FUEL EXPENDITURE IN CONSERVATIVE FLIGHT

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The behaviour of the conservative path of an aircraft, when the load factor (i.e. lift-to-weight ratio) is unity, is investigated by taking different angles of initial projection of the vehicle; it is found that in each case the trajectory is a loop and that it is symmetrical about a vertical line. Also the fuel expenditure for the two portions of the trajectory before and after reaching the highest point has been calculated and some numerical examples have been worked out.

Nomenclature

X, h = horizontal distance and altitude respectively from the starting-point,
V = velocity of the vehicle at any time,
γ = path inclination at any time,
W = instantaneous weight of the vehicle,
D = the aerodynamic drag,
L = the aerodynamic lift,
C = specific fuel consumption (assumed constant),
g = acceleration due to gravity (assumed constant),
E_{max} = maximum aerodynamic efficiency,
V_i = initial velocity of the vehicle,
W_i = initial weight of the vehicle,
γ_0 = initial path inclination, i.e. the initial angle of projection of the vehicle,

$$u = \frac{V}{V_i}, \quad \xi = \frac{xg}{V_i^2}, \quad \eta = \frac{hg}{V_i^2}, \quad \mu = \frac{W}{W_i},$$

ξ_1 = the value of ξ at the highest point of the trajectory,
μ_1 = the value of μ at the highest point of the trajectory,
μ_f = the value of μ at the final point of the trajectory,
W_p = propellant expenditure at any time = $W_i - W$,
W_{p_1} = propellant expenditure from the starting-point to the highest point of the trajectory,

W_{p_2} = propellant expenditure from the highest point to the final point of the trajectory,

$$\zeta = \frac{W_p}{W_i}, \quad \zeta_1 = \frac{W_{p_1}}{W_i}, \quad \zeta_2 = \frac{W_{p_2}}{W_i}, \quad \zeta_f = \zeta_1 + \zeta_2,$$

and a dot sign represents differentiation with respect to time.

Introduction

The conservative path when the load factor is constant and greater than unity has been studied by Miele (1962). It has been shown there that the trajectory is a loop when the load factor is greater than one. In Miele's analysis initial path inclination has been taken as zero. It appears from this analysis that with load factor unity and initial path inclination zero the trajectory reduces to the trivial case of a horizontal straight line. So, for the investigation of the case when the load factor is unity, it is necessary to start with a non-zero initial path inclination. This has been done in the present paper and it is found that the trajectory has a loop similar to that in Miele's case when load factor is greater than one. But the loop in the present case (when load factor = 1) depends strongly on the initial path inclination. The trajectory has been actually computed in five particular cases with initial path inclinations 20°, 30°, 40°, 50° and 60°.

The main point, however, in this paper is the calculation of fuel expenditure in the flight. As expected it is found that the fuel expenditure in the first half of the trajectory is greater than that for the second half. But this again depends strongly on the initial path inclination. The difference in fuel expenditure in the two parts diminishes as the initial path inclination increases. This has been shown by numerical calculation with suitable data for turbojet aircraft.

Equations of Flight

For a conservative path, the thrust always balances the aerodynamic drag. Therefore the path traced out by the vehicle depends only on the action of instantaneous aerodynamic lift and the instantaneous weight of the vehicle. The ratio $n = \frac{L}{W}$ is called the load factor and we shall consider the particular case when $n = 1$, i.e. when the lift balances the instantaneous weight. Then the equations of flight may be written as (Miele 1962, pp. 260, 268)

$$\dot{X} - V \cos \gamma = 0 \qquad (1)$$

$$\dot{h} - V \sin \gamma = 0 \qquad (2)$$

$$\dot{V} + g \sin \gamma = 0 \qquad (3)$$

$$\dot{\gamma} = \frac{g}{V}(1 - \cos \gamma) \qquad (4)$$

$$\dot{W} + CD = 0 \qquad (5)$$

Also

$$D = \frac{W}{2E_{\max}}\left(u^2 + \frac{1}{u^2}\right). \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Assuming the inclination of the trajectory at the starting-point to be γ_0, we get from (3) and (4)

$$u = \frac{1-\cos\gamma_0}{1-\cos\gamma}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Trajectory Computation

