ON SOME INTEGRALS INVOLVING MEIJER'S G-FUNCTION OF TWO VARIABLES

by R. U. VERMA, *Department of Mathematics and Astronomy, University of Lucknow, Lucknow*

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In this paper a number of infinite integrals and their related expansions, involving Meijer's G-function of two variables by using Mellin transforms, have been obtained. In the first instance, an integral is evaluated with the help of Mellin transform of one variable. Then double Mellin transform due to Reed (1944) has been employed in evaluating the integrals involving the product of two Meijer's G-functions of two variables.

1. INTRODUCTION

Recently in 1965, Agarwal defined a G-function of two variables which includes not only the Meijer's G-function of one variable as a particular case, but also most of the known functions of two variables. Thus the integrals which we obtain are very general. These integrals incorporate as special cases the results of Rathie (1965) which involve Appell's function F_4 and its various particular cases of interest.

The following notations will be used throughout the present paper.

The symbol $(\epsilon_{m,\,p})$ denotes the sequence of $(p-m+1)$ parameters $\epsilon_m, \epsilon_{m+1}, \ldots, \epsilon_p$ but when $m = 1$, we shall denote it by (ϵ_p) instead of by $(\epsilon_{1,\,p})$. As usual $\Gamma[(a_A)\,;\,(b_B)]$ denotes $\left\{\prod_{r=1}^{A}\Gamma(a_r)\right\}\left\{\prod_{r=1}^{B}\Gamma(b_r)\right\}^{-1}$. Also the symbol $\Delta_n[m;\alpha]$ denotes $\frac{\alpha}{m}, \frac{\alpha+1}{m}, \ldots, \frac{\alpha+n-1}{m}$ parameters, but when $n = m$ we shall denote it by $\Delta[m\,;\alpha]$ instead of by $\Delta_m[m\,;\alpha]$.

We shall use the following results:

$$\int_0^\infty z^{\rho-1}G^{m',\,n'}_{l,\,u}\left(\alpha z\,\middle|\,\begin{matrix}(a_l)\\(b_u)\end{matrix}\right)dz = \Gamma[(b_{m'})+\rho,\ 1-(a_{n'})-\rho;\ 1-(b_{m'+1,\,u})-\rho,\ (a_{n'+1,\,l})+\rho]\alpha^{\rho}, \quad \ldots \quad (1.1)$$

where

$$l+u < 2(m'+n'),\ |\arg\alpha| < \pi[m'+n'-\tfrac{1}{2}(l+u)]$$

and

$$-\min_{1<j<m'} R(b_j) < R(P) < 1-\max_{1\leqslant j\leqslant n'} R(a_j).$$

The Meijer's G-function of two variables due to Agarwal (1965) is defined by the relation

$$G^{n_1, \nu_1, \nu_2, m_1, m_2}_{p, [t:t'], s, [q:q']}\left[\begin{matrix} x \\ \\ y \end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);\ (\gamma'_{t'})\\ (\delta_s)\\ (\beta_q);\ (\beta'_{q'})\end{matrix}\right.\right] = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Phi(\xi+\eta)\psi(\xi, \eta)x^{\xi}y^{\eta}\,d\xi\,d\eta, \quad \text{.. (1.2)}$$

where

$$\Phi(\xi+\eta) = \Gamma\begin{bmatrix}1-(\epsilon_{n_1})+\xi+\eta; \\ (\epsilon_{n_1+1, p})-\xi-\eta,\ (\delta_s)+\xi+\eta\end{bmatrix},$$

$$\psi(\xi, \eta) = \Gamma\begin{bmatrix}(\beta_{m_1})-\xi,\ (\gamma_{\nu_1})+\xi,\ (\beta'_{m_2})-\eta,\ (\gamma'_{\nu_2})+\eta; \\ 1-(\beta_{m_1+1, q})+\xi,\ 1-(\gamma_{\nu_1+1, t})-\xi,\ 1-(\beta'_{m_2+1, q'})+\eta,\ 1-(\gamma'_{\nu_2+1, t'})-\eta\end{bmatrix},$$

$$0 \leqslant m_1 \leqslant q,\ 0 \leqslant m_2 \leqslant q',\ 0 \leqslant \nu_1 \leqslant t,\ 0 \leqslant \nu_2 \leqslant t',\ 0 \leqslant n_1 \leqslant p.$$

The integral (1.2) converges if

$$p+q+s+t < 2(m_1+\nu_1+n_1),\ \ p+q'+s+t' < 2(m_2+\nu_2+n_1),$$
$$|\arg x| < \pi[m_1+\nu_1+n_1-\tfrac{1}{2}(p+q+s+t)] \text{ and}$$
$$|\arg y| < \pi[m_2+\nu_2+n_1-\tfrac{1}{2}(p+q'+s+t')].$$

It is desirable to mention that the modified form due to A. Verma which has been taken here is slightly variant from that of Agarwal (1965), though in essence the function is the same.

Agarwal (1965) studied the behaviour of (1.2) only for smaller values of x and y; but the behaviour for larger values of x and y has not been studied even though it has a meaning as $x \to \infty$ and $y \to \infty$.

2. The First Main Result

In this section the main result has been obtained first and later some interesting expansions have been derived. We have

$$\int_0^{\infty} z^{\rho-1}G^{m', n'}_{l, u}\left(\alpha z\left|\begin{matrix}(a_l)\\ (b_u)\end{matrix}\right.\right) G^{n_1, \nu_1, \nu_2, m_1, m_2}_{p, [t:t'], s, [q:q']}\left[\begin{matrix} xz^n \\ \\ yz^n \end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);\ (\gamma'_{t'})\\ (\delta_s)\\ (\beta_q);\ (\beta'_{q'})\end{matrix}\right.\right] dz$$

$$= (2\pi i)^{n'}\alpha^{\rho}(2\pi)^{\frac{1}{2}(1-n)(2m'+2n'-u-l)}(n)^{\sum_{j=1}^{u} b_j-\sum_{j=1}^{l} a_j+(\rho-\frac{1}{2})(u-l)}$$

$$\times \sum_{r=1}^{\infty}\left\{e^{-i\pi(2r-1)\left[\sum_{j=1}^{n'} a_j+n'\rho\right]} G^{n_1+nm', \nu_1, \nu_2, m_1, m_2}_{p+nu, [t:t], s+nl, [q:q']}\right.$$

$$\left.\times\left[\begin{matrix}\alpha^n n^{n(u-l)}e^{-i\pi nn'(2r-1)}x \\ \\ \alpha^n n^{n(u-l)}e^{-i\pi nn'(2r-1)}y\end{matrix}\left|\begin{matrix}\Delta_n[-n;\ (b_{m'})+\rho-n],\ (\epsilon_p),\ \Delta[n;\ 1-(b_{m'+1, u})-\rho]\\ (\gamma_t);\ (\gamma'_{t'})\\ (\delta_s),\ \Delta[n;\ (a_l)+\rho]\\ (\beta_q);\ (\beta'_{q'})\end{matrix}\right.\right]\right\}, \quad \text{.. (2.1)}$$

where n is a positive integer.

The conditions of validity of (2.1) are:

$$p+q+s+t < 2(m_1+\nu_1+n_1),\quad p+q'+s+t' < 2(m_2+\nu_2+n_1),$$

$$|\arg x| < \pi[m_1+\nu_1+n_1-\tfrac{1}{2}(p+q+s+t)],$$

$$|\arg y| < \pi[m_2+\nu_2+n_1-\tfrac{1}{2}(p+q'+s+t')],$$

$$l+u < 2(m'+n'),\quad |\arg \alpha| < \pi[m'+n'-\tfrac{1}{2}(l+u)] \text{ and}$$

$$-\min_{1\leqslant j\leqslant m'} R(b_j) < R(\rho) < 1-\max_{1\leqslant j\leqslant n'} R(a_j).$$

Proof of (2.1):

Substituting (1.2) for the G-function of two variables in the left-hand side of (2.1) and changing the order of integration as the integrals involved are absolutely convergent under the conditions mentioned, we get

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\Phi(\xi+\eta)\psi(\xi,\eta)x^\xi y^\eta\int_0^\infty z^{\rho+n(\xi+\eta)-1}G_{l,u}^{m',n'}\left(\alpha z\,\middle|\,\begin{matrix}a_l\\ b_u\end{matrix}\right)dz\,d\xi\,d\eta.$$

On using (1.1) in the above expression, we see that it equals

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\psi(\xi,\eta)\Gamma\begin{bmatrix}1-(\epsilon_{n_1})+\xi+\eta\,;\\ (\epsilon_{n_1+1,p})-\xi-\eta,\ (\delta_s)+\xi+\eta\end{bmatrix}$$

$$\times\Gamma\begin{bmatrix}(b_{m'})+\rho+n(\xi+\eta),\ 1-(a_{n'})-\rho-n(\xi+\eta)\,;\\ 1-(b_{m'+1,u})-\rho-n(\xi+\eta),\ (a_{n'+1,l})+\rho+n(\xi+\eta)\end{bmatrix}\alpha^{\rho+n(\xi+\eta)}x^\xi y^\eta\,d\xi\,d\eta$$

$$=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\psi(\xi,\eta)\Gamma\begin{bmatrix}1-(\epsilon_{n_1})+\xi+\eta\,;\\ (\epsilon_{n_1+1,p})-\xi-\eta,\ (\delta_s)+\xi+\eta\end{bmatrix}$$

$$\times\Gamma\begin{bmatrix}(b_{m'})+\rho+n(\xi+\eta)\,;\\ 1-(b_{m'+1,u})-\rho-n(\xi+\eta),\ (a_l)+\rho+n(\xi+\eta)\end{bmatrix}\alpha^{\rho+n(\xi+\eta)}$$

$$\times\prod_{j=1}^{n'}\{\pi\operatorname{cosec}\pi[a_j+\rho+n(\xi+\eta)]\}x^\xi y^\eta\,d\xi\,d\eta.$$

Using the exponential definition of cosec $\pi[a_j+\rho+n(\xi+\eta)]$ and expanding it, we get on term-by-term integration the required result with the help of the multiplication formula for the Gamma function.

3. Certain Particular Cases

By giving suitable values to the parameters in (2.1), we get the following results. Many more results can be obtained from it, some may be even new.

(i) Taking $n_1 = 1$, $\nu_1 = 0$, $\nu_2 = 0$, $m_1 = 1$, $m_2 = 1$, $p = 2$, $t = 0$, $t' = 0$, $s = 0$, $q = 2$, $q' = 2$ and then using Agarwal (1965), we obtain

$$\int_0^\infty z^{\rho+n(\beta_1+\beta_1')-1} G_{l,u}^{m',n'}\left(\alpha z \left| \begin{matrix} (a_l) \\ (b_u) \end{matrix} \right.\right) \frac{\Gamma(1-\epsilon_1+\beta_1+\beta_1')x^{\beta_1}y^{\beta_1'}}{\Gamma(1-\beta_2+\beta_1)\Gamma(1-\beta_2'+\beta_1')\Gamma(\epsilon_2-\beta_1-\beta_1')}$$

$$\times F^{[4]}\left[1-\epsilon_1+\beta_1+\beta_1', 1-\epsilon_2+\beta_1+\beta_1'; 1-\beta_2+\beta_1, 1-\beta_2'+\beta_1'; xz^n, yz^n\right] dz$$

$$= (2\pi i)^{n'} \alpha^\rho (2\pi)^{\frac{1}{2}(1-n)(2m'+2n'-u-l)} (n)^{\sum_{j=1}^{u} b_j - \sum_{j=1}^{l} a_j + (\rho-\frac{1}{2})(u-l)}$$

$$\times \sum_{r=1}^{\infty} \left\{ e^{-i\pi(2r-1)\left[\sum_{j=1}^{n'} a_j + n'\rho\right]} G_{2+nu,[0:0],nl,[2:2]}^{1+nm',0,0,1,1} \right.$$

$$\left. \times \left[\begin{matrix} \alpha^n n^{n(u-l)} e^{-i\pi nn'(2r-1)} x \\ \\ \alpha^n n^{n(u-l)} e^{-i\pi nn'(2r-1)} y \end{matrix} \left| \begin{matrix} \Delta n[-n; (b_{m'})+\rho-n], (\epsilon_2), \Delta[n; 1-(b_{m'+1,u})-\rho] \\ \cdots \quad \cdots \\ \Delta[n; (a_l)+\rho] \\ (\beta_2); (\beta_2') \end{matrix} \right. \right] \right\}.$$

where n is a positive integer.

The conditions of validity are the same as in the main result after proper substitution.

This result is an extension of Rathie's main result.

(ii) Taking $n' = 0$, we obtain a very interesting case

$$\int_0^\infty z^{\rho-1} G_{l,u}^{m',0}\left(\alpha z \left| \begin{matrix} (a_l) \\ (b_u) \end{matrix} \right.\right) G_{p,[t:t'],s,[q:q']}^{n_1+\nu_1,\nu_2,m_1 m_2} \left[\begin{matrix} xz^n \\ \\ yz^n \end{matrix} \left| \begin{matrix} (\epsilon_p) \\ (\gamma_t); (\gamma_{t'}') \\ (\delta_s) \\ (\beta_q); (\beta_{q'}') \end{matrix} \right. \right] dz$$

$$= \alpha^\rho (2\pi)^{\frac{1}{2}(1-n)(2m'-u-l)} (n)^{\sum_{j=1}^{u} b_j - \sum_{j=1}^{l} a_j + (\rho-\frac{1}{2})(u-l)} G_{p+nu,[t:t'],s+nl,[q:q']}^{n_1+nm',\nu_1,\nu_2,m_1,m_2}$$

$$\times \left[\begin{matrix} \alpha^n n^{n(u-l)} x \\ \\ \alpha^n n^{n(u-l)} y \end{matrix} \left| \begin{matrix} \Delta n[-n; (b_{m'})+\rho-n], (\epsilon_p), \Delta[n; 1-(b_{m'+1,u})-\rho] \\ (\gamma_t); (\gamma_{t'}') \\ (\delta_s), \Delta[n, (a_l)+\rho] \\ (\beta_q); (\beta_{q'}') \end{matrix} \right. \right],$$

where n is a positive integer.

4. The Second Main Result

$$\int_0^\infty z^{\rho-1} G_{l,u}^{m',n'}\left(\alpha z \,\middle|\, \begin{matrix}(a_l)\\(b_u)\end{matrix}\right) G_{p+nn',[t:t'],s,[q:q']}^{n_1,\nu_1,\nu_2,m_1,m_2}\left[\begin{matrix} n^{nn'}xz^n \\ \\ n^{nn'}yz^n \end{matrix} \,\middle|\, \begin{matrix} (\epsilon_p),\ \Delta[n;\ 1-(a_{n'})-\rho] \\ (\gamma_t);\ (\gamma'_{t'}) \\ (\delta_s) \\ (\beta_q);\ (\beta'_{q'}) \end{matrix}\right] dz$$

$$= \alpha^\rho (2\pi)^{\frac{1}{2}(1-n)(2m'+2n'-u-l)} (n)^{\sum_{j=1}^{u} b_j - \sum_{j=1}^{l} a_j + (\rho-\frac{1}{2})(u-l)} G_{p+nu,[t:t'],s+nl,[q:q']}^{n_1+nm',\nu_1,\nu_2,m_1,m_2}$$

$$\times \left[\begin{matrix} \alpha^n n^{n(u-l+n')}x \\ \\ \alpha^n n^{n(u-l+n')}y \end{matrix} \,\middle|\, \begin{matrix} \Delta_n[-n;\ (b_{m'})+\rho-n],\ (\epsilon_p),\ \Delta[n;\ 1-(b_{m'+1,u})-\rho] \\ (\gamma_t);\ (\gamma'_{t'}) \\ (\delta_s),\ \Delta[n;\ (a_{n'+1,l})+\rho] \\ (\beta_q);\ (\beta'_{q'}) \end{matrix}\right], \tag{4.1}$$

where n is a positive integer.

Proof: In proving this result, we proceed as in (2.1) and get the required result by using the multiplication formula for the Gamma function and later (1.2).

The result is interesting in the sense that it does not consist of an infinite series of the G-functions of two variables as in (2.1).

However, if we replace x, y and p by $xn^{nn'}$, $yn^{nn'}$, $p+nn'$ respectively and replace $(\epsilon_{p+1,\,p+k})$ by $\dfrac{1-(a_{n'})-\rho'+k}{n'}$ for $k = 1, 2, \ldots, nn'$ in (2.1) and equate the left-hand sides of (2.1) and (4.1), we get the sum of an infinite series of G-functions of two variables as a G-function of two variables, viz.

$$G_{p+nu,[t:t'],s+nl,[q:q']}^{n_1+nm',\nu_1,\nu_2,m_1,m_2}\left[\begin{matrix} \alpha^n n^{n(u-l+n')}x \\ \\ \alpha^n n^{n(u-l+n')}y \end{matrix} \,\middle|\, \begin{matrix} \Delta_n[-n;\ (b_{m'})+\rho-n],\ (\epsilon_p),\ \Delta[n;\ 1-(b_{m'+1,u})-\rho] \\ (\gamma_t);\ (\gamma'_{t'}) \\ (\delta_s),\ \Delta[n;\ (a_{n'+1,l})+\rho] \\ (\beta_q);\ (\beta'_{q'}) \end{matrix}\right]$$

$$= (2\pi i)^{n'} \sum_{r=1}^{\infty} \left\{ e^{-i\pi(2r-1)\left[\sum_{j=1}^{n'} a_j + n'\rho\right]} G_{p+nu+nn',[t:t'],s+nl,[q:q']}^{n_1+nm',\nu_1,\nu_2,m_1,m_2} \right.$$

$$\left. \times \left[\begin{matrix} \alpha^n n^{n(u-l+n')} e^{-i\pi nn'(2r-1)}x \\ \\ \alpha^n n^{n(u-l+n')} e^{-i\pi nn'(2r-1)}y \end{matrix} \,\middle|\, \begin{matrix} \Delta_n[-n;\ (b_{m'})+\rho-n],\ (\epsilon_p),\ \Delta[n;\ 1-(b_{m'+1,u})-\rho],\ \Delta[n;\ 1-(a_{n'})-\rho] \\ (\gamma_t);\ (\gamma'_{t'}) \\ (\delta_s),\ \Delta[n;\ (a_l)+\rho] \\ (\beta_q);\ (\beta'_{q'}) \end{matrix}\right] \right\}, \quad \ldots \tag{4.2}$$

where n is a positive integer.

5. The Third Main Result

In this section we obtain a double integral analogue of (2.1) in the form (5.1) by using the double Mellin transform,

$$\int_0^\infty\int_0^\infty u^{\rho-1}v^{\lambda-1}G^{n',\,\nu_1',\,\nu_2',\,m_1',\,m_2'}_{p',\,[t_1:t_1'],\,s',\,[q_1:q_1']}\left[\begin{matrix}\alpha u\\ \\ \beta v\end{matrix}\middle|\begin{matrix}(a_{p'})\\ (c_{t_1});(c'_{t_1'})\\ (d_{s'})\\ (b_{q_1});(b'_{q_1'})\end{matrix}\right]$$

$$\times G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,[t:t'],\,s,\,[q:q']}\left[\begin{matrix}xu\\ \\ yv\end{matrix}\middle|\begin{matrix}(\epsilon_p)\\ (\gamma_t);(\gamma'_{t'})\\ (\delta_s)\\ (\beta_q);(\beta'_{q'})\end{matrix}\right]du\,dv$$

$$=(2\pi i)^{n'}\alpha^\rho\beta^\lambda\sum_{r=1}^{\infty}\left\{e^{-i\pi(2r-1)\left(\sum_{j=1}^{n'}a_j+n'(\rho+\lambda)\right)}\right.$$

$$\left.\times G^{n,\,\nu_1+m_1',\,\nu_2+m_2',\,m_1+\nu_1',\,m_2+\nu_2'}_{p+s',\,[t+q_1:t'+q_1'],\,s+p',\,[q+t_1:q'+t_1']}\left[\begin{matrix}e^{-i\pi n'(2r-1)}\alpha x\\ \\ e^{-i\pi n'(2r-1)}\beta y\end{matrix}\middle|\begin{matrix}(\epsilon_p),(d_{s'})\\ (\gamma_t),(b_{q_1});(\gamma'_{t'}),(b'_{q_1'})\\ (\delta_s),(a_{p'})\\ (\beta_q),(c_{t_1});(\beta'_{q'}),(c'_{t_1'})\end{matrix}\right]\right\},\qquad ..\ (5.1)$$

valid under the conditions:

$p+q+s+t<2(m_1+\nu_1+n)$, $p+q'+s+t'<2(m_2+\nu_2+n)$,

$|\arg x|<\pi[m_1+\nu_1+n-\frac{1}{2}(p+q+s+t)]$, $|\arg y|<\pi[m_2+\nu_2+n-\frac{1}{2}(p+q'+s+t')]$,

$p'+q_1+s'+t_1<2(m_1'+\nu_1'+n')$, $p'+q_1'+s'+t_1'<2(m_2'+\nu_2'+n')$,

$|\arg\alpha|<\pi\left[m_1'+\nu_1'+n'-\frac{1}{2}(p'+q_1+s'+t_1)\right]$,

$|\arg\beta|<\pi\left[m_2'+\nu_2'+n'-\frac{1}{2}(p'+q_1'+s'+t_1')\right]$,

$R(\rho+\lambda)<1-\max\limits_{1<j<n'}R(a_j)$, $-\min\limits_{1<j<m_1'}R(b_j)<R(\rho)<\max\limits_{1<j<\nu_1'}R(c_j)$ and

$-\min\limits_{1<j<m_2'}R(b_j')<R(\lambda)<\max\limits_{1<j<\nu_2'}R(c_j')$.

Proof: Taking the left side of (5.1) and changing the order of integration, as justified, we get

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\psi(\xi,\eta)\Phi(\xi+\eta)x^{\xi}y^{\eta}\int_0^\infty\int_0^\infty u^{\rho+\xi-1}v^{\lambda+\eta-1}$$

$$\times G^{n',\nu_1',\nu_2',m_1',m_2'}_{p',[t_1:t_1'],s',[q_1:q_1']}\left[\begin{matrix}\alpha u\\ \\ \beta v\end{matrix}\left|\begin{matrix}(a_{p'})\\ (c_{t_1});\ (c'_{t_1'})\\ (d_{s'})\\ (b_{q_1});\ (b'_{q_1'})\end{matrix}\right.\right]du\,dv\,d\xi\,d\eta.$$

Now, by solving the inner double integral involving Meijer's G-function of two variables with the help of the theory of double Mellin transform (Reed 1944) and then using $\Gamma(z)\Gamma(1-z)=\pi\operatorname{cosec}\pi z$, the required result is obtained.

Many more results can be obtained as particular cases, e.g. for $n'=0$ in (5.1) we get one of its very interesting special cases. We can get more general results by taking the variables in the G-functions as u^n and v^n and using the multiplication formula for Gamma function, n being a positive integer.

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