begin{aligned} \alpha_{z,n} &= p\alpha_{z+1,n-1} + q\beta_{z+1,n-1} \\ \beta_{z,n} &= q\alpha_{z-1,n-1} + p\beta_{z-1,n-1} \end{aligned}\right\} \quad \begin{aligned} -b+1 \leqslant z \leqslant a-1, \\ n \geqslant 1. \end{aligned} \qquad (10)$$

Writing

$$A_z(s) = \sum_{n=0}^{\infty} \alpha_{z,n} s^n \text{ and } B_z(s) = \sum_{n=0}^{\infty} \beta_{z,n} s^n,$$

we get the following difference equations from (10) and (9):

$$A_z(s) = ps\,A_{z+1}(s) + qs\,B_{z+1}(s)$$

and

$$B_z(s) = qs\,A_{z-1}(s) + ps\,B_{z-1}(s),$$

with boundary conditions

$$B_{-b+1}(s) = 0 \text{ and } A_{a-1}(s) = s. \qquad (11)$$

Hence

$$A_z(s) = C\mu^z \frac{\mu - ps}{qs} + D\mu^{-z-1} \frac{1 - \mu ps}{qs}$$

and

$$B_z(s) = C\mu^z + D\mu^{-z},$$

where

$$2ps\mu = 1 + cs^2 + \{(1-s^2)(1-c^2s^2)\}^{\frac{1}{2}},\ c \equiv p - q. \qquad (12)$$

Choosing the constants C and D so as to satisfy (11), we find on writing

$$1/M = (1 - \mu ps) - (\mu - ps)\mu^{2a+2b-3} \qquad (12a)$$

that

$$A_z(s) = Ms[1 - \mu ps - (\mu - ps)\mu^{2b+2z-1}]\mu^{-1+a-z}$$

and

$$B_z(s) = Mqs^2(1 - \mu^{2b+2z-2})\mu^{a-z}.$$

If $c_1 = 1 - c_2$ is the probability of the particle taking the first step to the right, then the PGF for unconditional absorption at a is

$$V_z(s) = c_1 A_z(s) + c_2 B_z(s)$$
$$= Ms\mu^{a-z-1}[c_1 - c_1 ps\mu + c_2 qs\mu - (c_1\mu - c_1 ps + c_2 qs)\mu^{2b+2z-1}]. \quad .. \quad (13)$$

$\bar{A}_z(s)$ and $\bar{B}_z(s)$, the corresponding PGF for absorption at $-b$, can be obtained similarly; or can alternatively be written down from $B_z(s)$ and $A_z(s)$ respectively by interchanging $a-z$ and $b+z$, as also c_1 and c_2. Thus the PGF for unconditional absorption at $-b$ is

$$W_z(s) = M\bar{s}\mu^{b+z-1}[(c_2 + c_1 qs\mu - c_2 ps\mu) - \mu^{2a-2z-1}(c_1 qs + c_2\mu - c_2 ps)]. \quad (13a)$$

The expressions $V_z(s)$ and $W_z(s)$ simplify respectively when $c_1 = q$ and $c_1 = p$. Then

$$V_z(s) = Msq\mu^{a-z-1}(1-\mu^{2z+2b})$$

and

$$W_z(s) = Msq\mu^{b+z-1}(1-\mu^{2a-2z}).$$

For $p = q = c_1 = c_2 = \frac{1}{2}$, these are in agreement with $P_z(s)$ and $Q_z(s)$ of §2, as they should be.

6. *PGF for the lth passage to the point r (> 0)*

Let $\rho^{(l)}_{z,n} \equiv$ probability of arriving at r for the lth time at the nth step, conditional on the particle starting from the position z ($< r$) and not reaching a or $-b$ earlier and the first step being to the right with probability $c_1 = 1 - c_2$

and $\phi^{(l)}_{z,n} \equiv$ the corresponding probability when $z > r$.

Clearly

$$R^{(1)}_z(s) \equiv \sum_{n=0}^{\infty} \rho^{(1)}_{z,n} s^n = [V_z(s)]_{a=r} \quad .. \quad .. \quad .. \quad (14)$$

and

$$\Phi^{(1)}_z(s) \equiv \sum \phi^{(1)}_{z,n} s^n = [W_z(s)]_{b=-r}, \quad .. \quad .. \quad .. \quad (15)$$

where $V_z(s)$ is given by (13) and $W_z(s)$ by (13a).

