SOME EXACT SOLUTIONS OF NAVIER-STOKES EQUATIONS FOR STEADY FLOW OF A VISCOUS LIQUID WITH AXIAL SYMMETRY

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This paper presents some exact solutions of the Navier-Stokes equations of viscous liquid motion in spherical polar coordinates with axial symmetry. The cases corresponding to the flows (i) in the annulus of a convergent tunnel with variable suction and injection at the walls, (ii) in the space bounded by porous concentric hemispheres and the common diametral plane with variable suction and injection and (iii) about a porous sphere, and the corresponding particular cases have been discussed.

1. INTRODUCTION

The object of this paper is to search for some new exact solutions of the Navier-Stokes equations of viscous liquid motion in spherical polar coordinates (r, θ, ϕ) with axial symmetry, the line OZ (i.e. $\theta = 0$) being the axis of symmetry. A few exact solutions of these equations are known, some of which are discussed in standard textbooks and some have been published recently (Squire 1952; Agarwal 1957).

In § 3 is given a general solution of the equations, from which only a few particular cases have been discussed. Two other solutions have been obtained in §§ 4 and 7, and it is shown that they correspond to the axially symmetric flows of a viscous liquid (i) in the annulus of a convergent tunnel bounded by two porous coaxial cones with variable suction and injection (§§ 5 and 6), (ii) in the space bounded by porous concentric hemispheres and the common diametral plane with variable suction and injection (§ 7A) and (iii) about a porous sphere when the velocity at infinity is parallel to the axis of symmetry (§ 7B).

The following interesting problems, namely the axially symmetric flow of a viscous liquid (i) in the space bounded by a porous cone and a porous circular plate perpendicular to the axis of the cone and passing through its vertex, the angle of inclination of the generator of the cone to the plane boundary being very small (§ 6), (ii) along an infinite plane boundary (perpendicular to OZ) which sucks in or ejects liquid with constant velocity (§ 7A), etc., have been discussed as particular cases. The results obtained in the case of infinite plane boundary have been found to be in agreement with those discussed in Schlichting's book (Schlichting 1960). Also the solution discussed by Agarwal (Agarwal 1957) and some others discussed in standard textbooks have been deduced as particular cases.

2. Simplification of Navier-Stokes Equations in Spherical Polar Coordinates

The Navier-Stokes vector equation of motion of a viscous liquid is (Milne-Thomson 1949)

$$\frac{\partial \vec{V}}{\partial t}-\vec{V}\times \text{rot}\ \vec{V} = -\text{grad}\ H-\nu\ \text{rot rot}\ \vec{V}, \qquad \ldots \quad \ldots \quad (2.1)$$

where $\vec{V}$ is the velocity vector, ν the kinematic coefficient of viscosity and

$$H = \frac{p}{\rho}+\tfrac{1}{2}V^2+\Omega,$$

p being the pressure, ρ the density and Ω the potential function of the external forces acting on the liquid element at the point considered.

Let us consider the spherical polar coordinates (r, θ, ϕ) with OZ as the line $\theta = 0$, and choose the velocity vector of the flow in the form

$$\vec{V} = \frac{1}{r^2 \sin\theta}\cdot\frac{\partial\psi}{\partial\theta}\cdot i_r-\frac{1}{r\sin\theta}\cdot\frac{\partial\psi}{\partial r}\cdot i_\theta+\frac{1}{r}\chi\,.\,i_\phi \qquad \ldots \quad \ldots \quad (2.2)$$

so as to satisfy the equation of motion (2.1), where ψ and χ are suitable functions of r and θ only, and i_r, i_θ, i_ϕ are the unit vectors at the point (r, θ, ϕ) in the directions of r, θ, ϕ increasing.

The equation of continuity of the flow, namely div $\vec{V} = 0$, is identically satisfied. Also we have

$$\text{rot}\ \vec{V} = \frac{1}{r^2}\left(\frac{\partial\chi}{\partial\theta}+\chi\cot\theta\right)i_r-\frac{1}{r}\cdot\frac{\partial\chi}{\partial r}\cdot i_\theta-\frac{1}{r\sin\theta}D^2\psi\,.\,i_\phi,$$

where

$$D^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\cdot\cot\theta\,.\,\frac{\partial}{\partial\theta}. \qquad \ldots \quad \ldots \quad (2.3)$$

Substituting in (2.1) we get the equations of motion in spherical polar coordinates with axial symmetry of the forms

$$\frac{\partial}{\partial t}\left(\frac{1}{r^2\sin\theta}\cdot\frac{\partial\psi}{\partial\theta}\right)-\frac{1}{r^2}\left(\frac{1}{\sin^2\theta}\cdot\frac{\partial\psi}{\partial r}\cdot D^2\psi+\chi\frac{\partial\chi}{\partial r}\right) = -\frac{\partial H}{\partial r}+\frac{\nu}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}(D^2\psi), \qquad (2.4)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\sin\theta}\cdot\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\left(\frac{1}{\sin^2\theta}\cdot\frac{\partial\psi}{\partial\theta}D^2\psi+\chi\frac{\partial\chi}{\partial\theta}+\chi^2\cot\theta\right) = \frac{\partial H}{\partial\theta}+\frac{\nu}{\sin\theta}\cdot\frac{\partial}{\partial r}(D^2\psi), \qquad (2.5)$$

$$\left(\frac{\partial}{\partial t}-\nu D^2\right)\chi-\frac{1}{r^2\sin\theta}\left\{\frac{\partial(\psi,\chi)}{\partial(r,\theta)}+\chi\cot\theta\,.\,\frac{\partial\psi}{\partial r}\right\} = \frac{\nu}{r^2\chi}\cdot\frac{\partial}{\partial\theta}(\chi^2\cot\theta), \qquad \ldots \quad (2.6)$$

assuming H to be independent of ϕ, where

$$\frac{\partial(x, y)}{\partial(a, b)} = \frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}.$$

