ON UNIFORM NÖRLUND SUMMABILITY OF FOURIER SERIES

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In this paper we introduce a new concept of summability, i.e. Uniform Nörlund Summability, and establish two theorems on it of the Fourier series corresponding to a function $f(x)$, periodic with period 2π and integrable (L) over $(-\pi, \pi)$.

1. Let the Fourier series corresponding to a function $f(x)$, periodic with period 2π and integrable (L), be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx). \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

Let $S_n(x)$ denote the partial sum of the series (1.1). We shall use the following notations:

$$\phi(t) = \phi(x, t) = f(x+t)+f(x-t)-2S,$$

$$\Phi(t) = \int_0^t |\phi(u)|\, du$$

$$\tau = \left[\frac{1}{t}\right],$$

where $[\lambda]$ denotes the integral part of λ.

2. The concept of uniform Harmonic summability as defined by the author (Saxena, 1965) earlier is as follows:

Let

$$u_0(x)+u_1(x)+u_2(x)+\ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

be any infinite series, and

$$U_\nu(x) = u_0(x)+u_1(x)+u_2(x)+\ldots+u_\nu(x).$$

If there exists a function $U = U(x)$ such that

$$\frac{1}{\log n}\sum_{k=0}^{n}\frac{1}{k+1}\{U_{n-k}(x)-U\} = o(1)$$

uniformly in a set E in which $U = U(x)$ is bounded, then we shall say that the series (2.1) is summable by Harmonic means uniformly in E to the sum U.

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3. Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0+p_1+p_2+\dots+p_n.$$

The sequence-to-sequence transformation

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu} S_\nu}{P_n} = \sum_{\nu=0}^{n} \frac{p_\nu S_{n-\nu}}{P_n} \; (P_n \neq 0) \qquad \dots \qquad (3.1)$$

defines the sequence $\{t_n\}$ of Nörlund means (Nörlund 1919) of the sequence $\{S_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series Σa_n is said to be summable (N, p_n) to the sum s if $\lim_{n\to\infty} t_n$ exists and equals s.

The conditions for regularity of the method of summability (N, p_n) defined by (3.1) are

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 \qquad \dots \qquad (3.2)$$

and

$$\sum_{k=0}^{n} |p_k| = O(|P_n|), \text{ as } n\to\infty. \qquad \dots \qquad (3.3)$$

4. The object of this paper is to introduce the concept of uniform Nörlund summability, which we define as follows:

Let

$$u_0(x)+u_1(x)+\dots \qquad \dots \qquad (4.1)$$

be any infinite series and

$$U_\nu(x) = u_0(x)+\dots+u_\nu(x).$$

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0+p_1+\dots+p_n.$$

If there exists a function $U = U(x)$ such that

$$\frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \{U_{n-\nu}(x) - U\} = o(1), \qquad \dots \qquad (4.2)$$

uniformly in a set E in which $U = U(x)$ is bounded, then we shall say that the series (4.1) is summable (N, p_n) uniformly in E to the sum U.

The conditions for regularity of the method of uniform (N, p_n) summability defined by (4.2) are the same as they are in the case of ordinary (N, p_n) summability because they are independent of x.

In this paper we take $\{p_n\}$ to be a real, non-negative, monotonic non-increasing sequence, such that $P_n \to \infty$, as $n \to \infty$, so that the regularity conditions (3.2) and (3.3) are automatically satisfied.

5. In 1965 the author (Saxena 1965) established the following theorem:

THEOREM A. If

$$\Phi(t) = o(t/\log(1/t)) \quad \ldots \quad (5.1)$$

uniformly in a set E in which $S = S(x)$ is bounded, as $t \to +0$, then the series (1.1) is summable by Harmonic means uniformly in E to the sum S.

In the present paper we generalize theorem A by replacing the special sequence of coefficients $p_n = \frac{1}{n+1}$ by a more general sequence of coefficients.

6. We establish the following theorems:

Theorem 1. If $\alpha(t)$ denotes a function of t, $\alpha(t)$ and $\frac{t}{\alpha(t)}$ ultimately increase steadily with t,

$$\log n = O(\alpha(P_n)), \text{ as } n \to \infty; \quad \ldots \quad (6.1)$$

and

$$\Phi(t) = o(t/\alpha(P_\tau)), \quad \ldots \quad (6.2)$$

uniformly in a set E in which $S = S(x)$ is bounded, as $t \to +0$, then the series (1.1) is summable (N, p_n) uniformly in E to the sum S.

