VISCOPLASTIC ANALYSIS OF CIRCULAR CYLINDRICAL SHELLS

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The present paper is concerned with the viscoplastic flow of circular cylindrical shells. The material of the shell is assumed to be rigid, viscoplastic and incompressible. A linearized theory of viscoplasticity and the associated flow rule are used. As an example, the viscoplastic behaviour of a clamped-free circular cylindrical shell under uniform radial pressure is considered. The velocities of the shell are computed and plotted for different values of λ, a positive constant depending on the viscosity of the material. It is found that the velocity of the shell diminishes with increasing values of λ. It is also found that as $\lambda \to 0$, the material tends to be rigid, perfectly plastic and the velocity of the shell is undetermined.

1. Introduction

The stress analysis of viscoplastic bodies is often based on the model of an incompressible Bingham Solid (Bingham 1922). When the stress is below a certain threshold which is specified by a yield condition, the solid is rigid; for larger values of the stress, the rate of deformation depends on how far the stress has exceeded the yield limit.

Appleby and Prager (1962) have shown that the use of Tresca's yield condition and the associated constitutive equations simplifies the analysis of boundary value problems of viscoplastic bodies. As an example, they considered a simply supported circular plate subjected to a uniformly distributed transverse load. In plate theory the yield condition assumes the same form for the generalized stresses as for the stress components. However, the yield surface in terms of the generalized stresses for the shell problem is in general non-linear. If we want to retain the advantages of using a linear criterion, the non-linear yield surface has to be replaced by a piecewise linear approximation. It may be noted that a similar approach has been successfully exploited in limit analysis and many solutions of practical importance have been obtained. For the analysis presented herein, it is assumed that the onset of yielding for the material of the shell is specified by the so-called square yield condition (Drucker 1954). Based on the square yield condition, a clamped-free circular cylindrical shell under uniform radial pressure is considered and the stresses and velocities are determined.

2. Basic Formulation

For a typical shell element (Fig. 1) the strains that are of interest are the axial and the circumferential components

$$\epsilon_x = \frac{dU}{dX} + z\frac{d^2W}{dX^2} \quad \epsilon_\phi = -\frac{W}{R}, \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

where U and W are the velocities in the axial and radial directions X and R respectively and Z is the distance through the shell thickness, measured positive outward from the median surface. As regards the stress resultants,

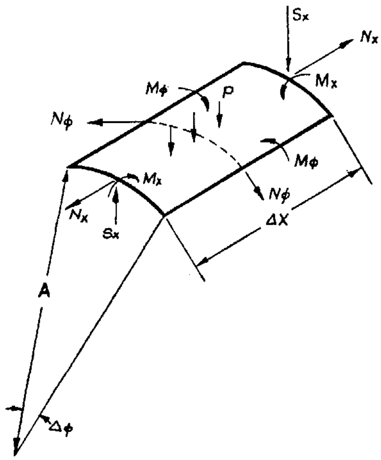

Fig. 1. Typical shell element.

the force N_x per unit length is zero for equilibrium in the axial direction. Since the circumferential curvature rate κ_ϕ is zero for the circular cylindrical shell, the circumferential moment M_ϕ per unit length becomes a reaction and hence it need not be considered in the analysis of the problem. The two other resultants, the axial moment M_x and the circumferential force N_ϕ per unit lengths, are related by the equilibrium equation

$$\frac{d^2M_x}{dX^2} + \frac{N_\phi}{R} + P = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

where P is the pressure intensity per unit area.

For the subsequent analysis, it is found convenient to use non-dimensional quantities. To this end, a shell of length $2L$, radius R and thickness $2H$ is considered and the following definitions are used:

$$\left.\begin{aligned} u = \frac{U}{L}, \quad w = \frac{W}{R}, \quad x = \frac{X}{L}, \quad z = \frac{Z}{H}, \quad c^2 = \frac{L^2}{RH} \\ m_x = \frac{M_x}{\sigma_0 H^2}, \quad n_\phi = \frac{N_\phi}{2\sigma_0 H}, \quad p = \frac{PR}{2\sigma_0 H} \end{aligned}\right\} \quad \ldots \quad (2.3)$$

Thus, the strain rate velocity and stress equilibrium eqns. (2.1) and (2.2) assume, respectively, the dimensionless forms

$$\epsilon_x = u' + z\frac{w''}{c^2} = u' - 2z\kappa_x,\ \epsilon_\phi = -w, \quad \ldots \quad (2.4)$$

$$\frac{m_x''}{2c^2} + n_\phi + p = 0, \quad \ldots \quad (2.5)$$

where primes denote differentiation with respect to x and $\kappa_x = -\frac{w''}{2c^2}$ is the axial curvature rate.

For the perfectly plastic shell, the onset of yielding is assumed to be specified by the square yield condition, Fig. 2. The mechanical behaviour of the shell is governed by the eight yield functions listed in Table I. The

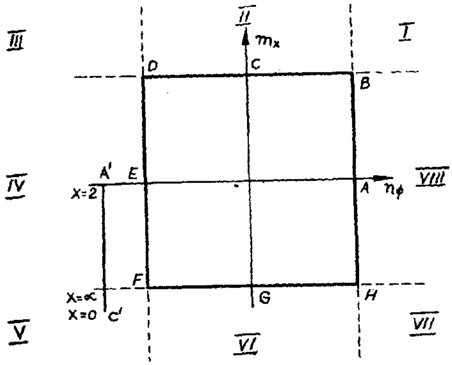

FIG. 2. Square yield condition.

TABLE I

Regime	Generalized stresses	Generalized strain rates
I	$m_x \geqslant 1;\ n_\phi \geqslant 1$	$\lambda\kappa_x = m_x - 1;\ \lambda\epsilon_\phi = n_\phi - 1$
II	$m_x \geqslant 1;\ n_\phi \leqslant 1;\ -n_\phi \leqslant 1$	$\lambda\kappa_x = m_x - 1;\ \lambda\epsilon_\phi = 0$
III	$m_x \geqslant 1;\ -n_\phi \geqslant 1$	$\lambda\kappa_x = m_x - 1;\ \lambda\epsilon_\phi = n_\phi + 1$
IV	$m_x \leqslant 1;\ -m_x \leqslant 1;\ -n_\phi \geqslant 1$	$\lambda\kappa_x = 0;\ \lambda\epsilon_\phi = n_\phi + 1$
V	$-m_x \geqslant 1;\ -n_\phi \geqslant 1$	$\lambda\kappa_x = m_x + 1;\ \lambda\epsilon_\phi = n_\phi + 1$
VI	$-m_x \geqslant 1;\ n_\phi \leqslant 1;\ -n_\phi \leqslant 1$	$\lambda\kappa_x = m_x + 1;\ \lambda\epsilon_\phi = 0$
VII	$-m_x \geqslant 1;\ n_\phi \geqslant 1$	$\lambda\kappa_x = m_x + 1;\ \lambda\epsilon_\phi = n_\phi - 1$
VIII	$-m_x \leqslant 1;\ m_x \leqslant 1;\ n_\phi \geqslant 1$	$\lambda\kappa_x = 0;\ \lambda\epsilon_\phi = n_\phi - 1$

viscoplastic flow rule of Hohenemser and Prager (1932) can be written in the general form

$$\lambda\epsilon_{ij} = \langle F \rangle \frac{\partial F}{\partial \sigma_{ij}}, \quad \ldots \quad (2.6)$$

where

$$\left.\begin{aligned} \epsilon_{ij} &= \text{velocity strain tensor} \\ \sigma_{ij} &= \text{stress tensor} \\ <F> &= F \text{ if } F \geqslant 0, \\ <F> &= 0 \text{ if } F < 0, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

and λ is a positive constant depending on the viscosity of the material, and F is the yield function. The non-vanishing strain-rate components associated with the yield functions are also listed in Table I.

In order to complete the formulation of the problem, the boundary and continuity conditions must be stated. At the built-in end of the shell, $x = 0$, the velocity and slope are zero.

$$w(o) = w'(o) = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

At the free end, $x = 2$, the shear and moment should vanish. Therefore

$$m_x'(2) = m_x(2) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

$$m_x,\ m_x',\ n_\phi,\ w \text{ and } w' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

are all continuous throughout the shell.

3. Solution

If the pressure applied to the shell is gradually increased from zero, the shell remains rigid so long as p is less than a certain critical value p_0, known as the static collapse load. Let us assume that at the instant of collapse the stress solution under this hypothesis is obtained by substituting the equation of side EF into the equation of equilibrium (2.5). Integration of the resulting equation subject to the boundary conditions (2.9) leads to

$$m_x = -c^2(p-1)(x-2)^2 ;\ n_\phi = -1 \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

$$p_0 = 1+1/4c^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

The corresponding velocity field is found to be

$$w = Ax. \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

It is easily verified that the solution defined by eqns. (3.1), (3.2) and (3.3) is statically and kinematically admissible provided A is non-negative.

It is now necessary to fix the stress profile for values of $p > p_0$. At the free end, $x = 2$, the bending moment is zero and the stress point lies on the n_ϕ axis. Hence it seems reasonable to assume that the stress points for $0 \leqslant x \leqslant 2$ form a curve such as $A'C'$ in Fig. 2.

The yield functions that have positive values lie in regime IV for $2 \geqslant x \geqslant \alpha$ and in regime V for $\alpha \geqslant x \geqslant 0$.

The shell is, therefore, divided into two regions that are in different viscoplastic regimes. The substitution of the relevant equations from Table I into the equation of equilibrium (2.5) subject to the continuity and boundary conditions (2.8 to 2.10) results in

$0 \leqslant x \leqslant \alpha$:

$$m_x = -1+2\beta^2\{(c_2 \cosh \beta x - c_3 \sinh \beta x) \cos \beta x - [c_3 \cosh \beta x + (p-1) \sinh \beta x] \sin \beta x\} \qquad (3.4)$$

$$n_\phi = -p+[(p-1) \cosh \beta x + c_3 \sinh \beta x] \cos \beta x + (c_2 \sinh \beta x - c_3 \cosh \beta x) \sin \beta x \qquad (3.5)$$

$$\lambda w = (p-1) - \{[(p-1) \cosh \beta x + c_3 \sinh \beta x] \cos \beta x + (c_2 \sinh \beta x - c_3 \cosh \beta x) \sin \beta x\} \qquad (3.6)$$

$\alpha \leqslant x \leqslant 2$:

$$m_x = 2c^2(x-2)^2 \left\{ \frac{B+1-p}{2} - \left(\frac{x+4}{\alpha+4}\right) \left[\frac{B+1-p}{2} + \frac{1}{2c^2(\alpha-2)^2} \right] \right\} \qquad (3.7)$$

$$n_\phi = \frac{6x}{\alpha+4} \left[\frac{B+1-p}{2} + \frac{1}{2c^2(\alpha-2)^2} \right] - (B+1) \qquad (3.8)$$

$$\lambda w = B - \frac{6x}{\alpha+4} \left[\frac{B+1-p}{2} + \frac{1}{2c^2(\alpha-2)^2} \right]. \qquad (3.9)$$

The continuity of w' at $x = \alpha$ gives the load intensity

$$p = 1 + \frac{1}{\beta c^2(\alpha+4)(\alpha-2)^2} \left[\frac{\beta Z_8 - 3Z_3Z_6}{Z_3Z_5 - Z_7} \right]. \qquad (3.10)$$

In eqns. (3.4) to (3.10), the following notation is used

$$\beta = \sqrt{c/\sqrt{2}} \qquad (3.11)$$

$$B+1-p = \left(\frac{\alpha+4}{Z_1}\right) \left[\frac{3Z_2}{c^2(\alpha+4)(\alpha-2)^2} - 2\beta(p-1)(\sin \alpha\beta)(\sinh \alpha\beta) \right] \qquad (3.12)$$

$$c_2 = (p-1)(\tan \alpha\beta)(\tanh \alpha\beta) + c_3[(\tan \alpha\beta) + (\tanh \alpha\beta)] \qquad (3.13)$$

$$c_3 = \frac{1}{Z} \left\{ \frac{3\left[B+1-p+\frac{1}{c^2(\alpha-2)^2} \right](\cos \alpha\beta)(\cosh \alpha\beta)}{\beta(\alpha+4)} + \frac{(p-1)}{2} \times [(\sin 2\alpha\beta) - (\sinh 2\alpha\beta)] \right\} \qquad (3.14)$$

$$Z = (\sin^2 \alpha\beta)(\cosh^2 \alpha\beta) + (\cos^2 \alpha\beta)(\sinh^2 \alpha\beta) \qquad (3.15)$$

$$Z_1 = \tfrac{3}{2}[(\sinh 2\alpha\beta) - (\sin 2\alpha\beta)] - 2\beta(\alpha-2)Z \qquad (3.16)$$

$$Z_2 = \alpha\beta Z - \tfrac{1}{2}[(\sinh 2\alpha\beta) - (\sin 2\alpha\beta)] \qquad (3.17)$$

$$Z_3 = 2\beta^3[(\sin \alpha\beta)(\operatorname{sech} \alpha\beta) + (\sec \alpha\beta)(\sinh \alpha\beta)] \qquad (3.18)$$

$$Z_4 = 2\beta^3[(\sin \alpha\beta)(\operatorname{sech} \alpha\beta) - (\sec \alpha\beta)(\sinh \alpha\beta)] \qquad (3.19)$$

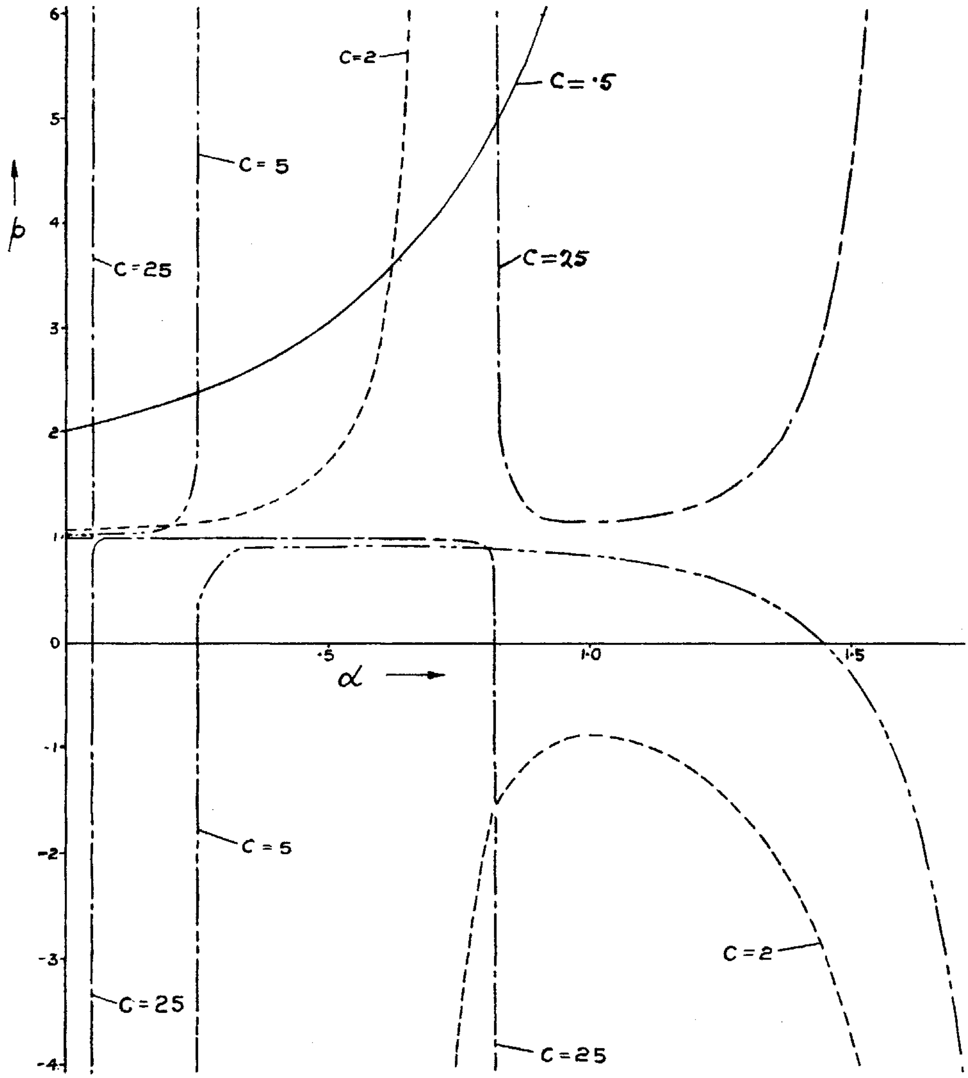

FIG. 3. p versus α.

$$Z_5 = (\tan \alpha\beta)(\tanh \alpha\beta) + \frac{1}{2Z}[(\tan \alpha\beta) + (\tanh \alpha\beta)]\left[(\sin 2\alpha\beta) - (\sinh 2\alpha\beta) - \frac{3(\sin 2\alpha\beta)(\sinh 2\alpha\beta)}{Z_1}\right] \quad .. \ (3.20)$$

$$Z_6 = \frac{1}{ZZ_1}(Z_1 + 3Z_2)[(\sin \alpha\beta)(\cosh \alpha\beta) + (\cos \alpha\beta)(\sinh \alpha\beta)] \quad .. \quad .. \ (3.21)$$

$$Z_7 = Z_4 - \frac{2\beta}{Z_1}[6\beta^2 + c^2(\alpha - 2)^2(\tan \alpha\beta)(\tanh \alpha\beta)](\sin \alpha\beta)(\sinh \alpha\beta) \quad .. \ (3.22)$$

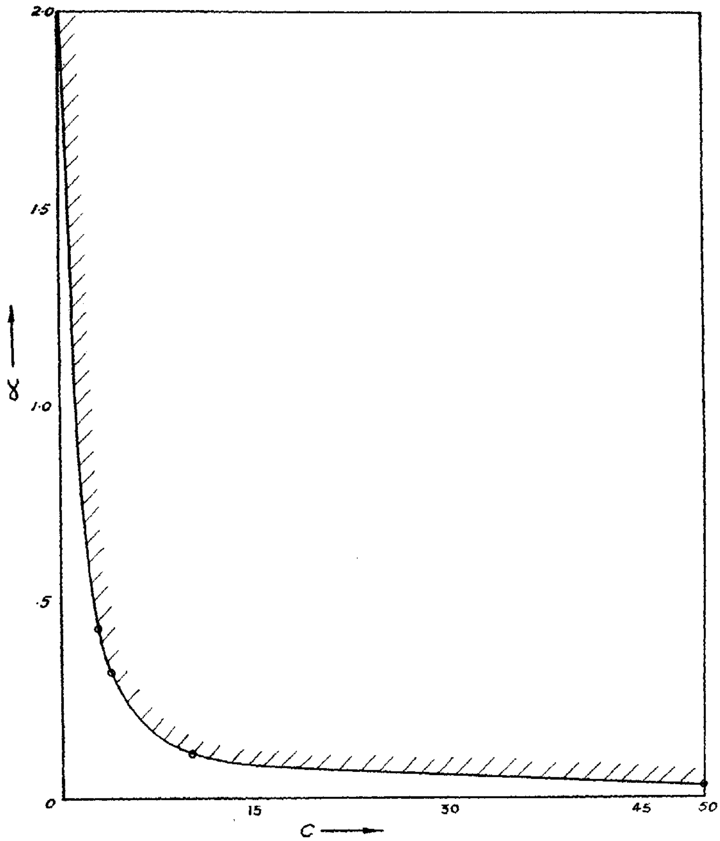

FIG. 4. c versus max. α.

$$Z_8 = 3c^2(\alpha^2-4)(\tan \alpha\beta)(\tanh \alpha\beta)+6\beta^2+\frac{3Z_2}{Z_1}[6\beta^2+c^2(\alpha-2)^2 \times (\tan \alpha\beta)(\tanh \alpha\beta)]. \quad .. \quad (3.23)$$

The solution given by eqns. (3.4) to (3.23) should satisfy the inequalities for regimes IV and V listed in Table I. The complexity of the eqns. (3.4) to (3.9) precludes a general analytical discussion of the inequalities and hence numerical verification is resorted to. Since $0 \leqslant \alpha \leqslant 2$, the load intensity p is evaluated as a function of α for different values of c. Fig. 3 shows p as a function of α for four values of c. For extremely short shells (for example, $c = 0.5$) p monotonically increases with α. As the value of c increases, it is found that there are more branches of the solution in the range $0 \leqslant \alpha \leqslant 2$. However, we restrict our attention to that branch of the solution in which p monotonically increases with α and hence there is a limiting value of α for any value c. The range of interest in the $\alpha - c$ plane is indicated in Fig. 4, and for such values the inequalities in regimes IV and V are verified numerically and it is found that all the inequalities are satisfied.

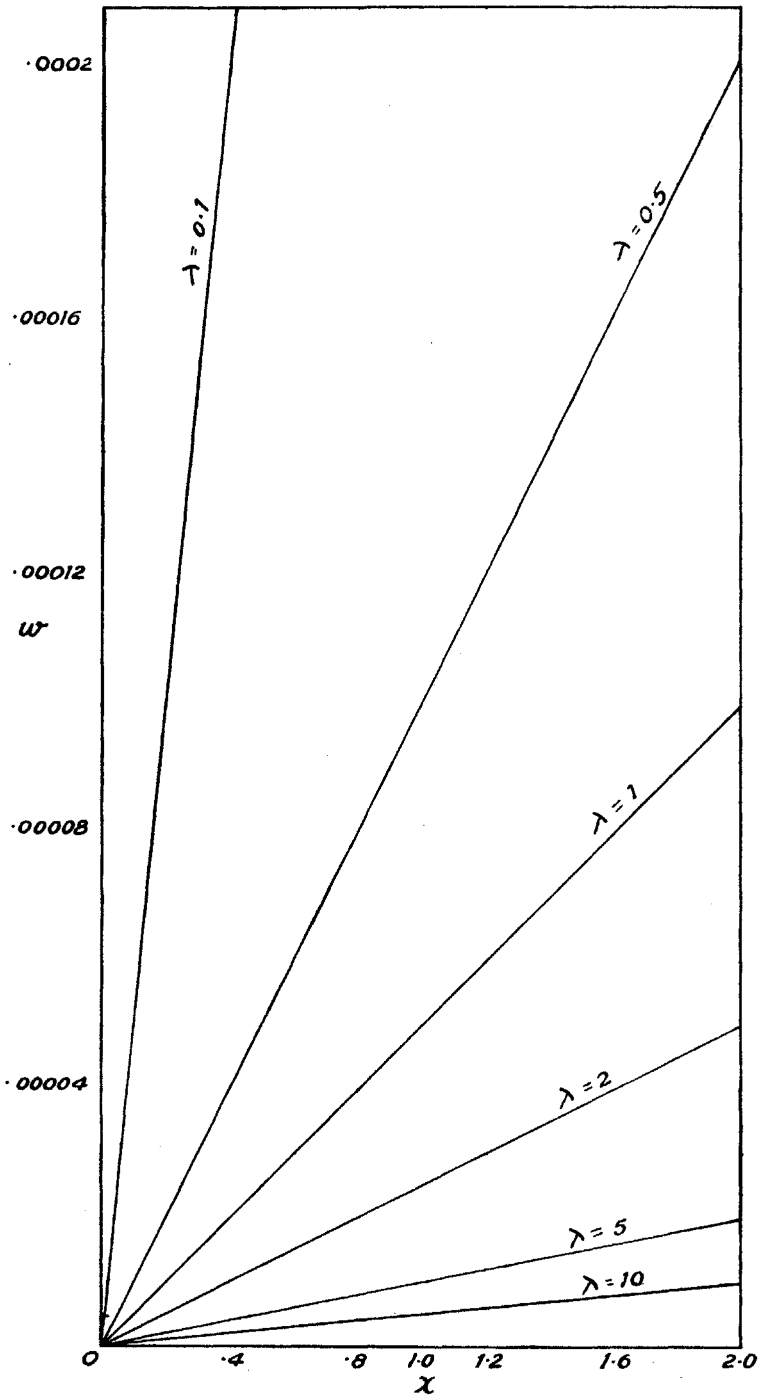

FIG. 5. w versus x; $c = 10$; $\alpha = \cdot 01$.

4. Discussion

Figure 3 shows that as α tends to zero, the load intensity p tends to the value of the collapse load p_0 of a rigid perfectly plastic shell given by eqn. (3.2).

Figure 5 shows the effect of viscosity on the velocity distribution for the case $c = 10$ and $\alpha = 0{\cdot}01$. The velocity of the shell diminishes with increasing values of λ. On the other hand, as $\lambda \to 0$, the material tends to be rigid, perfectly plastic and the velocity of the shell is undetermined.

References

Appleby, E. J., and Prager, W. (1962). A Problem in Viscoplasticity. *J. appl. Mech.*, 381–384.

Bingham, E. C. (1922). Fluidity and Plasticity. McGraw-Hill Book Company, Inc., New York, pp. 215–218.

Drucker, D. C. (1954). Limit Analysis of Cylindrical Shells under Axially Symmetric Loading. *Proc. midwest. Conf. Solid Mech.*, 158–163.

Hohenemser, K., and Prager, W. (1932). Uber die Ansätze der Mechanik der Kontinua. *Z. angew. Math. Mech.*, **12**, 216–226.