Eliminating H between (2.4) and (2.5) we get

$$\left(\frac{\partial}{\partial t} - \nu D^2\right)D_1^2\psi - \frac{1}{r^2 \sin\theta}\cdot\frac{\partial(\psi, D_1^2\psi)}{\partial(r, \theta)} - \frac{4\nu}{r}\left(\frac{\partial}{\partial r}+\frac{1}{r}\cot\theta\cdot\frac{\partial}{\partial\theta}\right)D_1^2\psi$$

$$= \frac{1}{r^2\sin\theta}\left\{\frac{2\chi}{r^3}\cdot\frac{\partial\chi}{\partial\theta} - \frac{\partial}{\partial r}\left(\frac{\chi^2\cot\theta}{r^2}\right)\right\}, \qquad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

where

$$D_1^2 \equiv \frac{1}{r^2\sin^2\theta}D^2.$$

If the velocity component V_ϕ in the direction of ϕ is zero, i.e. if $\chi = 0$, then eqn. (2.6) is identically satisfied. If further the motion is steady, then eqn. (2.7) reduces to

$$\nu\left(D^2+\frac{4}{r}\cdot\frac{\partial}{\partial r}+\frac{4}{r^2}\cot\theta\cdot\frac{\partial}{\partial\theta}\right)D_1^2\psi + \frac{1}{r^2\sin\theta}\cdot\frac{\partial(\psi, D_1^2\psi)}{\partial(r, \theta)} = 0, \qquad \ldots \quad (2.8)$$

which determines completely the steady motion of a viscous liquid with axial symmetry under given boundary conditions.

Eqn. (2.8) is identically satisfied if we put

$$D^2\psi = kr^2\sin^2\theta, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

where k is an arbitrary constant. Hence the solutions of (2.9) are the solutions of Navier-Stokes eqn. (2.1) under the above stated conditions.

Also in this case we have from (2.4) and (2.5)

$$\frac{\partial H}{\partial r} = k\frac{\partial\psi}{\partial r} + 2k\nu\cos\theta,$$

$$\frac{\partial H}{\partial\theta} = k\frac{\partial\psi}{\partial\theta} - 2k\nu r\sin\theta,$$

so that if the external forces are absent, then we obtain

$$\frac{p}{\rho} + \tfrac{1}{2}V^2 = \text{constant} + 2k\nu r\cos\theta + k\psi, \qquad \ldots \quad \ldots \quad (2.10)$$

where

$$V^2 = V_r^2 + V_\theta^2.$$

We now assume

$$\frac{\partial^2\psi}{\partial r^2} = r^n\cdot f_1(\theta),$$

where n is a positive integer; so that we have

$$\psi = \frac{r^{n+2}}{(n+1)(n+2)}f_1(\theta) + rf_2(\theta) + f_3(\theta), \qquad \ldots \quad \ldots \quad (2.11)$$

where f_1, f_2 and f_3 are arbitrary functions of θ only.

Substituting in (2.9) we get the identity

$$r^{n+2}\left\{\frac{1}{(n+1)(n+2)}(f_1''-\cot\theta\,.f_1')+f_1\right\}-kr^4\sin^2\theta$$
$$+r(f_2''-\cot\theta\,.f_2')+(f_3''-\cot\theta\,.f_3')=0 \quad \ldots \quad \ldots \quad (2.12)$$

for all values of r and θ in the region considered, where the primes denote differentiation with respect to θ. We thus have the following two cases:

(i) when $k = 0$ (§ 3) and
(ii) when $k \neq 0$ (§ 4).

3. Solution of Eqn. (2.9) when $k = 0$*

If $k = 0$, then we have from (2.12)

$$\left.\begin{aligned} &f_1''-\cot\theta\,.f_1'+(n+1)(n+2)f_1=0\\ &f_2''-\cot\theta\,.f_2'=0\\ &f_3''-\cot\theta\,.f_3'=0\end{aligned}\right\}. \quad \ldots \quad \ldots \quad (3.1)$$

The last two equations give

$$\left.\begin{aligned} f_2&=B_2-A_2\cos\theta\\ f_3&=B_3-A_3\cos\theta\end{aligned}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

where the A's and B's are arbitrary constants.

The first equation of (3.1) is transformed to

$$\frac{d^2f}{d\eta^2}+\frac{(n+1)(n+2)}{1-\eta^2}\,.f(\eta)=0, \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

where $f_1(\theta)=f(\eta)$ and $\eta = \cos\theta$.

Since $\eta = 0$ is an ordinary point of (3.3) and $\eta = \pm 1$ are its regular singularities, therefore the general solution of (3.3) in the neighbourhood of $\eta = 0$ is valid for $|\eta| < 1$ and it is†

$$f(\eta) = A_1D_{n+2}(\eta)+B_1\Phi(\eta), \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

where A_1 and B_1 are arbitrary constants, n a positive integer and $D_{n+2}(\eta)$ a polynomial of degree $n+2$, defined by

$$D_{n+2}(\eta)=\frac{1\,.\,3\,.\,5\ldots(2n+1)}{(n+2)\,!}\left[\eta^{n+2}-\frac{(n+2)(n+1)}{2\,.\,(2n+1)}\,.\,\eta^n\right.$$
$$\left.+\frac{(n+2)(n+1)n(n-1)}{2\,.\,4\,.\,(2n+1)(2n-1)}\,.\,\eta^{n-2}-\ldots\right], \quad \ldots \quad \ldots \quad (3.5)$$

* The case $k = 0$ reduces to one in which the vorticity is zero.