Now for $z < r$,

$$\rho^{(2)}_{z,n} = \sum_{k=1}^{n-2} \rho^{(1)}_{z,n-k-1}[p \times \text{(probability of first passage from } r+1 \text{ to } r \text{ at the } k\text{th step)} + q \times \text{(probability of first passage from } r-1 \text{ to } r \text{ at the } k\text{th step)}]$$

so that

$$R^{(2)}_z(s) = sR^{(1)}_z(s)\left\{p[W_z(s)]_{z=r+1,\, b=-r,\, c_1=p} + q[V_z(s)]_{z=r-1,\, a=r,\, c_1=q}\right\}.$$

In general,

$$R^{(l)}_z(s) = sR^{(1)}_z(s)\left[p\{\Phi^{(l-1)}_{r+1}(s)\}_{c_1=p} + q\{R^{(l-1)}_{r-1}(s)\}_{c_1=q}\right] \quad .. \quad .. \quad .. \quad (16)$$

and

$$\Phi_z^{(l)}(s) = s\Phi_z^{(1)}(s)\left[p\{R_{r-1}^{(l-1)}(s)\}_{c_1=q}+q\{\Phi_{r+1}^{(l-1)}(s)\}_{c_1=p}\right].$$

Now define

$$J_z(s, t) = \sum_{l>1}^{\infty} R_z^{(l)}(s)t^l \text{ and } H_z(s, t) = \sum_{l=1}^{\infty} \Phi_z^{(l)}(s)t^l.$$

Then from (16)

$$J_z(s, t) = tR_z^{(1)}+tsR_z^{(1)}\left[p\{H_{r+1}(s, t)\}_{c_1=p}+q\{J_{r-1}(s, t)\}_{c_1=q}\right] \quad \ldots \quad \ldots \quad (17)$$

and

$$H_z(s, t) = t\Phi_z^{(1)}+ts\Phi_z^{(1)}[q\{H_{r+1}(s, t)\}_{c_1=p}+p\{J_{r-1}(s, t)\}_{c_1=q}]. \quad \ldots \quad \ldots \quad (18)$$

Here $\{H_{r+1}(s, t)\}_{c_1=p}$ and $\{J_{r-1}(s, t)\}_{c_1=q}$ are determined by putting $z = r+1$ and $c_1 = p$ in (18) and $z = r-1$ and $c_1 = q$ in (17) and solving, we thus get on writing

$$\Theta = 1-qstA-qstB-cs^2t^2AB;\ c = p-q \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

with

$$A = \left\{R_{r-1}^{(1)}\right\}_{c_1=q} \text{ and } B = \left\{\Phi_{r+1}^{(1)}\right\}_{c_1=p} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

that

$$\{H_{r+1}(s, t)\}_{c_1=p} = tB(1+stcA)/\Theta$$

and

$$\{J_{r-1}(s, t)\}_{c_1=q} = tA(1+stcB)/\Theta.$$

Hence

$$J_z(s, t) = tR_z^{(1)}(s)(1+stcB)/\Theta \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

and

$$H_z(s, t) = t\Phi_z^{(1)}(s)(1+stcA)/\Theta. \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

$R_z^{(l)}(s)$ and $\Phi_z^{(l)}(s)$ are then obtained as the coefficients of t^l in (21) and (22) respectively, which can be expressed in explicit forms by using successively the results (19), (20), (14), (15), (13), (13*a*) and (12).

7. *PGF for unconditional return to the origin*

Let $h_{r,n}^{(1)}$ denote the unconditional probability of arriving (in the presence of absorbing barriers at a and $-b$) at the origin for the first time on the nth step from the starting position $r(>0)$. $h_{0,n}^{(1)}$ may be interpreted as the probability of first return to the origin; and for $-b < r < a$

$$h_{0,n}^{(1)} = c_1h_{1,n-1}^{(1)}+c_2h_{-1,n-1}^{(1)}$$

or

$$H_0^{(1)}(s) = c_1sH_1^{(1)}(s)+c_2sH_{-1}^{(1)}(s),$$

where

$$H_1^{(1)}(s) = \{W_z(s)\}_{z=1,\ b=0,\ c_1=p} = \frac{qs(1-\mu^{2a-2})}{1-\mu ps-\mu^{2a-3}(\mu-ps)}$$

and

$$H_{-1}^{(1)}(s) = \{V_z(s)\}_{z=-1,\ a=0,\ c_1=q} = \frac{qs(1-\mu^{2b-2})}{1-\mu ps-(\mu-ps)\mu^{2b-3}}.$$