From (1) and (4) we get

$$\frac{dX}{d\gamma} = \frac{V^2\cos\gamma}{g(1-\cos\gamma)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

which, on transformation by means of the relations

$$\xi = \frac{Xg}{V_i^2} \quad \text{and} \quad u = \frac{V}{V_i},$$

reduces to

$$\frac{d\xi}{d\gamma} = (1-\cos\gamma_0)^2\frac{\cos\gamma}{(1-\cos\gamma)^3} \quad \text{[by (7)].}$$

Integrating,

$$\xi = (1-\cos\gamma_0)^2\int_{\gamma_0}^{\gamma}\frac{\cos\gamma\, d\gamma}{(1-\cos\gamma)^3}$$

$$= \tfrac{1}{4}(1-\cos\gamma_0)^2\left[\left(\cot\frac{\gamma}{2} - \cot\frac{\gamma_0}{2}\right) - \tfrac{1}{5}\left(\cot^5\frac{\gamma}{2} - \cot^5\frac{\gamma_0}{2}\right)\right]. \quad \ldots \quad (9)$$

Again from (2) and (4) we get

$$\frac{dh}{d\gamma} = \frac{V^2\sin\gamma}{g(1-\cos\gamma)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

which, by means of the relations

$$\eta = \frac{hg}{V_i^2} \quad \text{and} \quad u = \frac{V}{V_i},$$

can be transformed to

$$\frac{d\eta}{d\gamma} = (1-\cos\gamma_0)^2\frac{\sin\gamma}{(1-\cos\gamma)^3} \quad \text{[by (7)].}$$

Integrating, we get

$$\eta = (1-\cos\gamma_0)^2\int_{\gamma_0}^{\gamma}\frac{\sin\gamma\, d\gamma}{(1-\cos\gamma)^3}$$

$$= \frac{1}{2(1-\cos\gamma)^2}\{\cos^2\gamma - \cos^2\gamma_0 + 2\,(\cos\gamma_0 - \cos\gamma)\}$$

$$= \tfrac{1}{2}\left\{1 - \left(\frac{1-\cos\gamma_0}{1-\cos\gamma}\right)^2\right\}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

The eqs. (10) and (11) give the parametric equation of the path in terms of γ. A graph is drawn showing the ξ, η relation for different values of γ_0 while γ changes from γ_0 to $2\pi-\gamma_0$ (Fig. 1). It is clear that in each case the trajectory is a loop as in the case when the load factor exceeds unity (Miele 1962). The area of the loop gradually tends to zero as γ_0 tends to zero.

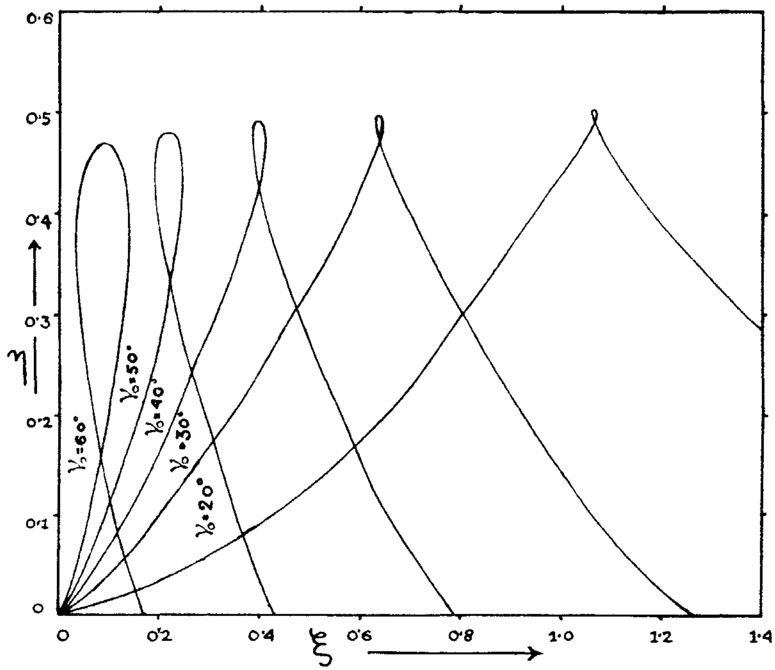

FIG. 1. Conservative trajectories for different initial path inclinations.