† Multiplying (3.3) by $1-\eta^2$ and differentiating with respect to η, eqn. (3.3) is transformed to Legendre's equation for $f'(\eta)$ of order $n+1$, and then solved.

the last term in the bracket being

$$(-1)^{\frac{n}{2}+1}\frac{(n+2)(n+1)n(n-1)\ldots 4.3.2.1}{2.4.6\ldots(n+2).(2n+1)(2n-1)\ldots(n+1)}$$

or

$$(-1)^{\frac{n+1}{2}}\frac{(n+2)(n+1)n(n-1)\ldots 5.4.3.2}{2.4.6\ldots(n+1).(2n+1)(2n-1)\ldots(n+2)}.\eta$$

according as n is an even or odd positive integer. The polynomial D_{n+2} is related to the Legendre's polynomial P_n by the relation

$$\frac{d}{d\eta}\{D_{n+2}(\eta)\} = P_{n+1}(\eta) \qquad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

for all positive integral values of n; and hence we have the property

$$D_{n+2}(\eta) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

when $\eta = \pm 1$. Also

$$\Phi(\eta) = \eta + \sum_{m=1}^{\infty}\left\{(-1)^m(n+1)(n-1)(n-3)\ldots(n-2m+3)\right.$$
$$\left.\times(n+2)(n+4)(n+6)\ldots(n+2m).\frac{\eta^{2m+1}}{(2m+1)!}\right\} \qquad \ldots \quad (3.8)$$

or

$$1+\sum_{m=1}^{\infty}\left\{(-1)^m(n+2)n(n-2)\ldots(n-2m+4)\right.$$
$$\left.\times(n+1)(n+3)(n+5)\ldots(n+2m-1).\frac{\eta^{2m}}{(2m)!}\right\} \qquad \ldots \quad (3.9)$$

according as n is even or odd. The polynomial $D_{n+2}(\eta)$ satisfies eqn. (3.3) for all $|\eta| \leqslant 1$, while $\Phi(\eta)$ is convergent for $|\eta| < 1$.

Further, from the general solutions of (3.3) in the neighbourhood of the regular singularities $\eta = \pm 1$ and using the property (3.7), we have

$$f(\eta) = B_1', \quad \text{when } \eta = 1,$$
$$= B_1'', \quad \text{when } \eta = -1,$$

where B_1' and B_1'' are arbitrary constants. Here $\eta = \pm 1$ represent the axis of symmetry and we assume that

$$f(1) = f(-1).$$

Thus since the flow is continuous, therefore assuming

$$B_1' = B_1'' = B_1,$$

we can adjust the solution (3.4) everywhere in $|\eta| \leqslant 1$ on the assumption that

$$\Phi(\eta) = 1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

when $\eta = \pm 1$, that is writing

$$\Phi(\eta) = 1+\sum_{m=1}^{n+2} (-1)^m \frac{(n+2)(n+1)n\ldots(n-m+3).(n+1)(n+2)(n+3)\ldots(n+m)}{2^m.\{(m-1)!\}^2.m}\tau^m$$

$$\times\left\{\log\tau+\sum_{s=1}^{m}\frac{1}{n+s}-\sum_{s=-2}^{m-3}\frac{1}{n-s}-2.\sum_{s=1}^{m}\frac{1}{s}+\frac{1}{m}\right\}$$

$$+\sum_{m=n+3}^{\infty}(-1)^{n+2}\frac{(n+1)(n+2).(m+1)(m+2)\ldots(m+n)}{2^m.(m-1)(m-2)\ldots(m-n-2)}.\tau^m \quad \ldots (3.11)$$

at the points $\eta = \pm 1$, where $\tau = 1 \mp \eta$ according as $\eta = \pm 1$.

Hence substituting from (3.2) and (3.4) in (2.11) we obtain in the present case

$$\psi = \frac{r^{n+2}}{(n+1)(n+2)}\{A_1 D_{n+2}(\eta)+B_1\Phi(\eta)\}+r(B_2-A_2\eta)+(B_3-A_3\eta), \quad (3.12)$$

where $\eta = \cos\theta$, A's and B's are arbitrary constants, $D_{n+2}(\eta)$ is defined by (3.5), and $\Phi(\eta)$ is defined by (3.8) and (3.9) for $|\eta| < 1$ and by (3.11) at $\eta = \pm 1$.

Using the property (3.6) of D_{n+2} we have from (2.2) the components of velocity as follows:

$$V_r = -\frac{1}{r^2}\frac{\partial\psi}{\partial\eta}$$

$$= -\frac{r^n}{(n+1)(n+2)}\{A_1 P_{n+1}(\eta)+B_1\Phi'(\eta)\}+\frac{A_2}{r}+\frac{A_3}{r^2}, \quad \ldots (3.13)$$

$$V_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$$= -\frac{r^n}{(n+1)\sqrt{1-\eta^2}}\{A_1 D_{n+2}(\eta)+B_1\Phi(\eta)\}-\frac{B_2-A_2\eta}{r\sqrt{1-\eta^2}}, \quad \ldots (3.14)$$

where $D_{n+2}(\eta)$ has a factor $1-\eta^2$.

The pressure distribution of the flow is then obtained from (2.10).

Assuming A_2, A_3, B_1 and B_2 to be zero and using (3.5) we have from the general solution (3.12) the following particular cases:

(i) Putting $n = 0$ we obtain

$$\psi = Ar^2\sin^2\theta+B, \quad \ldots \quad \ldots \quad \ldots \quad \ldots (3.15)$$

where A and B are arbitrary constants. The solution (3.15) is a well-known one and is discussed in standard textbooks. A discussion of this solution will also be found in § 7, defined by (7.10).

(ii) When $n = 1$ we have

$$\left.\begin{aligned} \psi &= Ar^3 \cos\theta \sin^2\theta + B \\ V_r &= Ar(3\cos^2\theta - 1) \\ V_\theta &= -3Ar \sin\theta \cos\theta \end{aligned}\right\}, \qquad \ldots \quad \ldots \quad \ldots \quad (3.16)$$

where A and B are arbitrary constants. The solution (3.16) represents the axially symmetric flow of a viscous liquid within or outside the convergent space bounded by the porous cones $\cos\theta = \pm 1/\sqrt{3}$, the velocities of injection and suction at the walls being proportional to r (r is finite).

