CERTAIN RECURRENCE RELATIONS FOR THE H-FUNCTION

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The aim of this note is to obtain some recurrence relations for the H-function. Since the G-function is a special case of the H-function, the recurrence relations for the G-function follow as particular cases of our findings; also, for some particular values of the parameters in the recurrence relations thus obtained for the G-function, we get the results given by Meijer.

1. Introduction

The H-function introduced by Fox (1961, p. 408) will be represented and defined in the following manner (Gupta and Jain *in press*):

$$H_{p,q}^{m,n}\left[x \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{1}^{m} \Gamma(b_j - \beta_j\xi) \prod_{1}^{n} \Gamma(1 - a_j + \alpha_j\xi)}{\prod_{m+1}^{q} \Gamma(1 - b_j + \beta_j\xi) \prod_{n+1}^{p} \Gamma(a_j - \alpha_j\xi)} x^{\xi} d\xi \quad \ldots \quad \ldots \quad (1.1)$$

where x is not equal to zero and an empty product is interpreted as having the value unity; p, q, m, n are integers satisfying $1 \leqslant m \leqslant q$; $0 \leqslant n \leqslant p$; $\alpha_j (j = 1, \ldots, p)$; $\beta_j (j = 1, \ldots, q)$ are positive numbers and $a_j (j = 1, \ldots, p)$, $b_j (j = 1, \ldots, q)$ are complex numbers such that no pole of $\Gamma(b_h - \beta_h\xi) (h = 1, \ldots, m)$ coincides with any pole of $\Gamma(1 - a_i + \alpha_i\xi) (i = 1, \ldots, n)$, i.e.

$$\alpha_i(b_h + \nu) \neq (a_i - \eta - 1)\beta_h \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

$(\nu, \eta = 0, 1, \ldots; h = 1, \ldots, m; i = 1, \ldots, n)$.

Further, the contour L runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points

$$\xi = (b_h + \nu)/\beta_h (h = 1, \ldots, m; \nu = 0, 1, \ldots;) \quad \ldots \quad \ldots \quad (1.3)$$

which are the poles of $\Gamma(b_h - \beta_h\xi)$ lie to the right of L and the points

$$\xi = (a_i - \eta - 1)/\alpha_i (i = 1, \ldots, u; \eta = 0, 1, \ldots;) \quad \ldots \quad \ldots \quad (1.4)$$

which are the poles of $\Gamma(1 - a_i + \alpha_i\xi)$ lie to the left of L. Such a contour is possible on account of (1.2).

In this note we have established some recurrence relations for the H-function. Since the G-function is a special case of the H-function, the recurrence relations for the G-function follow as particular cases of the relations given in this note. The results are believed to be new.

2. Recurrence Relations

(i)

$$\left.\begin{aligned}
&(b_1-b_2)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1-1,\alpha_1),(b_2-1,\alpha_2),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad=(a_1-b_1)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1-1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad\quad-(a_1-b_2)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]
\end{aligned}\right\}\qquad(2.1)$$

where $n \geqslant 1$ and $m \geqslant 2$.

(ii)

$$\left.\begin{aligned}
&(a_p-a_1)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad=H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(\alpha_p-1,\alpha_1)\\(b_1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad\quad+(a_p-b_1)(a_p-b_2)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1-1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad\quad+(1+b_1+b_2-2ap)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-1,\alpha_1)\\(b_1-1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad\quad+H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-2,\alpha_1)\\(b_1-1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]
\end{aligned}\right\}$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$. .. (2.2)

(iii)

$$\left.\begin{aligned}
&(b_1-a_1+1)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad=H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1+1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad\quad+H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-1,\alpha_1)\\(b_1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad\quad+(a_p-b_1)(a_p-b_2)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1-1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]\\
&\quad\quad+(b_2-a_p)H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-1,\alpha_1)\\(b_1-1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]
\end{aligned}\right\}\qquad(2.3)$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$.

(iv)

$$\left.\begin{aligned}
&(b_1-a_1)H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\\
&=H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-1,\alpha_1)\\(b_1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\\
&\quad+(b_1-a_p)(b_1-a_1)H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1-1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\\
&\quad+(1+b_2-a_p)H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-1,\alpha_1)\\(b_1-1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\\
&\quad+H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-2,\alpha_1)\\(b_1-1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]
\end{aligned}\right\}\tag{2.4}$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$.

(v)

$$\left.\begin{aligned}
&(1+b_1+b_2-a_1-a_p)H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_p,\beta_q)\end{matrix}\right.\right]\\
&=H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1+1,\alpha_1),(b_2,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\\
&\quad+(b_2-a_p)(b_2-a_1)H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p,\alpha_1)\\(b_1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\\
&\quad+H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1-1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_p-1,\alpha_1)\\(b_1,\alpha_1),(b_2-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]
\end{aligned}\right\}\tag{2.5}$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$.

Proofs: Since (Rainville 1963, p. 72)

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}a-1,\,b\\c\end{matrix};\,-sx\right)\\
&\quad={}_2F_1\left(\begin{matrix}a,\,b-1\\c\end{matrix};\,-sx\right)+\frac{b-a}{c}\,sx\,{}_2F_1\left(\begin{matrix}a,\,b\\c+1\end{matrix};\,-sx\right)
\end{aligned}\quad\ldots\quad(2.6)$$

we have

$$\begin{aligned}
&\frac{(a-1)\Gamma(a-1)\Gamma(b)}{\Gamma(c)}\int_0^\infty x^\lambda f(x)\,{}_2F_1\left(\begin{matrix}a-1,\,b\\c\end{matrix};\,-sx\right)dx\\
&=\frac{(b-1)\Gamma(a)\Gamma(b-1)}{\Gamma(c)}\int_0^\infty x^\lambda f(x)\,{}_2F_1\left(\begin{matrix}a,\,b-1\\c\end{matrix};\,-sx\right)dx\\
&\quad+\frac{(b-a)\Gamma(a)\Gamma(b)}{\Gamma(c+1)}\,s\,.\int_0^\infty x^{\lambda+1}f(x)\,{}_2F_1\left(\begin{matrix}a,\,b\\c+1\end{matrix};\,-sx\right)dx
\end{aligned}\quad\ldots\quad(2.7)$$

provided that the integrals involved exist.

