GEOMETRODYNAMICS OF THE NON-NULL FIELD

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Assuming a consistent non-holonomic form in terms of an analytic function ρ, of the metric tensor of the space-time V_4 in the case of the first and the second classes, the conditions for the existence of ρ satisfying Einstein-Maxwell equations are investigated.

§ 1. We consider a four-dimensional space-time of general relativity with the metric tensor $h_{\lambda\mu}$. Let R^{μ}_{λ} be its contracted curvature tensor and $R = R^{\lambda}_{\lambda}$ its scalar curvature. If the rank of the matrix $((h_{\lambda\mu}))$ be 4, there exists an inverse tensor $h^{\lambda\mu}$ such that

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^{\nu}_{\mu}. \qquad \text{.. .. .. ..} \quad (1.1)$$

In what follows Greek indices have the range I, II, III, IV, and Latin indices run in general from 1 to 4. The letters ξ, η run through I, II and III.

Let $k_{\lambda\mu} = -k_{\mu\lambda}$ be the electromagnetic tensor field. Then the stress tensor $F_{\mu\lambda}$ is defined by

$$F_{\mu\lambda} \overset{\text{def}}{=} k_{\mu}{}^{\alpha}k_{\alpha\lambda} - \tfrac{1}{4}k_{\alpha\beta}k^{\beta\alpha}h_{\mu\lambda}. \qquad \text{.. .. ..} \quad (1.2)$$

The Einstein field equations have the form

$$R_{\mu\lambda} - \left(\frac{R}{2} + p\right)h_{\mu\lambda} = \mu U_{\mu}U_{\lambda} - F_{\mu\lambda} \qquad \text{.. .. ..} \quad (1.3a)$$

with the Maxwell equations given by

$$\nabla_{[\omega}k_{\mu\nu]} = 0, \qquad \text{.. .. .. ..} \quad (1.3b)$$

$$\nabla_{\alpha}k^{\alpha\beta} = \epsilon u^{\beta}, \qquad \text{.. .. .. ..} \quad (1.3c)$$

where p is the pressure, μ is the mass and U_{μ} is a unit time-like vector field,

$$U_{\mu}U^{\mu} = -1, \qquad \text{.. .. .. ..} \quad (1.3d)$$

ϵ is the energy and ∇_{α} is the operator of covariant derivative.

The electromagnetic tensor field $k_{\lambda\mu}$ is said to be

of the first class if $k \overset{\text{def}}{=} \det\left((k_{\alpha}{}^{\beta})\right) \neq 0$,

of the second class if $k = 0$, $K \overset{\text{def}}{=} -\tfrac{1}{4}k_{\alpha}{}^{\beta}k_{\beta}{}^{\alpha} \neq 0$,

of the third class if $k = 0$, $K = 0$, $k_{\alpha}{}^{\beta}k_{\beta}{}^{\gamma} \neq 0$.

If λ's are the eigenvalues of $k_\alpha{}^\beta$ and a_β the corresponding eigenvectors, then

$$(k_\alpha{}^\beta - \lambda\delta_\alpha{}^\beta)a_\beta = 0$$

and the eigenvalues are given by

$$\lambda^4 + 2K\lambda^2 + k = 0 \qquad \text{(1.4)}$$

giving

$$\left.\begin{aligned} \lambda_1 &= -\lambda_2 = i\sqrt{K+\sqrt{D}} \\ \lambda_3 &= -\lambda_4 = \sqrt{D-K} \end{aligned}\right\}, \qquad \text{(1.5)}$$

where $D \overset{\text{def}}{=} K^2 - k$ and $\overset{k}{a}_\beta$ are then the corresponding eigenvectors.

§ 2. Hlavatý (1958) has shown that the non-holonomic components for the first and the second cases are given by

$$h_{ij} = \underset{i}{a}{}^\lambda \underset{j}{a}{}^\mu h_{\lambda\mu}; \quad h^{ij} = \overset{i}{a}{}_\lambda \overset{j}{a}{}_\mu h^{\lambda\mu} \qquad \text{(2.1)}$$

with

$$((h_{ij})) = ((h^{ij})) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad \text{(2.2)}$$

We consider a generalization of this form given by

$$(h_{ij}) = \begin{pmatrix} 0 & \alpha & 0 & \rho \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ \rho & 0 & \beta & 0 \end{pmatrix}, \qquad \text{(2.3)}$$

where the parameters α and β are determined by the Maxwell equations (1.3) and ρ is another parameter, the analytic conditions for whose existence we shall investigate.