(iii) Considering $n = 2$ we get

$$\left.\begin{aligned} \psi &= Ar^4 \sin^2\theta(1-5\cos^2\theta) + B \\ V_r &= 4Ar^2 \cos\theta(3-5\cos^2\theta) \\ V_\theta &= -4Ar^2 \sin\theta(1-5\cos^2\theta) \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad (3.17)$$

This solution can, therefore, be regarded as representing the flow between two porous cones defined by $\cos\theta = \pm\sqrt{3/5}$, or between the plane $\theta = \pi/2$ and one of the cones $\cos\theta = \pm\sqrt{3/5}$, or along the porous plane $\theta = \pi/2$, the velocities of injection and suction along the walls being proportional to r^2 (r is finite).

In general, the solution

$$\psi = Ar^{n+2}D_{n+2}(\eta) + B$$

represents flows between porous boundaries with injection and suction velocities proportional to r^n (r is finite), the boundaries being (a) only cones $\theta =$ constant when n is odd and (b) the plane $\theta = \pi/2$ and the cones $\theta =$ constant when n is even. Also the number of boundaries increases with n; for, the $n+1$ roots of $P_{n+1}(\eta) = 0$ are all real lying between ± 1.

4. Solution of (2.9) when $k \neq 0$

If $k \neq 0$, then from (2.12) we must have $n = 2$, and then (2.12) gives

$$f_1'' - \cot\theta \,.\, f_1' + 12 f_1 = 12k \sin^2\theta, \qquad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

$$\left.\begin{aligned} f_2'' - \cot\theta \,.\, f_2' &= 0 \\ f_3'' - \cot\theta \,.\, f_3' &= 0 \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

Equation (4.1) reduces to the form

$$\frac{d^2f}{d\eta^2} + \frac{12}{1-\eta^2} f(\eta) = 12k, \qquad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

where $f_1(\theta) = f(\eta)$ and $\eta = \cos\theta$.

Since $6k\eta^2(1-\eta^2)$ is the particular integral of (4.3) in the neighbourhood of its ordinary point $\eta = 0$, therefore, putting

$$f(\eta) = y(\eta) + 6k\eta^2(1-\eta^2), \qquad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

eqn. (4.3) is transformed to

$$\frac{d^2y}{d\eta^2}+\frac{12}{1-\eta^2}\cdot y = 0. \qquad \ldots \quad (4.5)$$

The general solution of (4.5) in the neighbourhood of its ordinary point $\eta = 0$, i.e. in the region $|\eta| < 1$, gives (cf. the solution of (3.3))

$$f(\eta) = A_1(1-\eta^2)(1-5\eta^2)+B_1F(\eta)+6k\eta^2(1-\eta^2), \qquad \ldots \quad (4.6)$$

where A_1 and B_1 are arbitrary constants, and

$$F(\eta) = \eta+\sum_{m=1}^{\infty} 2^m\cdot\frac{(-3).(-1).1.3.5\ldots(2m-5)}{(m+2)(m+3)(m+4)\ldots(2m+1)}\eta^{2m+1}, \qquad \ldots \quad (4.7)$$

which is convergent in $|\eta| < 1$.

Also from the general solutions of (4.5) in the neighbourhood of its regular singularities $\eta = \pm 1$ we have

$$f(\eta) = B_1', \quad \text{when } \eta = 1,$$

$$= B_1'', \quad \text{when } \eta = -1,$$

where B_1' and B_1'' are arbitrary constants. Thus, assuming as before (cf. § 3)

$$f(1) = f(-1) \text{ and } B_1' = B_1'' = B_1,$$

we can adjust the solution (4.6) everywhere in $|\eta| \leqslant 1$ on the assumption that

$$F(\eta) = 1, \quad \text{when } \eta = \pm 1; \qquad \ldots \quad (4.8)$$

that is, writing

$$F(\eta) = 3\tau\log\tau\left(-2+6\tau-5\tau^2+\tfrac{5}{4}\tau^3\right)+1+\tfrac{11}{2}\tau-45\tau^2$$

$$+\tfrac{109}{2}\tau^3-\tfrac{303}{16}\tau^4+\sum_{m=5}^{\infty}\frac{3}{2^{m-2}}\cdot\frac{(m+1)(m+2)}{(m-1)(m-2)(m-3)(m-4)}\cdot\tau^m \qquad \ldots \quad (4.9)$$

at the points $\eta = \pm 1$, where $\tau = 1\mp\eta$ according as $\eta = \pm 1$.

Hence substituting from (4.2) and (4.6) in (2.11) we obtain in this case

$$\psi = \frac{r^4}{12}\{A_1(1-\eta^2)(1-5\eta^2)+6k\eta^2(1-\eta^2)+B_1F(\eta)\}$$

$$+r(B_2-A_2\eta)+(B_3-A_3\eta), \qquad \ldots \quad (4.10)$$

where $\eta = \cos\theta$, A's, B's and k are arbitrary constants, and $F(\eta)$ is defined by (4.7) in $|\eta| < 1$ and by (4.9) at the points $\eta = \pm 1$.

The components of velocity are

$$V_r = \frac{r^2}{3}\left\{A_1\eta(3-5\eta^2)-3k\eta(1-2\eta^2)-\frac{B_1}{4}F'(\eta)\right\}+\frac{A_2}{r}+\frac{A_3}{r^2}, \qquad \ldots \quad (4.11)$$

$$V_\theta = -\frac{r^2}{3}\left[\{A_1(1-5\eta^2)+6k\eta^2\}\sqrt{1-\eta^2}+\frac{B_1}{\sqrt{1-\eta^2}}F(\eta)\right]-\frac{B_2-A_2\eta}{r\sqrt{1-\eta^2}}. \quad (4.12)$$

Equation (2.10) then gives the pressure distribution of the flow.