Now if we take

$$f(x) = H_{p,q}^{m,n}\left[zx^{\sigma}\middle|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, d_p)\\(b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

in (2.7) and evaluate the integrals involved therein with the help of the following result (Gupta and Jain *in press*):

$$\int_0^{\infty} x^{\lambda} H_{p,q}^{m,n}\left[zx^{\sigma}\middle|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right] {}_2F_1\left(\begin{matrix}c_1, c_2\\d\end{matrix}; -sx\right) dx$$

$$= \frac{\Gamma(d) \,.\, s^{-\lambda-1}}{\Gamma(c_1) \,.\, \Gamma(c_2)} H_{p+2,q+2}^{m+2,n+1}\left[\frac{z}{s^{\sigma}}\middle|\begin{matrix}(-\lambda, \sigma), (a_1, \alpha_1), \ldots, (a_p, \alpha_p), (d-\lambda-1, \sigma)\\(c_1-\lambda-1, \sigma), (c_2-\lambda-1, \sigma), (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

valid for

$$\sigma > 0,\ R\left(\lambda+1+\sigma \min \frac{b_h}{\beta_h}\right) > 0 (h = 1, \ldots, m),\ |\arg s| < \pi$$

$$R\left(\lambda+1+\sigma \max \frac{a_j-1}{\alpha_j} - \min c_i\right) < 0 (j = 1, \ldots, n;\ i = 1, 2)$$

$$\mu = \sum_{1}^{n} (\alpha_j) - \sum_{n+1}^{p} (\alpha_j) + \sum_{1}^{m} (\beta_j) - \sum_{m+1}^{q} (\beta_i) > 0 \text{ and } |\arg z| < \tfrac{1}{2}\mu\pi,$$

we get the recurrence relation (2.1) after some slight changes in the parameters and arguments.

The remaining recurrence relations can be proved in a similar manner if we start with (Erdelyi 1953, p. 103, eqns. 30, 34, 39, 33) respectively instead of (2.6).

3. Particular Cases

The following recurrence relations for the G-function (Erdelyi 1953, p. 207) are obtained if we take all the α's and the β's to be equal to unity in the recurrence relations (2.1) to (2.5) respectively.

(i)

$$\left.\begin{aligned}&(b_1-b_2)G_{p,q}^{m,n}\left[x\middle|\begin{matrix}a_1-1, a_2, \ldots, a_p\\b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]\\&= (a_1-b_1)G_{p,q}^{m,n}\left[x\middle|\begin{matrix}a_1, a_2, \ldots, a_p\\b_1-1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\&\quad-(a_1-b_2)G_{p,q}^{m,n}\left[x\middle|\begin{matrix}a_1, a_2, \ldots, a_p\\b_1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]\end{aligned}\right\} \quad \ldots \quad (3.1)$$

where $m \geqslant 2$ and $n \geqslant 1$.

(ii)

$$\left.\begin{aligned}
&(a_p-a_1)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\
&=G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p-1\\ b_1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+(a_p-b_1)(a_p-b_2)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p\\ b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+(1+b_1+b_2-2a_p)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p-1\\ b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p-2\\ b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]
\end{aligned}\right\} \quad \ldots \quad (3.2)$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$.

(iii)

$$\left.\begin{aligned}
&(b_1-a_1+1)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\
&=G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1+1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p-1\\ b_1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+(a_p-b_1)(a_p-b_2)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p\\ b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+(b_2-a_p)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p-1\\ b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]
\end{aligned}\right\} \quad \ldots \quad (3.3)$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$.

(iv)

$$\left.\begin{aligned}
&(b_1-a_1)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\
&=G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p-1\\ b_1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+(b_1-a_p)(b_1-a_1)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1-1, b_2, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+(1+b_2-a_p)G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p-1\\ b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]\\
&\quad+G_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p-2\\ b_1-1, b_2-1, b_3, \ldots, b_q\end{matrix}\right]
\end{aligned}\right\} \quad \ldots \quad (3.4)$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$.

(v)

$$\left.\begin{aligned}(1+b_1+b_2-a_1-a_p)G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1, b_2, b_3, \ldots, b_q\end{matrix}\right.\right]\\ = G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1+1, b_2, b_3, \ldots, b_q\end{matrix}\right.\right]\\ +(b_2-a_p)(b_2-a_1)G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1, a_2, \ldots, a_{p-1}, a_p\\ b_1, b_2-1, b_3, \ldots, b_q\end{matrix}\right.\right]\\ +G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1-1, a_2, \ldots, a_{p-1}, a_p-1\\ b_1, b_2-1, b_3, \ldots, b_q\end{matrix}\right.\right]\end{aligned}\right\} \quad \ldots \quad (3.5)$$

where $m \geqslant 2$ and $1 \leqslant n \leqslant p-1$.

On putting $a_p = b_2$ in (3.3) and (3.5), we arrive at a recurrence relation given by Meijer (Erdelyi 1953, p. 209).

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