The equation (1.3*b*) can be written as

$$\nabla_{[i}k_{jk]} = 0 \qquad \text{(2.4)}$$

and we see that α and β should satisfy

$$\partial_1(\alpha\beta\rho) + \partial_2(\alpha\beta\rho) = 0, \qquad \text{(2.5a)}$$

$$\partial_3(\alpha\beta\rho) - \partial_4(\alpha\beta\rho) = 0. \qquad \text{(2.5b)}$$

The second set of Maxwell equations

$$\nabla_i k_j^{\ i} = \epsilon u_j \qquad \text{(2.6)}$$

yields

$$\partial_2 \alpha = \frac{\epsilon(u_1+u_2)}{2}, \qquad \ldots \qquad (2.7a)$$

$$\partial_4 \beta = -\frac{\epsilon(u_3+u_4)}{2}. \qquad \ldots \qquad (2.7b)$$

§ 3. Now we calculate $k_\lambda^{\ \mu}$ and U_λ. Since k_{ij} are skew-symmetric we have the following relations:

$$\left.\begin{aligned}
&\alpha k_1^{\ 2}+\rho k_1^{\ 4} = 0\\
&\alpha k_1^{\ 1}+\alpha k_2^{\ 2}+\rho k_2^{\ 4} = 0\\
&\beta k_1^{\ 4}+\alpha k_3^{\ 2}+\rho k_3^{\ 4} = 0\\
&\rho k_1^{\ 1}+\beta k_1^{\ 3}+\alpha k_4^{\ 2}+\rho k_4^{\ 4} = 0\\
&\alpha k_2^{\ 1} = 0\\
&\beta k_2^{\ 4}+\alpha k_3^{\ 1} = 0\\
&\rho k_2^{\ 1}+\beta k_2^{\ 3}+\alpha k_4^{\ 1} = 0\\
&\beta k_3^{\ 4} = 0\\
&\rho k_3^{\ 1}+\beta k_3^{\ 3}+\beta k_4^{\ 4} = 0\\
&\rho k_4^{\ 1}+\beta k_4^{\ 3} = 0
\end{aligned}\right\}. \qquad \ldots \qquad (3.1)$$

These yield the following components for $((k_i^{\ j}))$.

$$((k_i^{\ j})) = \begin{pmatrix} 0 & -\rho & \alpha & \alpha \\ 0 & -\rho & -\alpha & \alpha \\ -\beta & -\beta & \rho & 0 \\ \beta & -\beta & -\rho & 0 \end{pmatrix}. \qquad \ldots \qquad (3.2)$$

The matrix $((k_{ij}))$ then becomes

$$((k_{ij})) = \begin{pmatrix} 0 & 0 & \alpha\beta & \alpha\beta \\ 0 & 0 & \alpha\beta & -\alpha\beta \\ -\alpha\beta & -\alpha\beta & 0 & 0 \\ -\alpha\beta & \alpha\beta & 0 & 0 \end{pmatrix}. \qquad \ldots \qquad (3.3)$$

Now

$$^{(2)}k_{ij} = k_i^{\ l} k_{lj} \qquad \ldots \qquad (3.4)$$

therefore

$$^{(2)}k_{ij} = \begin{pmatrix} -2\alpha^2\beta & 0 & -\alpha\beta\rho & \alpha\beta\rho \\ 0 & 2\alpha^2\beta & -\alpha\beta\rho & \alpha\beta\rho \\ -\alpha\beta\rho & -\alpha\beta\rho & -2\alpha\beta^2 & 0 \\ \alpha\beta\rho & \alpha\beta\rho & 0 & 2\alpha\beta^2 \end{pmatrix} \qquad \ldots \qquad (3.5)$$

$$^{(2)}k_{\mu\nu} \overset{\text{def}}{=} {}^{(2)}k_{ij} \overset{i}{a}_\lambda \overset{j}{a}_\mu . \qquad \ldots \qquad (3.6)$$

The basic vectors $\underset{i}{a}_\lambda$ and $\overset{i}{a}_\lambda$ are given by

$$\left.\begin{aligned} \underset{1}{a}_\lambda &= \alpha \overset{2}{a}_\lambda + \rho \overset{4}{a}_\lambda \\ \underset{2}{a}_\lambda &= \overset{1}{a}_\lambda \\ \underset{3}{a}_\lambda &= \beta \overset{4}{a}_\lambda \\ \underset{4}{a}_\lambda &= \rho \overset{1}{a}_\lambda + \beta \overset{3}{a}_\lambda \end{aligned}\right\} \quad \ldots \quad (3.7)$$

and

$$\left.\begin{aligned} \overset{1}{a}_\lambda &= \frac{1}{\alpha} \underset{2}{a}_\lambda \\ \overset{2}{a}_\lambda &= \frac{1}{\alpha}\left(\underset{1}{a}_\lambda - \frac{\rho}{\beta} \underset{3}{a}_\lambda\right) \\ \overset{3}{a}_\lambda &= \frac{1}{\beta}\left(\underset{4}{a}_\lambda - \frac{\rho}{\alpha} \underset{2}{a}_\lambda\right) \\ \overset{4}{a}_\lambda &= \frac{1}{\beta} \underset{3}{a}_\lambda \end{aligned}\right\}. \quad \ldots \quad (3.8)$$

From (3.7) and (3.8) we have

$$((h^{ij})) = \begin{pmatrix} 0 & \frac{1}{\alpha} & 0 & 0 \\ \frac{1}{\alpha} & 0 & \frac{-\rho}{\alpha\beta} & 0 \\ 0 & \frac{-\rho}{\alpha\beta} & 0 & \frac{1}{\beta} \\ 0 & 0 & \frac{1}{\beta} & 0 \end{pmatrix} \quad \ldots \quad (3.9)$$

giving

$$h^{ij} = h^{\lambda\mu} \overset{i}{a}_\lambda \overset{j}{a}_\mu . \quad \ldots \quad (3.10)$$

From (1.3a) we see that the left-hand side is of rank four; therefore the right-hand side must also be of rank four.
Therefore we can suppose U_λ to be given by

$$U_\lambda = A\overset{1}{a}_\lambda + B\overset{2}{a}_\lambda + C\overset{3}{a}_\lambda + D\overset{4}{a}_\lambda \quad \ldots \quad (3.11)$$

and $U_\lambda U^\lambda = -1$ gives

$$2(\beta AB - \rho BC + \alpha CD) + \alpha\beta = 0. \quad \ldots \quad (3.12)$$

Further,

$$\begin{aligned} h_{\mu\nu} &= h_{ij} \overset{i}{a}_\mu \overset{j}{a}_\nu \\ &= 2\left[\alpha \overset{(1}{a}_\mu \overset{2)}{a}_\nu + \rho \overset{(1}{a}_\mu \overset{4)}{a}_\nu + \beta \overset{(3}{a}_\mu \overset{4)}{a}_\nu\right]. \end{aligned} \quad \ldots \quad (3.13)$$

§ 4. We write

$$h_{ij} = \tilde{h}_{ij} + X_{ij}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

where

$$((\tilde{h}_{ij})) = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \beta & 0 \end{pmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

and

$$((X_{ij})) = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & 0 \end{pmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

$$\tilde{h}_{\lambda\mu} \overset{\text{def}}{=} \tilde{h}_{ij} \overset{i}{a}_\lambda \overset{j}{a}_\mu, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.4a)$$

$$X_{\lambda\mu} \overset{\text{def}}{=} X_{ij} \overset{i}{a}_\lambda \overset{j}{a}_\mu. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.4b)$$

Now we transform the equation (1.3*a*) using (3.5) and (3.6) and the relations (3.11), (3.13), (4.1) and (4.4). We use the method (Schmidt 1966) in order to apply the Cauchy existence theorem to our problem. Separating the second derivatives from the rest of the terms, we get

$$h^{\alpha\beta}(\partial_{\mu\alpha} X_{\nu\beta} + \partial_{\nu\alpha} X_{\mu\beta} - \partial_{\alpha\beta} X_{\mu\nu} - \partial_{\mu\nu} X_{\alpha\beta}) = Q_{\mu\nu}, \qquad \ldots \quad (4.5)$$

where $Q_{\mu\nu}$ denotes the rest of the equations not containing the second derivatives.

Now we separate $\partial_{\text{IV}}\, \partial_{\text{IV}}$ from the rest of the terms and let the indices ξ, η run through I, II and IV only. We get

$$\left.\begin{array}{ll} h^{\text{IV IV}}\, \partial_{\text{IV IV}} X_{\xi\eta} = M_{\xi\eta} & \text{for } \mu, \nu = \xi, \eta \\ h^{\text{IV}\,\eta}\, \partial_{\text{IV IV}} X_{\xi\eta} = M_{\xi\,\text{IV}} & \text{for } \mu, \nu = \text{IV}, \xi \\ h^{\xi\eta}\, \partial_{\text{IV IV}} X_{\xi\eta} = M_{\text{IV IV}} & \text{for } \mu, \nu = \text{IV}, \text{IV} \end{array}\right\}, \qquad \ldots \quad \ldots \quad (4.6)$$

where $M_{\mu\nu}$ stands for the rest of the terms not containing the second derivatives.

If $h^{\text{IV IV}} \neq 0$, then substituting from (4.3) and (4.4*b*) in the above equations we get

$$\left.\begin{array}{l} (\partial_{\text{IV IV}}\rho) \overset{1}{a}_\xi \overset{4}{a}_\eta + (\partial_{\text{IV IV}}\rho) \overset{4}{a}_\xi \overset{1}{a}_\eta = N_{\xi\eta} \\ h^{\text{IV}\,\eta} \left\{ (\partial_{\text{IV IV}}\rho) \overset{1}{a}_\xi \overset{4}{a}_\eta + (\partial_{\text{IV IV}}\rho) \overset{4}{a}_\xi \overset{1}{a}_\eta \right\} = N_{\xi\,\text{IV}} \\ h^{\xi\eta} \left\{ (\partial_{\text{IV IV}}\rho) \overset{1}{a}_\xi \overset{4}{a}_\eta + (\partial_{\text{IV IV}}\rho) \overset{4}{a}_\xi \overset{1}{a}_\eta \right\} = N_{\text{IV IV}} \end{array}\right\}, \qquad \ldots \quad \ldots \quad (4.7)$$

where $N_{\mu\nu}$ stands for the rest of the equations not containing the second derivatives.

If we denote the dual vectors to $\overset{r}{a}_\xi$ by $\underset{r}{a}^\xi$,

$$\left.\begin{aligned} \overset{r}{a}_\xi \underset{s}{a}^\xi &= \delta^r_s \\ \overset{r}{a}_\xi \underset{r}{a}^\eta &= \delta^\eta_\xi \end{aligned}\right\}, \qquad (4.8)$$

where $r = 1, 2, 3$.

Equation (4.7), on multiplying by $\underset{r}{a}^\xi \underset{s}{a}^\eta$, then becomes

$$\delta_r{}^1\delta_s{}^4(\partial_{\text{IV IV}}\rho)+\delta_r{}^4\delta_s{}^1(\partial_{\text{IV IV}}\rho) = N_{\xi\eta}\underset{r}{a}^\xi \underset{s}{a}^\eta, \qquad (4.9)$$

that is

$$(\partial_{\text{IV IV}}\rho) = N_{\xi\eta}\underset{1}{a}^\xi \underset{s}{a}^\eta \qquad (4.10)$$

and

$$N_{\xi\eta}\underset{r}{a}^\xi \underset{s}{a}^\eta = 0, \qquad (4.11)$$

for $(r, s) = (1, 1), (1, 2), (2, 2), (2, 3)$ and $(3, 3)$.

From (4.7) we get

$$h^{\text{IV}\,\eta}(\partial_{\text{IV IV}}\rho) = N_{\xi\,\text{IV}}\underset{1}{a}^\xi \underset{4}{a}^\eta, \qquad (4.12)$$

and using (4.10), (4.12) becomes

$$h^{\text{IV}\,\eta} N_{\xi\eta} = N_{\xi\,\text{IV}}, \qquad (4.13)$$

and similarly we get from (4.7) and (4.9)

$$h^{\xi\eta} N_{\xi\eta} = N_{\text{IV IV}}. \qquad (4.14)$$

Therefore to get a unique solution ρ, the boundary values for ρ and $\partial_{\text{IV}}\rho$ must satisfy the conditions (4.11), (4.12), (4.13) and (4.14) along a hypersurface $X^{\text{IV}} = 0$ (Mishra 1965).

Thus the necessary and sufficient condition for the existence of an analytic function ρ satisfying Einstein-Maxwell equations in the neighbourhood of $X^{\text{IV}} = 0$ is that there exist analytic boundary values of ρ and $\partial_{\text{IV}}\rho$ along $X^{\text{IV}} = 0$, satisfying (4.12), (4.13), (4.14) and that the solutions ρ of (4.10) having such boundary values satisfy (4.12), (4.13) and (4.14) in a neighbourhood of $X^{\text{IV}} = 0$.

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