DISTRIBUTION OF PERIODIC ORBITS OF THE THIRD KIND (SYMMETRIC PERIODIC ORBITS) IN THE RESTRICTED PROBLEM OF THREE BODIES

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(Communicated by P. L. Srivastava, F.N.I.)

(*Received December 12, 1965*)

In this paper the notion of the index of a curve is generalized for a higher dimensional curve and by means of this generalization the distribution of elliptic and hyperbolic periodic orbits of different orders in a small neighbourhood of the generating solution for sufficiently small values of μ has been studied. The results of the paper are valid not only for the circular case of the restricted problem of three bodies, but also for the elliptic case.

INTRODUCTION

This paper which is based on my earlier three papers (Choudhry 1964, *in press a*, *in press b* hereinafter referred to as I, II and III respectively) has many objectives. In I, we had established the existence of symmetric periodic orbits of Schwarzchild's type for a circular restricted problem of three bodies for a fixed μ. Here we have first examined (§ 2 and 3) how these orbits are distributed for a continuous increase or decrease in the values of μ, making use of Poincaré's theory of indices. Secondly, the notion of indices has been generalized for a three-dimensional space in order to simplify further study. Though Poincaré (1879) and Lefschetz (1963) had also generalized this notion, their generalizations were mostly topological as contrasted to our generalization (§ 1) which is for a geometrically higher dimensional space. Thirdly, we have studied in this paper the distributions of periodic orbits not only of the second order but also of higher orders, as distinct from Merman (1961) and Birkhoff (1936) who had considered only first order symmetric orbits. Also the problem has been studied for a three-dimensional space. Fourthly, it is shown that all the results obtained for the circular restricted problem of three bodies are equally valid for the elliptical case. In II, the periodic orbits of the third kind had been shown to be equivalent to symmetric periodic orbits. Finally, it is suggested that the distribution of symmetric periodic orbits plays an important role in the existence of almost periodic solution in a

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restricted problem of three bodies (circular or elliptic). (This latter point will be the subject of the author's next paper).

For the sake of convenience, we have used the Delaunay variables with the usual notations. These variables simplify the study a lot. Although the symmetric periodic orbits of different types and of different orders have not been distinguished, these terms have been freely used everywhere. We use the words 'frequency of right-hand side' to mean $2\pi/T$ where T is the period of r.h.s. expressions of the differential equations giving a periodic motion. The meanings of all other concepts used in this paper are the same as those given in I, II and III.

Index of Poincaré

We denote by $\vec{V}(M)$ the vector whose initial point is $M(x, y)$ and whose components are $P(x, y)$ and $Q(x, y)$. The totality of such vectors is the vector field defined by the system

$$\frac{dx}{dt} = P(x, y), \ \frac{dy}{dt} = Q(x, y).$$

The point $M(x, y)$ is a singular point if and only if $\vec{V}(M) = 0$.

Consider a closed curve C not passing through a singular point. Take a point M on C. As M describes C once, say in positive sense, the vector $\vec{V}(M)$ may go through a number of complete revolutions, some in the positive and some in the negative sense. In the process, the angle which $\vec{V}(M)$ makes with a fixed vector changes by an integral multiple $2\pi J_c$ of 2π. The integer $J_c(+ve$, zero or $-ve)$ is called the index of C relative to the vector field $V = \left\{\vec{V}(M)\right\}$.

If M be a point, then we shall choose a circular neighbourhood $S(M, \epsilon)$ containing no singular points with the possible exception of M itself. Let C be the circumference of S. Then J_c is called the index of M.

In a three-dimensional generalization, let the components of the vector $\vec{V}(M)$ be $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$. A point $M(x, y, z)$ will be a singular point if and only if

$$P(x, y, z) = Q(x, y, z) = R(x, y, z) = 0.$$

If M be a singular point in the three-dimensional space, then its projection m on the plane $z = 0$ is a singular point on the z-plane. It is clear from the study of two-dimensional curves that the index of a curve or of a point depends on the singular points enclosed under it. Let us extend the definition of the index for a plane to a three- or higher-dimensional space; but, for the given curve, we shall consider its orthogonal projection on an arbitrary plane. Since the index of a point is independent of the orientation of the plane

(Lefschetz 1963), the word 'arbitrary' can be replaced by a suitable or a fixed plane. The properties of indices were studied in detail by Poincaré (1879), Merman (1961), Lefschetz (1963) and many others. These properties were applied to the plane circular restricted problem of three bodies by Birkhoff (1936) and Merman (1961). We shall assume the following theorem (Nemytskii and Stepanov 1960) as it easily yields an extension suitable for the solution of the present problem.

THEOREM.—The index of the invariant point of a transformation τ^k corresponding to a periodic solution of a canonical system of two equations, whose right side depends periodically on t, is equal to:

(i) -1 in the hyperbolic case,

(ii) $+1$ in the elliptic case, if the characteristic exponents are not either integral multiples of the frequency of the right-hand side or zero, and

(iii) zero in the parabolic case.

DISTRIBUTION OF SYMMETRIC PERIODIC ORBITS OF ELLIPTIC AND HYPERBOLIC TYPES OF THE FIRST ORDER

To study the distribution we shall establish the following theorems which will explain how these periodic orbits are distributed for different values of μ.

Theorem 1.—For sufficiently small $|\mu| \neq 0$ the symmetric periodic orbits of the first order, holomorphic in μ, and belonging to one of the two types of each category, are elliptic with purely imaginary characteristic exponents and the orbits belonging to the other type of the same category with the same characteristic numbers will be hyperbolic.

PROOF: Replacing m' by μ, the differential equations of the restricted problem of three bodies can be written as (II):

$$\frac{dG}{d\tau} = \mu \frac{\partial \Phi_1}{\partial g}, \quad \frac{dg}{d\tau} = -\mu \frac{\partial \Phi_1}{\partial G},$$

$$\frac{dH}{d\tau} = \mu \frac{\partial \Phi_1}{\partial h}, \quad \frac{dh}{d\tau} = -\frac{d\Phi_0}{dH} - \mu \frac{\partial \Phi_1}{\partial H},$$

where

$$\Phi_0(C, H) = \frac{k^2}{\sqrt{2C-2n'H}}, \quad \Phi_1 = \Phi_1(C, G, H, \tau, g, h)$$

and G, g, H, h and τ have the usual meanings. The generating solution may be written as

$$G^{(0)} = G_0, \quad H^{(0)} = H_0$$

$$g^{(0)} = g_0, \quad h^{(0)} = -\frac{n'}{n^{(0)}}\tau + h_0$$

where $n^{(0)} = \dfrac{(2C-2n'H^{(0)})^{\frac{3}{2}}}{k^2}$. Let T_0 be the period of this solution, then

$$\tau(T_0)-\tau(0) = 2k\pi \text{ and so } h^{(0)}(T_0)-h^{(0)}(0) = -\frac{n'}{n^{(0)}}2k\pi = -2l\pi.$$

$\therefore \quad n^{(0)}(H_0) = n'\dfrac{k}{l}$, where (k, l) are the characteristic numbers of this periodic solution. The general solution for $\mu \neq 0$ in the neighbourhood of the generating solution may be given as

$$G(\tau) = G_0+\beta_1+x_1, \quad g(\tau) = g_0+\gamma_1+y_1$$

$$H(\tau) = H_0+\beta_2+x_2, \quad h(\tau) = -\frac{n'}{n^{(0)}}\tau+h_0+\gamma_2+y_2$$

and

$$G(0) = G_0+\beta_1, \quad g(0) = g_0+\gamma_1$$

$$H(0) = H_0+\beta_2, \quad h(0) = h_0+\gamma_2.$$

Let

$$x_1(\beta, \gamma, \mu) = G(T_0)-G(0), \quad y_1(\beta, \gamma, \mu) = g(T_0)-g(0)$$

$$x_2(\beta, \gamma, \mu) = H(T_0)-H(0), \quad y_2(\beta, \gamma, \mu) = h(T_0)-h(0)+2k\pi;$$

and so by Taylor's theorem up to the first order of β, γ and μ,

$$x_1 = 2k\pi\mu\left(\frac{\partial[\Phi_1]}{\partial g_0}+\dots\right)$$

$$x_2 = 2k\pi\mu\left(\frac{\partial[\Phi_1]}{\partial h_0}+\dots\right)$$

$$y_1 = -2k\pi\mu\left(\frac{\partial[\Phi_1]}{\partial G_0}+\dots\right)$$

$$y_2 = -2k\pi\frac{d^2\Phi_0(H_0)}{dH_0^2}-\mu\left[\int_0^{2k\pi}\frac{\partial\Phi_1(G_0, H_0, g^{(0)}, h^{(0)})}{\partial H_0}d\tau\right.$$
$$\left.+\frac{d^2\Phi_0(H_0)}{dH_0^2}\int_0^{2k\pi}\left(\int_0^t\frac{\partial\Phi_1}{\partial h_0}d\tau\right)dt\right]+\dots$$

The condition of periodicity, viz. $x_1 = x_2 = y_1 = y_2 = 0$, gives the equations

$$\frac{\partial[\Phi_1]}{\partial g_0} = \frac{\partial[\Phi_1]}{\partial h_0} = \frac{\partial[\Phi_1]}{\partial G_0} = 0$$

which determine g_0, h_0 and G_0. The characteristic exponents of the corresponding periodic solution holomorphic in μ are given by the equation

$$\begin{vmatrix} \frac{\partial x_1}{\partial \beta_1}-S & \frac{\partial x_1}{\partial \beta_2} & \frac{\partial x_1}{\partial \gamma_1} & \frac{\partial x_1}{\partial \gamma_2} \\ \frac{\partial x_2}{\partial \beta_1} & \frac{\partial x_2}{\partial \beta_2}-S & \frac{\partial x_2}{\partial \gamma_1} & \frac{\partial x_2}{\partial \gamma_2} \\ \frac{\partial y_1}{\partial \beta_1} & \frac{\partial y_1}{\partial \beta_2} & \frac{\partial y_1}{\partial \gamma_1}-S & \frac{\partial y_1}{\partial \gamma_2} \\ \frac{\partial y_2}{\partial \beta_1} & \frac{\partial y_2}{\partial \beta_2} & \frac{\partial y_2}{\partial \gamma_1} & \frac{\partial y_2}{\partial \gamma_2}-S \end{vmatrix} = 0$$

where $S = e^{2k\pi\alpha}-1$.

Neglecting the terms of $O(\mu^2)$, it is found that

$$S = \pm 2k\pi\mu \frac{\partial^2[\Phi_1]}{\partial g_0 \partial G_0},\ \pm 2k\pi \sqrt{-\frac{d^2\Phi_0}{dH_0^2}\cdot\frac{\partial^2[\Phi_1]}{\partial h_0^2}\mu}+O(\mu).$$

As S is very small, we may replace S by $2k\pi\alpha$, then

$$\alpha = \pm\,\mu \frac{\partial^2[\Phi_1]}{\partial g_0 \partial G_0},\ \pm \frac{\sqrt{3}n'k}{(2C-2n'H_0)^{\frac{5}{4}}}\sqrt{-\mu \frac{\partial^2[\Phi_1]}{\partial h_0^2}}+O(\mu) \quad \text{.. (1)}$$

where

$$[\Phi_1] = \sum_{\substack{i_1,\, i_2,\, i_3 \\ i_1 n^{(0)} = i_3 n'}} C_{(i_1,\, i_2,\, i_3)}(C, G_0, H_0) \cos(i_2 g_0 + i_3 h_0).$$

It is clear that $[\Phi_1]$ is a continuous function of h_0 and g_0; and so, it must attain its maximum and minimum in the closed interval $[0, 2\pi]$ for each h_0 and g_0, i.e. in the open interval $(-\epsilon, 2\pi+\epsilon)$ for an arbitrary $\epsilon > 0$. Thus $[\Phi_1]$ has got in the interval $[0, 2\pi]$ for each g_0 and h_0, at least one maximum and one minimum. The corresponding values h_0 will satisfy the periodicity conditions

$$\frac{\partial[\Phi_1]}{\partial h_0} = \frac{\partial[\Phi_1]}{\partial g_0} = 0 \text{ and } \frac{\partial^2[\Phi_1]}{\partial h_0^2} \neq 0.$$

It will have maximum value when $\frac{\partial^2[\Phi_1]}{\partial h_0^2} < 0$ and minimum value when $\frac{\partial^2[\Phi_1]}{\partial h_0^2} > 0$. The condition $\frac{\partial^2[\Phi_1]}{\partial h_0^2} = 0$ is not possible. Corresponding to the maximum value the characteristic exponents α will all be real and corresponding to the minimum value they will all be purely imaginary. It is clear that the elementary divisors are simple and that if the characteristic exponents are restricted up to $O(\sqrt{\mu})$, then the maximum value would correspond to an elliptic periodic orbit and the minimum value to an hyperbolic periodic orbit.

Remembering the condition that $\frac{n^{(0)}}{n'} = \frac{k}{l}$, where k, l are the characteristic numbers of the solution, we can write down the necessary conditions of the periodicity in the form

$$\frac{\partial[\Phi_1]}{\partial g_0} = \frac{\partial[\Phi_1]}{\partial h_0} = -\sum_{i_1 k = i_3 l} i_2 C_{(i_1, i_2, i_3)}(C, G_0, H_0) \sin(i_2 g_0 + i_3 h_0)$$

$$= -\sum_{i_1 k = i_3 l} i_3 C_{(i_1, i_2, i_3)}(C, G_0, H_0) \sin(i_2 g_0 + i_3 h_0) = 0. \quad (2)$$

The conditions (2) show that either

$$g_0 = 0 = h_0 \text{ or } h_0 = 0,\ g_0 = \pi \text{ or}$$

$$g_0 = 0,\ h_0 = \pi \text{ or } h_0 = \pi,\ g_0 = \pi.$$

Let us consider them separately.

We shall classify our study according to the various values of the characteristic exponents (k, l) and of g_0 and h_0.

(A) *Let us consider the case when k is even and l is odd*

(i) $g_0 = 0 = h_0$

Let us assume that $k = 2^m k_1$ where m is an integer $\geqslant 1$, and k_1 is an odd integer. In this case in the series written above the terms corresponding to the indices i_3, containing the factor $2^p (p < m)$ must be absent, as $i_3 l$ will contain 2^p (for l is odd) and $i_1 k$ will contain 2^q where $q \geqslant m$ which is not possible since $i_1 k = i_3 l$. Initially $\tau = 0$ and so $h_0 = 0$ means that the moving point lies at the ascending node which coincides with the perihelion and that it is in conjunction with Jupiter, i.e. the intersection is of $(\gamma\delta)$ type. After $k/2 (= 2^{m-1} k_1)$ revolutions the point is again at the perihelion and Jupiter completes $\frac{l}{2} = \frac{l-1}{2} + \frac{1}{2} = n + \frac{1}{2}$ revolutions, i.e. Jupiter is in opposition with the point and so the intersection is of the type $(\alpha\beta)$. So we shall get periodic orbit of one of the types $(\alpha\beta, \gamma\delta)$ of the second category.

(ii) $g_0 = 0,\ h_0 = \pi$

In this case, i_3 will have, as in case (i), the factor 2^m; and so, writing h_0' for $2^m h_0$ and afterwards dropping the dash, we shall have $h_0 = 2^{-m}\pi$. Let us express k/l as a continued fraction and let $\frac{P_{n-1}}{Q_{n-1}}$, $\frac{P_n}{Q_n}$ be the penultimate and the last convergents of the corresponding continued fraction. As k and l are mutually prime numbers, so $\frac{k}{l}$ is in the simplest form, $P_n = k$, $Q_n = l$. Let us put $x = (-1)^n P_{n-1} k_1$, $y = (-1)^n Q_{n-1} k_1$. Then, by virtue of the relation $P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}$, we shall have $lx - ky = k_1$, whence $lx = ky + k_1$ and so x is an odd number for k is even and k_1 is odd.

Initially (i.e. when $\tau = 0$), the moving point lies at the ascending node which coincides with the perihelion by virtue of $g_0 = 0$ and its angular distance from Jupiter is $2^{-m}\pi$. After $\frac{x}{2}\left(=\frac{x-1}{2}+\frac{1}{2}\right)$ revolutions, the point lies at the aphelion and during the same time Jupiter completes $\frac{l}{k}\cdot\frac{x}{2}$ revolutions. Since $\frac{l}{k}\cdot\frac{x}{2}=\frac{y}{2}+\frac{1}{2^{m+1}}$ therefore Jupiter will complete $\frac{y}{2}+\frac{1}{2^{m+1}}$ revolutions. As $\frac{1}{2^{m+1}}$ revolutions $= 2^{-m}\pi = h_0$, so after $\frac{1}{2^{m+1}}$ revolutions Jupiter appears at the conjunction with the perihelion and if y is even, then even after $y/2$ revolutions Jupiter continues to be in conjunction with the perihelion. If y is odd, then Jupiter is in conjunction with the aphelion. Since the point lies at the aphelion, the orthogonal intersection will either be of the type $(\gamma'\delta')$ or be of the type $(\alpha'\beta')$. In the first case after $\frac{k}{2}(= 2^{m-1}k_1)$ revolutions and correspondingly after $\frac{l}{2}\left(=\frac{l-1}{2}+\frac{1}{2}=n+\frac{1}{2}\right)$ revolutions of Jupiter, we shall get another orthogonal intersection of the type $(\alpha'\beta')$ and in the second case of the type $(\gamma'\delta')$. In either cases, we get periodic orbits of the type $(\alpha'\beta', \gamma'\delta')$, i.e. the second type of the second category.

(iii) $g_0 = \pi$, $h_0 = 0$

Initially $\tau = 0$ and $h_0 = 0$ means that the moving point lies at the perihelion and that it is in opposition with Jupiter, i.e. the intersection is of the type $(\alpha\beta)$. After $\frac{k}{2}(= 2^{m-1}k_1)$ revolutions, the point will again appear at the perihelion and Jupiter completes $\frac{l}{2}\left(=\frac{l-1}{2}+\frac{1}{2}=n+\frac{1}{2}\right)$ revolutions, where n is an integer. Thus the point is in conjunction with Jupiter and the intersection is of the type $(\gamma\delta)$. We thus get one of the types $(\alpha\beta, \gamma\delta)$ of the second category, corresponding to k even and l odd.

(iv) $g_0 = \pi$, $h_0 = \pi$

As in case (ii), we shall have $h_0 = 2^{-m}\pi$ and x is an odd number. Initially (i.e. when $\tau = 0$), the moving point lies at the perihelion and its angular distance from Jupiter is $(2^{-m}+1)\pi$. After $\frac{x}{2}\left(=\frac{x-1}{2}+\frac{1}{2}\right)$ revolutions the point lies at the aphelion and during the same time Jupiter completes $\frac{l}{k}\cdot\frac{x}{2}$ revolutions. Since $\frac{l}{k}\cdot\frac{x}{2}=\frac{y}{2}+\frac{1}{2^{m+1}}$, Jupiter will complete $\frac{y}{2}+\frac{1}{2^{m+1}}$ revolutions. As $\frac{1}{2^{m+1}}$ revolutions $= 2^{-m}\pi = h_0$, so, after $\frac{1}{2^{m+1}}$ revolutions, Jupiter is in conjunction with the aphelion and, if y is even, then even after $y/2$ revolutions Jupiter continues to be in conjunction with the aphelion. If y is odd, then the Jupiter is in opposition with the aphelion. As the point lies at the aphelion, so the orthogonal intersection will be of the type $(\alpha'\beta')$ or of the type $(\gamma'\delta')$. In the first case after $\frac{k}{2}$ $(= 2^{m-1}k_1)$ revolutions and correspondingly after $\frac{l}{2}\left(=\frac{l-1}{2}+\frac{1}{2}=n+\frac{1}{2}\right)$ revolutions of Jupiter we shall get orthogonal intersections of the type $(\gamma'\delta')$; and in the second case the intersections will be of the type $(\alpha'\beta')$. In either case we get periodic orbits of the type $(\alpha'\beta', \gamma'\delta')$, i.e. the second type of the second category.

(B) *Let us consider now the case when k is odd, l is even*

(i) $g_0 = 0 = h_0$

Here initially the point lies at the perihelion and at the same time it is in conjunction with Jupiter (orthogonal intersection is of the type $(\gamma\delta)$). After $\frac{k}{2}\left(=\frac{k-1}{2}+\frac{1}{2}=n+\frac{1}{2}\right)$ revolutions the point lies at the aphelion and as Jupiter completes $\frac{l}{2}$ revolutions, the point is in opposition with Jupiter (orthogonal intersection is of the type $(\gamma'\delta')$). Thus we get periodic orbits of one of the types $(\gamma\delta, \gamma'\delta')$ of the third category corresponding to k odd and l even.

(ii) $g_0 = 0$, $h_0 = \pi$

In this case initially the point lies at the perihelion and the perihelion is in opposition with the Jupiter (orthogonal intersection is of the type $(\alpha\beta)$). After $\frac{k}{2} = n+\frac{1}{2}$ revolutions the point will lie at the aphelion and Jupiter will be in conjunction with the point and thus the intersection will be of the type of $(\alpha'\beta')$. Thus we get the periodic orbit of the second of the types $(\alpha\beta, \alpha'\beta')$ of the third category.

(iii) $g_0 = \pi$. $h_0 = 0$

Initially the point lies at the perihelion and the perihelion is in opposition with the Jupiter (orthogonal intersection is of the type $(\alpha\beta)$). After $\frac{k}{2} = n+\frac{1}{2}$ revolutions the point will lie at the aphelion and during these revolutions Jupiter will complete $\frac{l}{2}$ revolutions and thus Jupiter will be in conjunction with the point and the intersection will be of the type of $(\alpha'\beta')$. Thus in this case we get the periodic orbits of the second of the types $(\alpha\beta, \alpha'\beta')$ of the third category.

(iv) $g_0 = \pi$, $h_0 = \pi$

Here it may be seen that this case corresponds to the periodic orbit of the first of the type $(\gamma\delta, \gamma'\delta')$ of the third category.

(C) *We shall now consider the case when k is odd, l is odd*

(i) $h_0 = 0$, $g_0 = 0$

Here initially the point lies at the perihelion and it is in conjunction with Jupiter and so the corresponding orthogonal intersection is of the type $(\gamma\delta)$. After $\frac{k}{2}$ revolutions, the point will lie at the aphelion and correspondingly after $\frac{l}{2}$ revolutions Jupiter will be again in conjunction with it. Thus the orthogonal intersection is of the type $(\alpha'\beta')$. We shall get the periodic orbit of one of the two types of $(\gamma\delta, \alpha'\beta')$ of the fourth category corresponding to both k and l odd.

Similarly in the cases (ii) $g_0 = 0$, $h_0 = \pi$, (iii) $g_0 = \pi$, $h_0 = 0$ and (iv) $g_0 = \pi$, $h_0 = \pi$, the periodic orbits correspond to one of the two types of $(\gamma\delta, \alpha'\beta')$ or $(\alpha\beta, \gamma'\delta')$ of the fourth category corresponding to both k and l odd.

Thus, for all the three conditions for all the values $g_0 = 0, \pi$; $h_0 = 0, \pi$, the periodic solutions of the first order and of the two types are given not depending upon the coefficients C_{i_1, i_2, i_3}, corresponding to the minimum and maximum value for R, as the preceding ones always exist. Now suppose that there are solutions of the equations $\frac{\partial R}{\partial g_0} = 0 = \frac{\partial R}{\partial h_0}$ depending on the coefficients $C_{(i_1, i_2, i_3)}$. Here again for suitable variation of $C_{(i_1, i_2, i_3)}$ we can make $\frac{\partial^2 R}{\partial g_0^2}$ positive or negative which will correspond to the maximum or minimum value of R. Thus, corresponding to the two signs of $\frac{\partial^2 R}{\partial g_0^2}$, we shall have two types of periodic orbits of each category.

Theorem 2.—In the general restricted problem of three bodies for a fixed C and the initial moment, each invariant point of the transformation τ^k, corresponding to some periodic solution, holomorphic in μ, is isolated at the point $\mu = 0$ for all $\mu = \mu_0 \neq 0$.

PROOF: By virtue of the holomorphism at the point $\mu = 0$, the initial conditions of our periodic solution satisfy the necessary and sufficient conditions of periodicity $x_i(\beta_i, \gamma_i, \mu) = y_i(\beta_i, \gamma_i, \mu) = 0$ written in the preceding theorem, if $|\mu|$ (and consequently $|\beta|$ and $|\gamma|$) is so small, that $|\mu|$, $|\beta(\mu)|$, $|\gamma(\mu)|$ do not exceed the corresponding radii of convergence. The condition $\frac{d^2\Phi_0(H_0)}{dH_0^2} \neq 0$ which holds good for the problem under consideration guarantees the solution of the equation $y_2 = 0$ with respect to β_2 as a holomorphic function in μ, β_1, γ_1, γ_2 reducing to zero for $\mu = \beta_1 = \gamma_1 = \gamma_2 = 0$. As the unknown quantities γ_1, γ_2 enter in the equations together with g_0 and h_0 in the form of $g(0) = g_0 + \gamma_1$, $h(0) = h_0 + \gamma_2$, then without losing any generality we can put $\gamma_1 = \gamma_2 = 0$, taking g_0 and h_0 to be the unknowns. Then for $\mu = 0$, we have $\beta_1(0) = 0 = \beta_2(0)$ and for $\mu \neq 0$ the equations for the determination of G_0 and H_0 take the form

$$\frac{x_1[\beta_1(\mu), \beta_2(\mu), 0, 0, \mu]}{\mu} = 0 = \frac{x_2[\beta_1(\mu), \beta_2(\mu), 0, 0, \mu]}{\mu}.$$

Taking the limit when $\mu \to 0$, we shall find that $\frac{\partial[\Phi_1]}{\partial g_0}$ and $\frac{\partial[\Phi_1]}{\partial h_0}$ are infinitely small together with μ whence it follows that $\frac{\partial[\Phi_1]}{\partial g_0} = \frac{\partial[\Phi_1]}{\partial h_0} = 0$. This necessary condition permits us to calculate the possible values of g_0 and h_0. As $\frac{\partial[\Phi_1]}{\partial g_0}$ and $\frac{\partial[\Phi_1]}{\partial h_0}$ are not identically zero, then, by virtue of the analytic dependence of $\frac{\delta[\Phi_1]}{\partial g_0}$ and $\frac{\partial[\Phi_1]}{\partial h_0}$ on g_0 and h_0, it is clear that the roots of the equations $\frac{\partial[\Phi_1]}{\partial g_0} = 0$ and $\frac{\partial[\Phi_1]}{\partial h_0} = 0$ will be isolated. A root is said to be isolated if in a small neighbourhood of the root there is no other root. Thus our periodic solution defined by the initial conditions β_{10}, β_{20}, g_0, h_0 for $|\mu| \neq 0$, but sufficiently small, corresponds to isolated invariant point of the transformation τ^k (Merman 1961, pp. 71-72). During the course of the increase of $|\mu|$ to $|\mu_0|$ the functions $\beta_1(\mu)$, $\beta_2(\mu)$, $g_0(\mu)$ and $h_0(\mu)$ can pass through singular point and then they will cease to be holomorphic. However, by virtue of the analytic continuation on the corresponding branch of Riemann surface, each branch of these functions will be single-valued and all the points of the periodic solution (correspondingly invariant point of the transformation τ^k) will remain isolated, which is what is required to be proved.

Theorem 3.—To each $\mu \neq 0$ and the characteristic numbers (k, l) satisfying the conditions of theorem 2 (I), there corresponds an invariant point of the transformation τ^k with the index, either equal to $+1$ or to -1 for an ordinary point and to an odd number for a multiple point.

Since the concept of the index for a higher dimensional space is not different from that for a plane, so this theorem will hold good for a space of any dimension. For this plane case the proof may be referred to Merman (1961).

Theorem 4.—For all $\mu \neq 0$, excepting countable values of μ and for all integers k and l satisfying the conditions of Theorem 2 (Choudhry, I) except a finite number of values of l, corresponding to a given k, the symmetric periodic orbits of the first order, belonging to one of the two types of each category, are elliptic with purely imaginary characteristic exponents, not being an integral multiple of the frequency of the right-hand side, and the orbit belonging to the other type of the same category is hyperbolic. Symmetric periodic orbits of the first order and of the first of the types mentioned corresponding to excluded set of values of μ, k and l are elliptic.

PROOF: Let us take arbitrary numbers k, l and $\mu \neq 0$ satisfying the conditions of Theorem 2 (*see* I) and let us consider any two periodic orbits of two distinct types with the characteristic numbers k and l whose existence shows that they are functions of μ (Theorem 1). For sufficiently small $|\mu|$, according to Poincaré-Schwarzschild's result, both the orbits will be holomorphic in μ; and by virtue of Theorem 1, one of them will be elliptic with purely imaginary characteristic exponents, not equal to the integral multiple of the frequency of right-hand side and the other will be hyperbolic.

We know from § 1 that the index of the invariant point of the transformation τ^k, corresponding to the elliptic periodic orbit, will be equal to $+1$ and corresponding to the hyperbolic orbit it will be equal to -1. By virtue of the preceding theorem, this result will not change so long till they do not coincide with other invariant points. Also it is well known that the sum of indices of the invariant points before their coincidence will be equal to the index of the singular point formed by coincidence and it will be equal to the sum of indices of the invariant points formed after coincidence.