FUEL EXPENDITURE

Combining (5) and (6) we get

$$\dot{W} = -\frac{CW}{2E_{\max}}\left(u^2+\frac{1}{u^2}\right). \qquad (12)$$

From (4) and (12) we get

$$\frac{dW}{d\gamma} = -\frac{CWV(u^4+1)}{2gE_{\max}(1-\cos\gamma)u^2}$$

or

$$\frac{dW}{d\gamma} = -\frac{CWV(u^4+1)}{2gE_{\max}(1-\cos\gamma_0)u} \quad \text{[by (7)].} \qquad (13)$$

Transforming this by means of

$$\frac{W}{W_i} = \mu \quad \text{and} \quad \frac{V}{V_i} = u,$$

we get

$$\frac{d\mu}{d\gamma} = -\frac{CV_i\mu(u^4+1)}{2gE_{\max}(1-\cos\gamma_0)}.$$

$$\therefore \quad \frac{d\mu}{\mu} = -\frac{ad\gamma}{(1-\cos\gamma)^4} - bd\gamma \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where

$$a = \frac{CV_i(1-\cos\gamma_0)^3}{2gE_{\max}}, \; b = \frac{CV_i}{2gE_{\max}(1-\cos\gamma_0)}.$$

Integrating (14) from $\gamma = \gamma_0$ to $\gamma = \pi$ (which corresponds to the highest point of the trajectory) we get,

$$\log \mu\Big]_1^{\mu_1} = -a\int_{\gamma_0}^{\pi} \frac{d\gamma}{(1-\cos\gamma)^4} - b(\pi-\gamma_0)$$

or

$$\log \mu_1 = -aJ - b(\pi-\gamma_0) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where

$$J = \int_{\gamma_0}^{\pi} \frac{d\gamma}{(1-\cos\gamma)^4}.$$

Integrating (14) from $\gamma = \pi$ to $\gamma = 2\pi - \gamma_0$ we get

$$\log \mu\Big]_{\mu_1}^{\mu_f} = -a\int_{\pi}^{2\pi-\gamma_0} \frac{d\gamma}{(1-\cos\gamma)^4} - b(\pi-\gamma_0)$$

$$= -a\int_{\gamma_0}^{\pi} \frac{d\gamma}{(1-\cos\gamma)^4} - b(\pi-\gamma_0)$$

or

$$\log \mu_f - \log \mu_1 = -aJ - b(\pi-\gamma_0) = \log \mu_1 \quad [\text{by } (15)]$$

or

$$\log \frac{\mu_f}{\mu_1} = \log \mu_1$$

$$\therefore \quad \frac{\mu_f}{\mu_1} = \mu_1$$

or

$$\mu_f = \mu_1^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Now

$$\mu_1 = \frac{W_i - W_{p1}}{W_i} = 1 - \frac{W_{p1}}{W_i} = 1 - \zeta_1 \quad \ldots \quad \ldots \quad (17)$$

$$\mu_f = \frac{W_i - W_{p1} - W_{p2}}{W_i} = 1 - \frac{W_{p1}}{W_i} - \frac{W_{p2}}{W_i} = 1 - \zeta_1 - \zeta_2. \quad \ldots \quad (18)$$

From (16), (17) and (18) we get

$$\left.\begin{aligned} 1-\zeta_1-\zeta_2 &= (1-\zeta_1)^2 \\ \text{or} \quad \zeta_1(1-\zeta_1) &= \zeta_2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

or

$$\frac{\zeta_1}{\zeta_2} = \frac{1}{1-\zeta_1} > 1$$

$$\therefore \quad \zeta_1 > \zeta_2 \qquad \qquad (20)$$

which shows that the fuel expenditure during the first half of the manoeuvre is greater than that during the second half.

It is also evident that when the fuel expenditure for any one part is known that for the other part may be obtained from (19).

Now

$$J = \int_{\gamma_0}^{\pi} \frac{d\gamma}{(1-\cos\gamma)^4}$$

$$= \tfrac{1}{8}\left[\cot\frac{\gamma_0}{2} + \cot^3\frac{\gamma_0}{2} + \tfrac{3}{5}\cot^5\frac{\gamma_0}{2} + \tfrac{1}{7}\cot^7\frac{\gamma_0}{2}\right].$$