The solution (4.10) is quite different from that obtained by Agarwal (1957), and the particular case discussed by Agarwal is obtained from (4.10) assuming the A's, B_1 and B_2 to be zero, and is given by

$$\psi = \frac{k}{2} r^4 \cos^2\theta \sin^2\theta + B_3. \qquad \ldots \quad \ldots \quad \ldots (4.13)$$

5. Axially Symmetric Flow of a Viscous Liquid in the Annulus of a Convergent Tunnel with Variable Suction and Injection at the Walls

Let us investigate the axially symmetric flow of a viscous liquid through a convergent tunnel bounded by the porous walls

$$\theta = \alpha \text{ and } \theta = \beta \quad (0 < \beta < \alpha < \pi/2)$$

between the sections $r = a$ and $r = b$, where $0 < b < a$ and a is finite.

Since $r \neq 0$ and $\eta = \cos\theta \neq \pm 1$ for the present problem, therefore the velocity components V_r and V_θ are given by (4.11) and (4.12) when the function $F(\eta)$ is defined by (4.7). Let the tunnel suck or eject liquid at the outer and inner walls $\theta = \alpha$ and $\theta = \beta$ $(0 < b \leqslant r \leqslant a)$ with velocities $k_1 r^2 + k_2/r$ and $k_3 r^2 + k_4/r$ respectively. Therefore the boundary conditions are

$$\left.\begin{aligned} V_r &= 0, && \text{when } \theta = \alpha \text{ and } \theta = \beta \\ V_\theta &= k_1 r^2 + k_2/r, && \text{when } \theta = \alpha \\ &= k_3 r^2 + k_4/r, && \text{when } \theta = \beta \end{aligned}\right\}, \qquad \ldots (5.1)$$

where k_i ($i = 1$, 2, 3 and 4) are the given constants.

Applying the boundary conditions (5.1) to (4.11) and (4.12) we obtain

$$A_2 = A_3 = 0,$$

$$B_2 = -k_2 \sin\alpha,$$

$$A_1 = \frac{6k_1 \sin\alpha}{\Delta}\left\{\eta_2(1-2\eta_2^2)F'(\eta_1) - \eta_1(1-2\eta_1^2)F'(\eta_2)\right\},$$

$$B_1 = -\frac{24k_1 \sin\alpha}{\Delta} \cdot \eta_1\eta_2(\eta_2^2 - \eta_1^2)$$

and

$$k = \frac{2k_1 \sin\alpha}{\Delta}\left\{\eta_2(3-5\eta_2^2)F'(\eta_1) - \eta_1(3-5\eta_1^2)F'(\eta_2)\right\}, \qquad \ldots \quad \ldots (5.2)$$

where

$$\eta_1 = \cos\alpha,$$

$$\eta_2 = \cos\beta,$$

$$\begin{aligned} \Delta &= 8\eta_1\eta_2(\eta_2^2 - \eta_1^2)F(\eta_1) + 2\eta_1(1-\eta_1^2)^2 F'(\eta_2) \\ &\quad - 2\eta_2(1-\eta_1^2)(1+\eta_1^2-2\eta_2^2)F'(\eta_1), \end{aligned} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (5.3)$$

and the relations between k_i are

$$k_3 = \frac{2k_1 \sin\alpha}{\Delta \sin\beta}\Big\{4\eta_1\eta_2(\eta_2^2-\eta_1^2)F(\eta_2)-\eta_2(1-\eta_2^2)^2F'(\eta_1)$$
$$+\eta_1(1-\eta_2^2)(1+\eta_2^2-2\eta_1^2)F'(\eta_2)\Big\}, \quad \ldots \quad (5.4)$$

$$k_4 = k_2 \sin\alpha/\sin\beta. \quad \ldots \quad (5.5)$$

The last two relations show that k_i are not all independent.

Substituting from (5.2) in (4.11) and (4.12) we get the velocity distribution of the flow, given by

$$V_r = \frac{2k_1 r^2 \sin\alpha}{\Delta}\Big\{\eta_1\eta_2(\eta_2^2-\eta_1^2)F'(\eta)-\eta\eta_2(\eta_2^2-\eta^2)F'(\eta_1)$$
$$-\eta\eta_1(\eta^2-\eta_1^2)F'(\eta_2)\Big\}, \quad \ldots \quad (5.6)$$

$$V_\theta = \frac{2k_1 r^2 \sin\alpha}{\Delta \sin\theta}\Big\{4\eta_1\eta_2(\eta_2^2-\eta_1^2)F(\eta)-\eta_2(1-\eta^2)(1+\eta^2-2\eta_2^2)F'(\eta_1)$$
$$+\eta_1(1-\eta^2)(1+\eta^2-2\eta_1^2)F'(\eta_2)\Big\}+\frac{k_2 \sin\alpha}{r \sin\theta}, \quad \ldots \quad (5.7)$$

where $0 < \eta_1 \leqslant \eta \leqslant \eta_2 < 1$ and the function $F(\eta)$ is defined by (4.7).

Substituting in (4.10) and (2.10) we obtain the pressure distribution of the flow as follows:

$$\frac{p}{\rho}+\tfrac{1}{2}V^2 = C+kr(2\nu\eta-k_2\sin\alpha)$$
$$-\frac{kk_1r^4\sin\alpha}{2\Delta}\{4\eta_1\eta_2(\eta_2^2-\eta_1^2)F(\eta)-\eta_2(1-\eta^2)(1+\eta^2-2\eta_2^2)F'(\eta_1)$$
$$+\eta_1(1-\eta^2)(1+\eta^2-2\eta_1^2)F'(\eta_2)\}, \quad \ldots \quad (5.8)$$

where C is an arbitrary constant and $V^2 = V_r^2+V_\theta^2$.

If Q be the total flux of the liquid through a section $r =$ constant, then from (5.6)

$$Q = 2\pi r^2 \int_{\eta_1}^{\eta_2} V_r \, d\eta$$
$$= \frac{\pi k_1 r^4 \sin\alpha}{\Delta}(\eta_2^2-\eta_1^2)[4\eta_1\eta_2\{F(\eta_2)-F(\eta_1)\}$$
$$-(\eta_2^2-\eta_1^2)\{\eta_2F'(\eta_1)+\eta_1F'(\eta_2)\}], \quad \ldots \quad (5.9)$$

where $b \leqslant r \leqslant a$.

If we now assume that $k_2 = 0 = k_4$, then the above results give the flow of a viscous liquid with axial symmetry in the convergent annular space bounded by the porous cones $\theta = \alpha$ and $\theta = \beta$ $(0 < \beta < \alpha < \pi/2)$ between $r = 0$

and $r = a$. In this case the velocities of injection and suction at the outer and inner walls are respectively $k_1 r^2$ and $k_3 r^2$. It is interesting to note that the flux Q in this case remains unchanged and is given by (5.9), where $0 \leqslant r \leqslant a$.

Again, from (4.7) and (4.9) we have $F(0) = 0$, $F'(0) = 1$, $F(1) = 1$. Thus assuming $k_2 = 0$ and proceeding to the limit as $\eta_1 \to 0$ and then to the limit as $\eta_2 \to 1$ we get from the above results the flow of a viscous liquid with axial symmetry along a plane boundary ($\theta = \pi/2$), which ejects liquid with velocity $k_1 r^2$, and in which the velocity along the axis of symmetry is zero. The velocity and pressure distributions in this case are

$$\left.\begin{aligned} V_r &= -k_1 r^2 \cos\theta \sin^2\theta \\ V_\theta &= k_1 r^2 \sin^3\theta \\ p &= C - 4k_1 \rho\nu r \cos\theta \end{aligned}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad (5.10)$$

where C is an arbitrary constant, $0 \leqslant r \leqslant a$ and $0 \leqslant \theta \leqslant \pi/2$.

6. Calculations of the Results of § 5 for Small Values of η

When η is small enough, i.e. when $0 \leqslant \beta \leqslant \theta \leqslant \alpha < \pi/2$, then using the approximate formula (from 4.7)

$$F(\eta) \simeq \eta - 2\eta^3 + \tfrac{3}{5}\cdot\eta^5,$$

we obtain from (5.3) that

$$\Delta \simeq -2(\eta_2 - \eta_1)[1 - (\eta_2 - \eta_1)^2\{2 + \eta_1(3\eta_2 - \eta_1)\}]. \quad \ldots \quad (6.1)$$

Hence from eqns. (5.6) to (5.8) the velocity and pressure distributions of the flow correct to the order η^4 are obtained as follows:

$$V_r = k_1 r^2 \sin\alpha(\eta_2 - \eta)(\eta - \eta_1)(\eta + \eta_1 + \eta_2), \quad \ldots \quad \ldots \quad \ldots \quad (6.2)$$

$$V_\theta = \frac{k_2 \sin\alpha}{r \sin\theta} + \frac{k_1 r^2 \sin\alpha}{\sin\theta}[1 + (\eta_2 + \eta_1)(\eta_2 - \eta_1)^3 - (\eta_2 - \eta)^2\{(\eta_2 + \eta)^2 - 2\eta_1(\eta_2 + \eta_1)\}], \quad \ldots \quad \ldots \quad (6.3)$$

$$\frac{p}{\rho} + \tfrac{1}{2}V^2 = C + kr(2\nu\eta - k_2 \sin\alpha) - \frac{kk_1 r^4 \sin\alpha}{4}[1 + (\eta_2 + \eta_1)(\eta_2 - \eta_1)^3 - (\eta_2 - \eta)^2\{(\eta_2 + \eta)^2 - 2\eta_1(\eta_2 + \eta_1)\}], \quad \ldots \quad (6.4)$$

where $0 < b \leqslant r \leqslant a$, $0 < \eta_1 \leqslant \eta \leqslant \eta_2 \leqslant 1$, and from (5.2)

$$k = -k_1 \sin\alpha\Big[3 + (\eta_2^2 + \eta_2\eta_1 + \eta_1^2) + \Big\{3\eta_2^2\eta_1^2 + (\eta_2 - \eta_1)^2(2\eta_2^2 + 2\eta_2\eta_1 - \eta_1^2)\Big\}\Big]. \quad \ldots \quad \ldots \quad (6.5)$$

From (5.9) the total flux is obtained as

$$Q = \frac{\pi k_1 r^4 \sin\alpha}{2}(\eta_2+\eta_1)(\eta_2-\eta_1)^3, \quad \ldots \quad (6.6)$$

where $b \leqslant r \leqslant a$.

And the relations between the constants k_i are given by

$$\left.\begin{aligned} k_3 &= \frac{k_1 \sin\alpha}{\sin\beta}\{1+(\eta_2+\eta_1)(\eta_2-\eta_1)^3\} \\ k_4 &= k_2 \sin\alpha/\sin\beta \end{aligned}\right\}. \quad \ldots \quad (6.7)$$

Assuming $k_2 = 0$ we obtain from (6.2) to (6.7) the flow of a viscous liquid with axial symmetry in the convergent annular space bounded by the porous walls $\theta = \alpha$ and $\theta = \beta$ $(0 \leqslant \beta < \alpha < \pi/2)$ between $r = 0$ and $r = a$, when the velocities of injection and suction at the outer and inner walls are respectively $k_1 r^2$ and $k_3 r^2$. From (6.7) it is clear that k_1 and k_3 are of the same sign and the magnitude of k_3 is always greater than that of k_1 so that the medium ejects liquid at the outer wall $\theta = \alpha$ and sucks in at the inner wall $\theta = \beta$ when $k_1 > 0$.