Under the conditions of Theorem 2 (Choudhry, I), there exist an odd number of symmetric periodic orbits from which it follows that under the variation of μ they can appear or disappear only under the coincidence of even number of periodic solutions of the first order and of the same type.

Now there may occur, one after another, many coincidences of the invariant points under consideration with others. Let us take up the first coincidence. This coincidence may be of the following two kinds: the formation of new invariant points after the coincidence of an even number of invariant points and the coincidence of the given point with an even number of invariant points arising for less values of μ, i.e. up to the coincidence.

In the first kind the sum of the indices of all invariant points after the coincidence must be equal to $+1$, as, up to the coincidence, in a sufficiently small neighbourhood of the invariant point under consideration, with the index $+1$, there is no other invariant point. By Theorem 3, for each interval without coincidence, the index of each invariant point will be equal either to $+1$ or to -1 and so without difficulty we can conclude that the number of the invariant points with the index $+1$ formed after coincidence must be one more than that of the invariant points with the index -1.

In the second kind, up to the coincidence there are other invariant points. They can exist only in groups, each of which consisting of an even number of invariant points formed simultaneously. Up to the moment of the formation of such a group there existed an ordinary point of the domain, whose index is equal to zero. Consequently, the sum of the indices of the points of such a group is equal to zero, and as, after the formation of such groups, the index of a point of such groups must be equal either to $+1$ or to -1, so each such group can be divided into two sub-groups of which one consists of invariant points with the index $+1$ and the other with the index -1, under which the number of the points in both of the sub-groups must be the same.

Thus, all the invariant points, distinct from the point under consideration, are divided equally into two parts consisting of the points with the index $+1$ and of the points with the index -1. Let each part consist of m invariant points. Now assume that p points with the index $+1$ and q points with the index -1 coincide with the invariant point under consideration and let there remain r points with the index $+1$ and s points with the index -1 after the coincidence. By the properties of indices it is clear that $1+p-q = r-s$. The points with index $+1$ not coinciding with others are $m-p$ and the non-coincident points with the index -1 are $m-q$. Thus after coincidence, the number of the points with index $+1$ will be $m-p+r$ and the points with the index -1 will be $m-q+s$. The difference $(m-p+r)-(m-q+s) = (q-p)+(r-s) = q-p+(1+p-q) = 1$. In other words, similar to the coincidence of the first kind, the number of invariant points with the index $+1$, formed after the coincidence is one more than the invariant points with the index -1.

If now we take an arbitrary invariant point with the index $+1$, then all the remaining invariant points may be divided into two parts equally, consisting of points with the index $+1$ and of the points with the index -1. Thus the succeeding possible coincidence can be only of the two kinds similar to the first coincidence and the situation will go on repeating. Consequently, the conclusion on the relation of the points with indices $+1$ and -1 holds good for the succeeding moments prior to an arbitrary coincidence. Thus for all μ, k and l satisfying the conditions of Theorem 2 (Choudhry, I), there will exist at least one invariant point with the index $+1$, if those pairs (k, l) are excluded for which the invariant points for the given μ coincide. Under this, amongst the excluded multiple points, there is at least one point with positive index, as the sum of indices of all invariant points of the given types, multiple and simple, is always equal to 1 and so the corresponding periodic orbit will be elliptic.

Exactly in the same way we may prove the existence of at least one invariant point with the index -1 with the same restriction on the possibility of the exclusion of those pairs (k, l) for which the invariant points coincide for a given μ.

By virtue of Theorem 1, these singular points will correspond to elliptic periodic solution with the characteristic exponents, neither equal to zero, nor to the integral multiple of the frequency of the right-hand side and hyperbolic periodic solution if we exclude the elliptic periodic solution with the characteristic exponents equal to zero or the integral multiple of the frequency of the right-hand side from our consideration.

For the latter the root of the characteristic equation is equal to $+1$, which by virtue of the analyticity in μ can hold good only for a finite number of values of μ. This argument is valid for arbitrary pairs of integers (k, l), on which the periodic solution depends and which forms a countable set. As finite number of countable sets is a countable set, so the set of values of μ, excluded from our consideration, is not more than countable. As the number of the pairs of integers (k, l) excluded earlier for each μ, which corresponds to a multiple invariant point of the transformation τ^k corresponding to elliptic solution, is finite and so the theorem may be considered to be completely proved for a dynamical system with one degree of freedom.

By the introduction of the transformations of the paper (Choudhry, III, § 6) we can always reduce a dynamical system with two degrees of freedom to one with one degree of freedom. Here all the conditions for the above transformations are satisfied, i.e. the characteristic exponents are zero and finite and the elementary divisors are simple. Thus the distributions taking place for one degree of freedom will hold for two degrees as well as we can pass by means of the above transformations to a dynamical system with two degrees of freedom.