Also

$$\mu_1 = e^{-aJ-b(\pi-\gamma_0)} \quad \text{[by (15)]}$$

$$= e^{-\frac{a}{8}\left[\cot\frac{\gamma_0}{2}+\cot^3\frac{\gamma_0}{2}+\frac{3}{5}\cot^5\frac{\gamma_0}{2}+\frac{1}{7}\cot^7\frac{\gamma_0}{2}\right]-b(\pi-\gamma_0)} \qquad (21)$$

$$\therefore \quad \zeta_1 = 1-\mu_1 = 1-e^{-\frac{a}{8}\left[\cot\frac{\gamma_0}{2}+\cot^3\frac{\gamma_0}{2}+\frac{3}{5}\cot^5\frac{\gamma_0}{2}+\frac{1}{7}\cot^7\frac{\gamma_0}{2}\right]-b(\pi-\gamma_0)}. \qquad (22)$$

From (18) we get

$$\begin{aligned}\zeta_2 &= 1-\zeta_1-\mu_f \\ &= \mu_1-\mu_1^2 \qquad \text{[by (17) and (18)]} \\ &= \mu_1(1-\mu_1) \\ &= \zeta_1\mu_1 \\ &= \zeta_1 e^{-\frac{a}{8}\left[\cot\frac{\gamma_0}{2}+\cot^3\frac{\gamma_0}{2}+\frac{3}{5}\cot^5\frac{\gamma_0}{2}+\frac{1}{7}\cot^7\frac{\gamma_0}{2}\right]-b(\pi-\gamma_0)} \quad \text{[by (21)].}\end{aligned} \qquad (23)$$

Now

$$\begin{aligned}\zeta_f &= \zeta_1+\zeta_2 \\ &= 1-e^{-\frac{a}{4}\left[\cot\frac{\gamma_0}{2}+\cot^3\frac{\gamma_0}{2}+\frac{3}{5}\cot^5\frac{\gamma_0}{2}+\frac{1}{7}\cot^7\frac{\gamma_0}{2}\right]-2b(\pi-\gamma_0)} \quad \text{[by (22 and (23)].}\end{aligned} \qquad (24)$$

Putting $\dfrac{CV_t}{2gE_{\max}} = \lambda$ we get

$$a = \lambda(1-\cos\gamma_0)^2$$

and

$$b = \frac{\lambda}{1-\cos\gamma_0}$$

$$\therefore \quad \zeta_1 = 1 - e^{-\frac{\lambda}{8}(1-\cos\gamma_0)^3\left(\cot\frac{\gamma_0}{2}+\cot^3\frac{\gamma_0}{2}+\frac{3}{5}\cot^5\frac{\gamma_0}{2}+\frac{1}{7}\cot^7\frac{\gamma_0}{2}\right)-\frac{\lambda(\pi-\gamma_0)}{1-\cos\gamma_0}}$$

$$\zeta_2 = \zeta_1 e^{-\frac{\lambda}{8}(1-\cos\gamma_0)^3\left(\cot\frac{\gamma_0}{2}+\cot^3\frac{\gamma_0}{2}+\frac{3}{5}\cot^5\frac{\gamma_0}{2}+\frac{1}{7}\cot^7\frac{\gamma_0}{2}\right)-\frac{\lambda(\pi-\gamma_0)}{1-\cos\gamma_0}}$$

$$\zeta_f = 1 - e^{-\frac{\lambda}{4}(1-\cos\gamma_0)^3\left(\cot\frac{\gamma_0}{2}+\cot^3\frac{\gamma_0}{2}+\frac{3}{5}\cot^5\frac{\gamma_0}{2}+\frac{1}{7}\cot^7\frac{\gamma_0}{2}\right)-\frac{2\lambda(\pi-\gamma_0)}{1-\cos\gamma_0}}.$$

Numerical Examples

Taking $V_t = 750$ ft sec^{-1},
$g = 32{\cdot}174$ ft sec^{-2},
$E_{\max} = 19{\cdot}92031$, and for a simple turbojet aircraft
$C = 2$ hr^{-1},
we get $\lambda = 0{\cdot}00033$.

Then the fuel expenditure for describing the different branches of the curve are given in the following table.

	ζ_1	ζ_2	ζ_f
20°	0·01547	0·01523	0·03070
30°	0·00662	0·00658	0·01320
40°	0·00359	0·00358	0·00717
50°	0·00224	0·00223	0·00447
60°	0·00150	0·00150	0·00300

It is evident from the above that the excess of fuel expenditure for the first half gradually decreases and ultimately tends to zero as the initial path inclination increases.

Reference

Miele, A. (1962). Flight Mechanics, Vol. 1, pp. 268–271. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts.