UNSTEADY VISCOUS FLOW PAST A FLAT PLATE WITH SUCTION

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The unsteady viscous flow past a flat plate at zero incidence with uniform suction in the presence of a time varying pressure gradient has been investigated by the application of the Laplace transform. The case of steady flow when the pressure gradient is zero has been deduced as a special case.

1. Introduction

Mithal (1960) has discussed the unsteady flow of a viscous incompressible fluid in a tube of circular cross-section under a time varying pressure gradient. In the present paper, the author considers the case of unsteady viscous flow past a porous flat plate in the presence of a time varying pressure gradient. As a special case, the time varying pressure gradient has been taken to be the Dirac delta function. The case of steady flow in the absence of a pressure gradient has been obtained by taking the time from the start of the motion to be infinite.

2. Equations of Motion and Their Boundary Conditions

Consider the two-dimensional unsteady flow of an incompressible viscous fluid past an infinite plate at zero incidence with uniform suction. Take the

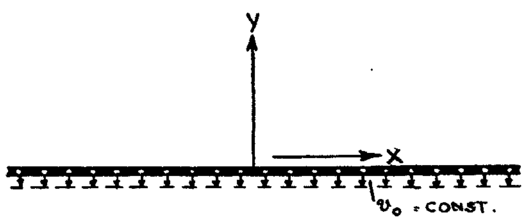

Fig. 1. Flat plate with uniform suction at zero incidence.

axes as shown in Fig. 1. Let u and v be the components of velocity at a point (x, y) in the directions of the x- and the y-axes respectively. Then the

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equation of continuity and the two equations of motion are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \ldots \quad (1)$$

and

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\end{aligned}\right\} \quad \ldots \quad (2)$$

Here ρ is the density of the fluid, ν is the coefficient of kinematic viscosity, t is the time measured from the start of the motion and p is the pressure at the point (x, y).

Suppose that the initial and the boundary conditions for the problem under consideration are

$$\left.\begin{aligned}&t \leqslant 0: u = 0 \text{ and } v = 0 && \text{for } y \geqslant 0\\ &t > 0: u = 0 \text{ and } v = v_0 = \text{const} < 0 && \text{for } y = 0\end{aligned}\right\} \quad \ldots \quad (3)$$

In view of the conditions (3) it is evident that the velocity distribution is independent of x. This implies that $\partial u/\partial x = 0$. Substituting $\partial u/\partial x = 0$ in eqn. (1) and using the second of the conditions (3) we obtain $v = v_0$. Putting $v = v_0$, the eqns. (2) become

and

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+v_0\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}\\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\} \quad \ldots \quad (4)$$

From the second of the eqns. (4), it is obvious that p is independent of y and from the first of the eqns. (4) it follows that $\partial p/\partial x$ is a function of t alone.

Let $-\frac{1}{\rho}\frac{\partial p}{\partial x} = f(t)$. Hence the first of the eqns. (4) becomes

$$\frac{\partial u}{\partial t}+v_0\frac{\partial u}{\partial y} = f(t)+\nu\frac{\partial^2 u}{\partial y^2}. \quad \ldots \quad (5)$$

3. Solution of the Equations

Applying the Laplace transform to eqn. (5) (i.e. multiplying the equation by $e^{-\lambda t}$ and then integrating from zero to infinity) and using the first of the conditions (3), we get, after simplification,

$$\frac{d^2\bar{u}}{dy^2}-\frac{v_0}{\nu}\frac{d\bar{u}}{dy}-\frac{\lambda}{\nu}\bar{u} = -\frac{\bar{f}(\lambda)}{\nu}, \quad \ldots \quad (6)$$

where

$$\bar{u} = \int_0^\infty u e^{-\lambda t}\,dt \quad \ldots \quad (7)$$

and

$$\bar{f}(\lambda) = \int_0^\infty f(t) e^{-\lambda t}\, dt. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

The solution of eqn. (6) is

$$\bar{u} = c_1 e^{y \frac{v_0 + \sqrt{v_0^2 + 4\lambda\nu}}{2\nu}} + c_2 e^{y \frac{v_0 - \sqrt{v_0^2 + 4\lambda\nu}}{2\nu}} + \frac{\bar{f}(\lambda)}{\lambda}, \qquad \ldots \quad \ldots \quad (9)$$

where c_1 and c_2 are arbitrary constants.

The second of the boundary conditions (3) becomes

$$\bar{u} = 0 \text{ at } y = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

As $\bar{u}$ should be finite at $y = \infty$, we must have $c_1 = 0$. Hence eqn. (9) becomes

$$\bar{u} = c_2 e^{y \frac{v_0 - \sqrt{v_0^2 + 4\lambda\nu}}{2\nu}} + \frac{\bar{f}(\lambda)}{\lambda}. \qquad \ldots \quad \ldots \quad \ldots \quad (11)$$

Applying the condition (10), we obtain

$$0 = c_2 + \frac{\bar{f}(\lambda)}{\lambda}.$$

Hence eqn. (11) reduces to

$$\bar{u} = \frac{\bar{f}(\lambda)}{\lambda} \left[1 - e^{y \frac{v_0 - \sqrt{v_0^2 + 4\lambda\nu}}{2\nu}} \right]. \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

Hence, by the inversion formula, we have

$$u = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{u} e^{\lambda t}\, d\lambda, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where γ is greater than the real part of all the singularities of $\bar{u}$.