Distribution of the Symmetric Periodic Orbits of the Second Order for Small Values of μ

In the previous section we have examined the distribution of symmetric periodic orbits of the first order. It is seen there that one of the two types of periodic orbits of each category is elliptic, while the other one is hyperbolic and only symmetric periodic orbits of the second, third and fourth category exist and that of the first category does not exist. All the results of Theorem 1 of § 1 will hold good except the difference in category. For this we shall prove the following theorem:

Theorem 5—There exists no symmetric periodic orbit of the second order except that of the first category.

PROOF: Let k and l be the characteristic numbers where $(k, l) = 2$. Suppose that $k = 2k_1$ and $l = 2l_1$ where $(k_1, l_1) = 1$. Let us consider the different cases according to the different values of g_0 and h_0.

(i) $g_0 = 0 = h_0$

Here initially, i.e. corresponding to $\tau = 0$, the point is at the perihelion which is in conjunction with Jupiter and this corresponds to the orthogonal intersection of the type $(\gamma\delta)$. After $k/2 = k_1$ revolutions the point will again lie at the perihelion and Jupiter in the same time will complete $\frac{l}{2} = l_1$ revolutions and thus Jupiter will again be in conjunction with the perihelion. Hence in this case we find that the symmetric periodic orbit is of the first category.

(ii) $g_0 = 0$, $h_0 = 0$

Here initially the orthogonal intersection corresponds to the case $(\alpha\beta)$ and after $\frac{k}{2}$ revolutions of the moving point and $\frac{l}{2}$ revolutions of Jupiter the orthogonal intersection will again correspond to $(\alpha\beta)$. Thus the symmetric periodic orbit will be of the type $(\alpha\beta, \alpha\beta)$, i.e. of the first category.

(iii) $g_0 = \pi$, $h_0 = 0$

Here initially the orthogonal intersection corresponds to the case $(\alpha'\beta')$ and the consecutive orthogonal intersection, i.e. after $\frac{k}{2}$ revolutions of the moving point and $\frac{l}{2}$ revolutions of Jupiter the orthogonal intersection will again correspond to $(\alpha'\beta')$. Thus the symmetric periodic orbit will be of the type $(\alpha'\beta', \alpha'\beta')$, i.e. of the first category.

(iv) $g_0 = \pi$, $h_0 = \pi$

Similarly, here the consecutive orthogonal intersection will be of the type $(\gamma'\delta', \gamma'\delta')$ and thus the symmetric periodic orbit will be of the first category.

Now let us show that the symmetric periodic orbit of the other categories do not exist at all. Let us prove that the periodic orbits of the second category do not exist. On the contrary, let us assume that they exist. Suppose that $g_0 = h_0 = 0$. Initially the orthogonal intersection will correspond to the type $(\gamma\delta)$. For the second category the consecutive orthogonal intersection should be of the type $(\alpha\beta)$, i.e. the perihelion should be in opposition with the Jupiter. Let to this position there correspond the numbers of revolution k' and l' and so $2k'$ and $2l'+1$ will be the characteristic numbers which clearly do not have 2 as a common divisor and it contradicts the statement that the symmetric periodic orbits are of the second order. Thus the symmetric periodic orbits of the second order and of the second category do not exist. Similarly, it can be shown that neither periodic orbits of the third category nor of the fourth category exist. The same result can be examined in all the remaining cases corresponding to the values $g_0 = 0$, $h_0 = \pi$; $g_0 = \pi$, $h_0 = 0$ and $g_0 = \pi = h_0$.

Also it may be seen under the conditions of Theorem 2 (Choudhry, I) similar to the periodic orbits of the first order that there always exists an odd number of such orbits for a small μ for symmetric periodic motions of the second order and they appear or disappear in pairs for a continuous increase or decrease of μ.

So far as the distribution of these orbits is concerned, we find that it is given by Theorem 4 of the previous section for in the whole proof we never take the help of the periodic orbit of the first order and so the theorem will hold good for symmetric periodic orbits of the second order as well.

Distribution of the Symmetric Periodic Orbits of the Higher Order

Here also similar to the symmetric periodic orbits of the first order and of the second order the following theorem can be established.

If $k = sk_1$ and $l = sl_1$ $(s > 1)$ be the two characteristic integers where $(k_1, l_1) = 1$. When s is odd, the symmetric periodic orbits will be of the second category, third category and of the fourth category according as (i) k_1 even, l_1 odd, (ii) k_1 odd, l_1 even and (iii) k_1 odd, l_1 odd. When s is even, all the symmetric periodic orbits will be of the first category alone.

In all the cases whether s is odd or even, there will exist odd number of periodic orbits and for an increase or decrease in μ these orbits can appear or disappear in pairs only.

For different values of μ the distribution of these orbits will be similar as in Theorem 4 of section 2 for the proof is independent of the order of the symmetric periodic orbits.

Acknowledgements

I take this opportunity to express my heartiest thanks to Dr. N. S. Nagendra Nath, F.N.I., for his valuable suggestions for the improvement of the paper and to Dr. P. L. Shrivastava, F.N.I., for his help in the publication of the paper.

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