(i) If $h_{0,n}^{(l)}$ denotes the probability of the lth return to the origin at the nth step, then

$$h_{0,n}^{(l)} = \sum_{k=1}^{n-2} c_1 h_{1,k}^{(1)}\{h_{0,n-k-1}^{(l-1)}\}_{c_1=q} + c_2 h_{-1,k}^{(1)}\{h_{0,n-k-1}^{(l-1)}\}_{c_1=p}.$$

Hence

$$H_0^{(l)}(s) = c_1 s H_1^{(1)}(s)\{H_0^{(l-1)}\}_{c_1=q} + c_2 s H_{-1}^{(1)}(s)\{H_0^{(l-1)}\}_{c_1=p}. \qquad (23)$$

To obtain the expression for $H_0^{(l)}(s)$, put first $c_1 = p$ and again $c_1 = q$ in (23). Thus, writing

$$\{H_0^{(l)}(s)\}_{c_1=p} \equiv A_l, \quad \{H_0^{(l)}(s)\}_{c_1=q} \equiv B_l,$$

$$H_1^{(1)}(s) \equiv \sigma_1 \text{ and } H_{-1}^{(1)}(s) = \sigma_2,$$

we have

$$A_l = ps\sigma_1 B_{l-1} + qs\sigma_2 A_{l-1}$$

and

$$B_l = qs\sigma_1 B_{l-1} + ps\sigma_2 A_{l-1}.$$

Hence

$$A_l = P\lambda_1^l + Q\lambda_2^l \text{ and } B_l = \{A_{l+1} - qs\sigma_2 A_l\}/ps\sigma_1,$$

where λ_1, λ_2 [different from those in (2)] are the roots of the quadratic

$$(\lambda - qs\sigma_2)(\lambda - qs\sigma_1) = p^2 s^2 \sigma_1 \sigma_2$$

and P, Q are determined so as to satisfy the relations

$$P\lambda_1 + Q\lambda_2 = ps\sigma_1 + qs\sigma_2$$

and

$$P\lambda_1^2 + Q\lambda_2^2 = ps\sigma_1(qs\sigma_1 + ps\sigma_2) + qs\sigma_2(ps\sigma_1 + qs\sigma_2).$$

Finally $H_0^{(l)}(s)$ is obtained from the relation

$$H_0^{(l)}(s) = c_1 s\sigma_1 B_{l-1} + c_2 s\sigma_2 A_{l-1}.$$

(ii) *Probability of at least one return*

The events 'first return to 0 at the ith step, $i = 1, 2, 3, \ldots$,' being mutually exclusive with probabilities $h_{0,i}^{(1)}$,

$$\sum_{i=1}^{\infty} h_{0,i}^{(1)} = H_0^{(1)}(1) = q\left[\frac{c_1(a-1)}{2p+aq-1} + \frac{c_2(b-1)}{2p+bq-1}\right]$$

is the probability of the particle returning to the origin at least once before absorption at a barrier. In the special case $c_1 = \frac{1}{2} = p$ it agrees with the value (8) as it should.

(iii) *Starting position at the middle of the range*

When the middle point between the barriers is the starting position, i.e. when $a = b$, these expressions become much simpler; then

$$\sigma_1 = \sigma_2, \text{ so that } \lambda_1 = s\sigma_1 \text{ and } \lambda_2 = -cs\sigma_1.$$

Consequently $A_l = (s\sigma_1)^l = B_l$
and the PGF for the lth return to the origin is

$$H_0^{(l)}(s) = (\sigma_1 s)^l = \frac{q^l s^{2l}(1-\mu^{2a-2})^l}{\{1-\mu ps-(\mu-ps)\mu^{2a-3}\}^l}$$

whence

$$H_0^{(l)}(1) = \{q(a-1)/(2p+aq-1)\}^l$$

is the probability of the particle returning at least l times to the origin before being absorbed at one of the barriers $\pm a$ in a correlated walk. This may be compared with the corresponding expression (7) with $a = b$ for the case of a non-correlated walk. For $p = \frac{1}{2}$, the two, of course, have a common value.

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