If, further, we put $\alpha = \pi/2$, i.e. $\eta_1 = 0$, then we get the axially symmetric flow of a viscous liquid in the space bounded by the porous walls $\theta = \pi/2$ and $\theta = \beta$ $(0 \leqslant \beta < \pi/2)$ between $r = 0$ and $r = a$ when the velocities of injection and suction at the two walls are respectively $k_1 r^2$ and $k_3 r^2$.

In this case, the velocity, pressure, etc., in the flow are obtained as follows:

$$V_r = k_1 r^2(\eta_2^2-\eta^2)\eta, \quad \ldots \quad (6.8)$$

$$V_\theta = \frac{k_1 r^2}{\sin\theta}(1+2\eta_2^2\eta^2-\eta^4), \quad \ldots \quad (6.9)$$

$$\frac{p}{\rho} = C - 2k_1\nu(3+\eta_2^2)\,.\,r\eta + \frac{k_1^2 r^4}{4}\left\{1+(\eta_2^2-2\eta^2)+(2\eta_2^4-2\eta_2^2\eta^2-\eta^4)\right\} \quad \ldots \quad (6.10)$$

and

$$Q = \frac{\pi k_1}{2}\,.\,\eta_2^4 r^4, \quad \ldots \quad (6.11)$$

where $0 \leqslant r \leqslant a$, $0 \leqslant \eta \leqslant \eta_2 \leqslant 1$ and

$$k_3 = \frac{k_1}{\sin\beta}(1+\eta_2^4). \quad \ldots \quad (6.12)$$

The magnitude of k_3 is greater than that of k_1 and they are of the same sign. If k_1 is positive, then both V_r and V_θ are positive and the liquid flows out of the space considered at the free section $r = a$. On a surface $\eta =$ constant in the liquid both V_r and V_θ increase with r; and on any section $r =$ constant in the liquid the component V_r (defined by 6.8) attains its maximum when $\eta = \eta_2/\sqrt{3}$. In the former case, on any section $r =$ constant in the liquid the component V_r (defined by 6.2) has its maximum ($k_1 > 0$) when $\eta = \left\{\frac{1}{3}\left(\eta_2^2+\eta_2\eta_1+\eta_1^2\right)\right\}^{1/2}$.

7. Axially Symmetric Flow (A) in the Space bounded by Porous Concentric Hemispherical Walls and the Common Diametral Plane and (B) about a Porous Sphere

Let us now assume

$$\psi = R \sin^2\theta, \qquad \cdots \quad (7.1)$$

where R is a function of r only. Eqn. (2.9) then reduces to

$$r^2 \cdot \frac{d^2R}{dr^2} - 2R = kr^4. \qquad \cdots \quad (7.2)$$

The solution of (7.2) gives

$$\psi = \left(Ar^2 + \frac{B}{r} + \frac{kr^4}{10}\right)\sin^2\theta, \qquad \cdots \quad (7.3)$$

where A, B and k are arbitrary constants.

Hence from (2.2) the components of velocity of the flow are

$$\left.\begin{aligned} V_r &= \left(2A + \frac{2B}{r^3} + \frac{kr^2}{5}\right)\cos\theta \\ V_\theta &= -\left(2A - \frac{B}{r^3} + \frac{2kr^2}{5}\right)\sin\theta \end{aligned}\right\} \qquad \cdots \quad (7.4)$$

which correspond to a problem for which $V_r = 0$ when $\theta = \pi/2$.

A. The solution (7.3) corresponds to the axially symmetric flow of a viscous liquid in the hemispherical space bounded by the porous walls $r = a$, $r = b$ and $\theta = \pi/2$ with variable suction and injection along the walls, where $0 < b < a$ and $0 \leqslant \theta \leqslant \pi/2$.

We now assume that the medium sucks in or ejects liquid with velocities $k_1 \cos\theta$ and $k_2 \cos\theta$ at the walls $r = a$ and $r = b$ respectively. Hence the boundary conditions are

$$\left.\begin{aligned} V_\theta &= 0, && \text{when } r = a \text{ and } r = b \\ V_r &= k_1\cos\theta, && \text{when } r = a \\ &= k_2\cos\theta, && \text{when } r = b \end{aligned}\right\} \qquad \cdots \quad (7.5)$$

for all arbitrary values of θ satisfying $0 \leqslant \theta \leqslant \pi/2$; so that eqn. (7.4) gives

$$\left.\begin{aligned} A &= k_1 \cdot \frac{(a+b)(a^3+b^3)+a^2b^2}{(a+b)(a^3+6b^3)+a^2b^2} \\ B &= \frac{2k_1 a^3 b^3 (a+b)}{(a+b)(a^3+6b^3)+a^2b^2} \\ k &= -\frac{5k_1(a^2+ab+b^2)}{(a+b)(a^3+6b^3)+a^2b^2} \\ k_2 &= k_1 \cdot \frac{(a+b)(6a^3+b^3)+a^2b^2}{(a+b)(a^3+6b^3)+a^2b^2} \end{aligned}\right\} . \qquad \ldots \quad \ldots \quad (7.6)$$

Substituting in (7.4), (7.3) and (2.10) we get the velocity and pressure distributions of the flow. In this case the plane boundary $\theta = \pi/2$ ejects liquid with velocity

$$-(2A - B/r^3 + 2kr^2/5)$$

which represents suction when $k_1 > 0$.

The flux of liquid across a hemispherical section $r =$ constant is

$$Q = \pi r^2 \left(2A + \frac{2B}{r^3} + \frac{kr^2}{5}\right), \qquad \ldots \quad \ldots \quad \ldots \quad (7.7)$$

where $0 < b \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi/2$ and A, B, k are given by (7.6).