Theorem 2. If $\alpha(t)$ stands for a function of t and ultimately increases steadily with t,

$$\int_{\frac{1}{n}}^{\delta} \frac{P_\tau}{\alpha(P_\tau)} \cdot \frac{1}{t}\, dt = O(P_n), \text{ as } n \to \infty; \quad \ldots \quad (6.3)$$

and

$$\Phi(t) = o(t/\alpha(P_\tau)), \quad \ldots \quad (6.4)$$

uniformly in a set E in which $S = S(x)$ is bounded, as $t \to +0$, then the series (1.1) is summable (N, p_n) uniformly in E to the sum S.

7. In order to prove these theorems we require the following lemmas:

Lemma 1 (McFadden, 1942): If p_n is non-negative and non-increasing, then, for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$, and any n,

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| < KP_\tau,$$

where K is an absolute constant and $P_m = p_0 + p_1 + \ldots + p_m$.

Lemma 2 (Hardy and Rogosinski, 1944): Suppose that f and g have period 2π; that f is L and g is V; that λ and α are real; and that $-\pi \leqslant a \leqslant b \leqslant \pi$. Then

$$J(a, b, \alpha, \lambda) = \int_a^b f(\theta+\alpha)\, g(\theta)\, e^{-\lambda i\theta}\, d\theta \to 0$$

uniformly in a, b and α, when $|\lambda| \to \infty$.

8. *Proof of Theorem 1.* It is well known that

$$S_n(x)-S = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{2\pi}\left(\int_0^{\delta}+\int_{\delta}^{\pi}\right)\phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{2\pi}\int_0^{\delta}\phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$+\frac{1}{2\pi}\int_{\delta}^{\pi}f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$+\frac{1}{2\pi}\int_{\delta}^{\pi}f(x-t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$-\frac{S}{\pi}\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{2\pi}\int_0^{\delta}\phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$+L_1+L_2+L_3, \text{ say.} \qquad \qquad (8.1)$$

Now

$$L_1 = o(1), \text{ uniformly in } E \text{ (by Lemma 2)} \qquad \qquad (8.2)$$

$$L_2 = \frac{1}{2\pi}\int_{\delta}^{\pi}f(x-t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= -\frac{1}{2\pi}\int_{-\delta}^{-\pi}f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\delta}f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

$$= o(1), \text{ uniformly in } E \text{ (by Lemma 2)} \qquad \qquad (8.3)$$

And

$$L_3 = -\frac{S}{\pi}\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

or

$$|L_3| = \frac{|S|}{\pi}\left|\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt\right|$$

$$< \frac{M}{\pi}\left|\int_{\delta}^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt\right|$$

since $|S| = |S(x)| \leqslant M$ for every $x \in E$, where M is some constant.

Therefore

$$|L_3| = o(1), \text{ uniformly in } E \text{ (by Lemma 2).}$$

Hence

$$L_3 = o(1), \text{ uniformly in } E. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.4)$$

Now from (8.1), (8.2), (8.3) and (8.4), we have

$$S_n(x) - S = \frac{1}{2\pi}\int_0^\delta \phi(t) \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} dt + o(1), \text{ uniformly in } E.$$

And therefore

$$\frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu\{S_{n-\nu}(x)-S\} = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \frac{1}{2\pi}\int_0^\delta \phi(t)\frac{\sin (n-\nu+\frac{1}{2})t}{\sin \frac{1}{2}t} dt$$

$$+o(1), \text{ uniformly in } E,$$

$$= \int_0^\delta \phi(t) \frac{1}{2\pi P_n}\left\{\sum_{\nu=0}^{n} p_\nu \frac{\sin (n-\nu+\frac{1}{2})t}{\sin \frac{1}{2}t}\right\} dt + o(1), \text{ uniformly in } E,$$

$$= \int_0^\delta \phi(t)\, \mu_n(t)\, dt + o(1), \text{ uniformly in } E,$$

where

$$\mu_n(t) = \frac{1}{2\pi P_n}\left\{\sum_{\nu=0}^{n} p_\nu \frac{\sin (n-\nu+\frac{1}{2})t}{\sin \frac{1}{2}t}\right\}$$

$$= \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta}\right) \phi(t)\, \mu_n(t)\, dt + o(1), \text{ uniformly in } E,$$

$$= F_1(x, n) + F_2(x, n) + o(1), \text{ uniformly in } E, \text{ say.} \qquad \ldots \quad \ldots \quad \ldots \quad (8.5)$$