4. A Special Case

Suppose that the pressure gradient is the Dirac delta function multiplied by a constant, i.e.

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = U \delta(t), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

then

$$\bar{f}(\lambda) = \int_0^\infty U \delta(t) e^{-\lambda t}\, dt = U.$$

Hence from eqn. (12) we have in this case

$$\bar{u} = \frac{U}{\lambda}\left[1 - e^{\frac{y\left(v_0 - \sqrt{v_0^2 + 4\lambda\nu}\right)}{2\nu}}\right]$$

$$= U\left[\frac{1}{\lambda} - \frac{e^{\frac{v_0 y}{2\nu}} e^{\frac{-y\sqrt{\lambda + v_0^2/4\nu}}{\sqrt{\nu}}}}{\lambda}\right].$$

To evaluate the inverse Laplace transform of $e^{\frac{-y\sqrt{\lambda+v_0^2/4\nu}}{\sqrt{\nu}}}\Big/\lambda$ when $y > 0$, we proceed as follows:

From the table of Laplace transforms of Thomson (1950), we have:

$$L^{-1}\left(e^{-y\sqrt{\frac{\lambda}{\nu}}}\right) = \frac{y e^{\frac{-y^2}{4\nu t}}}{2\sqrt{\nu\pi t^3}} \quad (y > 0).$$

By the first shifting theorem,

$$L^{-1}\left(e^{\frac{-y\sqrt{\lambda+v_0^2/4\nu}}{\sqrt{\nu}}}\right) = \frac{y e^{-\frac{y^2}{4\nu}\left(\frac{1}{t}+\frac{v_0^2 t}{y^2}\right)}}{2\sqrt{\nu\pi t^3}}.$$

By the integration theorem,

$$L^{-1}\left(\frac{e^{\frac{-y\sqrt{\lambda+v_0^2/4\nu}}{\sqrt{\nu}}}}{\lambda}\right) = \int_0^t \frac{y e^{\frac{-y^2}{4\nu}\left(\frac{1}{z}+\frac{v_0^2 z}{y^2}\right)}}{2\sqrt{\nu\pi z^3}}\, dz.$$

Hence for $y > 0$, u, which is the inverse Laplace transform of $\bar{u}$, is given by:

$$u = U\left[1 - e^{\frac{v_0 y}{2\nu}} \int_0^t \frac{y e^{\frac{-y^2}{4\nu}\left(\frac{1}{z}+\frac{v_0^2 z}{y^2}\right)}}{2\sqrt{\nu\pi z^3}}\, dz\right]. \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

And at $y = 0$, $\bar{u}$ is clearly zero. Therefore in this case u, the inverse Laplace transform of $\bar{u}$, is also zero, i.e.

$$u = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Thus the solution of the unsteady problem is given by eqns. (15) and (16). Since the motion starts impulsively, the whole mass of the fluid will start moving with velocity U while the fluid in contact with the plate at $y = 0$ is at rest. Therefore there is some sort of discontinuity in the flow for small values of the time t. This is exhibited by eqns. (15) and (16).

Now after time $t = \infty$, i.e. after an infinite time has elapsed since the start of the motion, the flow will become steady and the pressure gradient will be equal to zero. Hence the steady flow in the absence of a pressure gradient for $y > 0$ is given by:

$$u = U\left[1 - e^{\frac{v_0 y}{2\nu}} \int_0^\infty \frac{y e^{\frac{-y^2}{4\nu}\left(\frac{1}{z} + \frac{v_0^2 z}{y^2}\right)}}{2\sqrt{\nu\pi z^3}} dz\right].$$

Substituting $z^{-\frac{1}{2}} = \theta$, we have

$$u = U\left[1 - e^{\frac{v_0 y}{2\nu}} \int_0^\infty \frac{y e^{\frac{-y^2}{4\nu}\left(\theta^2 + \frac{v_0^2}{y^2\theta^2}\right)}}{\sqrt{\nu\pi}} d\theta\right].$$

From Edwards (1922),

$$\int_0^\infty \frac{y e^{\frac{-y^2}{4\nu}\left(\theta^2 + \frac{v_0^2}{y^2\theta^2}\right)}}{\sqrt{\nu\pi}} d\theta = e^{\frac{v_0 y}{2\nu}}.$$

Hence we finally obtain

$$u = U\left(1 - e^{\frac{v_0 y}{\nu}}\right). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (17)$$

Now u, as given by eqn. (17), vanishes at $y = 0$. Therefore u as given by eqn. (17) is the solution of the steady problem for $y \geqslant 0$. It may be asked why eqn. (17) should hold for $y = 0$, when it has been derived from eqn. (15) which is valid for values of $y > 0$ and not for $y = 0$. The answer to this question is available if we look upon the process of the start of the motion. At the time of start of the motion, the velocity is $u = U$ throughout the liquid except at $y = 0$. But as time passes, the velocity u in the extremely close vicinity of the plate decreases and tends towards zero due to viscosity. Therefore at time $t = \infty$, the velocity u in the extremely close vicinity of the plate (i.e. for very small values of y) will be almost zero. And by continuity it follows that u should be zero for $y = 0$ at time $t = \infty$.

The result (17) is one which is identical with the result of Schlichting (1955) if $U = U_\infty$.

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