Clearly, the present case is the same as when the flow is within the spherical shell bounded by the porous spheres $r = a$ and $r = b$ $(b < a)$ in the absence of any source or sink at the centre $r = 0$; for, the inflow and outflow through the porous boundary of the inner sphere $r = b$ are the same.

(i) Proceeding to the limit as $b \to 0$ we see from (7.6) that $B \to 0$, and hence from the above results we obtain the flow of a viscous liquid with axial symmetry in the hemispherical space bounded by the porous walls $r = a$ and $\theta = \pi/2$, the injection velocity at $r = a$ being $k_1 \cos\theta$. The components of velocity in this case are

$$\left.\begin{aligned} V_r &= k_1(2 - r^2/a^2)\cos\theta \\ V_\theta &= -2k_1(1 - r^2/a^2)\sin\theta \end{aligned}\right\}, \qquad \ldots \quad \ldots \quad \ldots \quad (7.8)$$

where $0 \leqslant r \leqslant a$ and $0 \leqslant \theta \leqslant \pi/2$. In this particular case the plane boundary $\theta = \pi/2$ sucks in with velocity $2k_1(1 - r^2/a^2)$ whose magnitude is maximum at the centre $r = 0$.

The solution (7.8) also represents the flow inside a porous sphere $r = a$, which sucks in along the hemispherical part, where $\pi/2 < \theta \leqslant \pi$, and ejects liquid along the other part, where $0 \leqslant \theta \leqslant \pi/2$, with velocity $k_1 \cos\theta$.

(ii) Again, considering the limit as $a \to \infty$ we see from (7.6) that $k \to 0$, and therefore eqn. (7.3) gives the flow of a viscous liquid with axial symmetry along an infinite plane boundary $(\theta = \pi/2)$ having a hemispherical boundary

$r = b$, where $0 \leqslant \theta \leqslant \pi/2$, the velocity of suction along the part $r = b$ being $k_2 \cos\theta$. The velocity distribution in this case is

$$\left.\begin{aligned} V_r &= \frac{k_2}{3}(1+2b^3/r^3)\cos\theta \\ V_\theta &= -\frac{k_2}{3}(1-b^3/r^3)\sin\theta \end{aligned}\right\}, \qquad (7.9)$$

where $r \geqslant b$ and $0 \leqslant \theta \leqslant \pi/2$. The plane boundary $\theta = \pi/2$ sucks in with velocity $\frac{k_2}{3}(1-b^3/r^3)$, and k_2 depends on the velocity at infinity.

This solution (7.9) also gives the flow past a porous sphere $r = b$, which sucks in along the part, where $0 \leqslant \theta \leqslant \pi/2$, and ejects along the other part, where $\pi/2 < \theta \leqslant \pi$, with velocity $k_2 \cos\theta$.

(iii) Finally, proceeding to the limit as $a \to \infty$ in (i), or as $b \to 0$ in (ii) and adjusting the constant arising due to the porosity of the boundary we get the flow of a viscous liquid with axial symmetry along an infinite plane boundary ($\theta = \pi/2$) which sucks in with constant velocity $2k_1$, defined by

$$\left.\begin{aligned} V_r &= 2k_1\cos\theta \\ V_\theta &= -2k_1\sin\theta \end{aligned}\right\}, \qquad (7.10)$$

where $0 \leqslant \theta \leqslant \pi/2$ and r is arbitrary. In this case, the resultant velocity at every point of the liquid is normal to the plane boundary and it is $V_z = 2k_1$, which agrees with the result given by Schlichting in his book (Schlichting 1960: eqn. 13.6, p. 272) when the velocity at infinity parallel to the plane is absent.

B. The axially symmetric flow of a viscous liquid about a porous sphere $r = a$ can also be determined from the solution (7.3) assuming the thickness of the shell to be infinitesimal, i.e. assuming the flow across the porous boundary $r = a$ to be continuous. Thus if U be the velocity of the liquid at infinity in the direction of z-increasing, and if the velocity of suction or injection along $r = a$ be $k_1 \cos\theta$, then the boundary conditions are

$$\left.\begin{aligned} V_r &= U\cos\theta \\ V_\theta &= -U\sin\theta \end{aligned}\right\} \text{ at } r \to \infty \qquad (7.11)$$

and

$$\left.\begin{aligned} V_\theta]_{r=a-0} &= V_\theta]_{r=a+0} = 0 \\ V_r]_{r=a-0} &= V_r]_{r=a+0} = k_1\cos\theta \end{aligned}\right\} \qquad (7.12)$$

for all arbitrary θ lying between 0 and π, where k_1 is a constant depending on U. The flows inside and outside the sphere $r = a$ are therefore obtained as follows:

$$\left.\begin{aligned} \psi &= k_1 r^2(1-r^2/2a^2)\sin^2\theta \\ V_r &= k_1(2-r^2/a^2)\cos\theta \\ V_\theta &= -2k_1(1-r^2/a^2)\sin\theta \end{aligned}\right\} \quad (r < a) \qquad (7.13)$$

and

$$\left.\begin{aligned} \psi &= \frac{k_1}{6}(r^2+2a^3/r)\sin^2\theta \\ V_r &= \frac{k_1}{3}(1+2a^3/r^3)\cos\theta \\ V_\theta &= -\frac{k_1}{3}(1-a^3/r^3)\sin\theta \end{aligned}\right\} \quad (r \geqslant a) \qquad \ldots \quad \ldots (7.14)$$

where from the condition at infinity we have

$$k_1 = 3U. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (7.15)$$

Since the inflow and outflow through the porous boundary $r = a$ are the same, there exists no source or sink at the centre of the sphere. Further, since in this case $k = 0$ outside the sphere and $k \neq 0$ inside it (cf. (7.3)), therefore from (2.10) it is clear that the pressure p is not continuous across the boundary $r = a$, and this phenomenon is responsible for the flow across the porous boundary.

Complete discussions of the physical properties of the flows considered in different articles have not been given as they follow directly from the functions ψ and V.

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