Now

$$F_1(x, n) = \int_0^{\frac{1}{n}} \phi(t)\, \mu_n(t)\, dt$$

$$= O\left(\int_0^{\frac{1}{n}} |\phi(t)|\, |\mu_n(t)|\, dt\right)$$

$$= O\left(n \int_0^{\frac{1}{n}} |\phi(t)|\, dt\right)$$

since $\mu_n(t) = O(n)$, as $n \to \infty$, uniformly in $0 < t < \dfrac{1}{n}$ (Pati, 1961, p. 88)

$$= O\left(n \cdot \frac{n^{-1}}{\alpha(P_n)}\right), \text{ uniformly in } E,$$

$$= O\left(\frac{1}{\alpha(P_n)}\right), \text{ uniformly in } E,$$

$$= o(1), \text{ uniformly in } E \text{ (by hypothesis).} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.6)$$

Again

$$F_2(x, n) = \int_{\frac{1}{n}}^{\delta} \phi(t)\,\mu_n(t)\,dt$$

$$= O\left(\int_{\frac{1}{n}}^{\delta} |\phi(t)|\,|\mu_n(t)|\,dt\right)$$

$$= O\left(\frac{1}{P_n}\int_{\frac{1}{n}}^{\delta} |\phi(t)|\,\frac{P_\tau}{t}\,dt\right)$$

since, for $\frac{1}{n} \leqslant t \leqslant \delta < \pi$, $\mu_n(t) = O\left(\frac{P_\tau}{P_n t}\right)^*$.

Now

$$\frac{1}{P_n}\int_{\frac{1}{n}}^{\delta} |\phi(t)|\,\frac{P_\tau}{t}\,dt = \frac{1}{P_n}\left[\Phi(t)\,\frac{P_\tau}{t}\right]_{\frac{1}{n}}^{\delta} + \frac{1}{P_n}\int_{\frac{1}{n}}^{\delta} \Phi(t)\,\frac{P_\tau}{t^2}\,dt + o(1), \text{ uniformly in } E,$$

$$= \left\{O\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(P_n)}\right) + o\left(\frac{1}{P_n}\int_{\frac{1}{n}}^{\delta}\frac{P_\tau}{\alpha(P_\tau)}\cdot\frac{1}{t}\,dt\right) + o(1)\right\}, \text{ uniformly in } E,$$

$$= \left\{o(1) + o\left(\frac{1}{P_n}\cdot\frac{P_n}{\alpha(P_n)}\int_{\frac{1}{n}}^{\delta}\frac{1}{t}\,dt\right)\right\}, \text{ uniformly in } E,$$

by the hypothesis that $\alpha(t)$ and $t/\alpha(t)$ ultimately increase steadily with t,

$$= \left\{o(1) + o\left(\frac{1}{\alpha(P_n)}\int_{\frac{1}{n}}^{\delta}\frac{1}{t}\,dt\right)\right\}, \text{ uniformly in } E,$$

$$= \left\{o(1) + o\left(\frac{1}{\alpha(P_n)}\right) + o\left(\frac{\log n}{\alpha(P_n)}\right)\right\}, \text{ uniformly in } E,$$

$$= o(1), \text{ uniformly in } E \text{ (by hypothesis).}$$

Hence

$$F_2(x, n) = o(1), \text{ uniformly in } E. \qquad \ldots \quad \ldots \quad \ldots \quad (8.7)$$

Now from (8.5), (8.6) and (8.7) the theorem follows.

Proof of Theorem 2. The proof of theorem 2 runs parallel to the proof of theorem 1.

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* By a technique similar to that followed in Pati (1961, p. 89) and by Lemma 1.

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