ON CURVATURE RESTRICTIONS OF A CERTAIN KIND OF CONFORMALLY-FLAT RIEMANNIAN SPACE OF CLASS ONE

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It is known that all the principal normal curvatures, say ρ, except one, say $\tilde{\rho}$, of a conformally-flat space of class one are equal to one another. Since $\rho \neq 0$, these spaces may be of two kinds: (*a*) those for which $\tilde{\rho} \neq 0$ and (*b*) those for which $\tilde{\rho} = 0$. In the present paper some properties of a space of kind (*b*) are obtained. These properties relate to the curvature tensor of the space and a vector tangential to the line of curvature corresponding to $\tilde{\rho}$. It is shown how these properties, and therefore the nature of the space, change as the scalar curvature of the space changes from being non-constant to being constant.

An n-dimensional conformally-flat Riemannian space of class one, denoted by C_n^1, say, is characterized by the property that at least $n-1$ of its principal normal curvatures are equal to one another. Denote this $(n-1)$ equal values by $\rho \neq 0$ and the remaining one by $\tilde{\rho}$. It was shown in a previous paper (Sen 1966) that a conformally-flat Riemannian space $C_n (n \geqslant 3)$ is a C_n^1, i.e. is of class one, if it satisfies the equations

$$R_{hijk} = \frac{R_{hj}R_{ik} - R_{hk}R_{ij}}{(n-2)\{(n-2)\rho^2 + \rho\tilde{\rho}\}} - \frac{\rho\tilde{\rho}}{n-2}(g_{hj}g_{ik}g_{hk}g_{ij}),$$

where $R = -(n-1)\{(n-2)\rho^2 + 2\rho\tilde{\rho}\}$. As usual g_{ij}, R_{hijk}, R_{ij} and R denote the fundamental tensor, the curvature tensor, the Ricci tensor and the scalar curvature respectively.

In this paper we shall be concerned with those C_n^1 for which $\tilde{\rho} = 0$, i.e. those C_n^1 which satisfy the equations

$$R_{hijk} = -\frac{n-1}{(n-2)R}(R_{hj}R_{ik} - R_{hk}R_{ij}), \quad (R \neq 0). \qquad (1)$$

Evidently this kind of a C_n^1 cannot be a space of constant curvature. Put

$$\lambda_i = \frac{1}{2}\frac{\partial}{\partial x^i}(\log R), \quad e_i = \lambda_i / \{g^{ij}\lambda_i\lambda_j\}^{\frac{1}{2}}. \qquad (2)$$

When R is not a constant, e_i is a unit vector in the direction of λ_i. Even when R becomes a constant, the unit vector e_i exists as a limit. In fact, e_i is tangential to the line of curvature corresponding to $\tilde{\rho}$ (Sen 1964). The object of this paper is to prove the following theorem (comma denoting co-variant derivative):

Theorem: A C_n^1 which is characterized by the equations (1) has the following properties in respect of its curvature tensor R_{hijk} and the vectors λ_i, e_i defined by (2):

(a) $R_{hijk}\lambda^h = 0.$

(b) $R_{hijk} = \dfrac{R}{(n-1)(n-2)}[(g_{hk}g_{ij}-g_{hj}g_{ik})+(g_{hj}e_ie_k+g_{ik}e_he_j-g_{hk}e_ie_j-g_{ij}e_he_k)].$

(c) $R_{hijk,\,l} = 2R_{hijk}\lambda_l+R_{lijk}\lambda_h+R_{hljk}\lambda_i+R_{hilk}\lambda_j+R_{hijl}\lambda_k.$

If, in particular, R is a constant

(d) $R_{hijk,\,l} = 0,\ e_{i,\,j} = 0.$

(e) R_{hijk} is a product tensor.

Proof: Since the space is of class one, the second fundamental tensor, say b_{ij}, must satisfy Gauss-Codazzi equations. So

$$b_{ij,\,k}-b_{ik,\,j} = 0, \text{ where } b_{ij} = \sqrt{-\frac{(n-1)}{(n-2)R}}\,R_{ij}.$$

Or, by (2), $R_{ij,\,k}-R_{ik,\,j}-(R_{ij}\lambda_k-R_{ik}\lambda_j) = 0.$ .. .. (3)

Multiplying (3) by g^{ij} we get, by (2),

$$R_{ik}\lambda^i = 0.$$

Hence, by virtue of (1), we get the property (*a*) of the theorem.

Again, since the space is a C_n, we get by (2)

$$R_{ij,\,k}-R_{ik,\,j}-\frac{R}{n-1}(g_{ij}\lambda_k-g_{ik}\lambda_j) = 0.$$

Comparing this with (3) we find

$$\left(R_{ij}-\frac{R}{n-1}g_{ij}\right)\lambda_k = \left(R_{ik}-\frac{R}{n-1}g_{ik}\right)\lambda_j. \quad .. \quad .. \quad (4)$$

Whence $R_{ij}-\dfrac{R}{n-1}g_{ij} = S\lambda_i\lambda_j$, where S is a scalar.

By (2), we can write this as

$$R_{ij} = \frac{R}{n-1}(g_{ij}-e_ie_j). \quad .. \quad .. \quad .. \quad (5)$$

Substituting (5) in (1), we get the property (*b*) of the theorem.

Further, differentiating (5) covariantly we get

$$R_{ij,\,k} = 2R_{ij}\lambda_k-\frac{R}{(n-1)}(e_ie_j)_{,\,k}. \quad .. \quad .. \quad .. \quad (6)$$

Applying (3) we find

$$R_{ij}\lambda_k-R_{ik}\lambda_j = \frac{R}{n-1}\{(e_ie_j)_{,\,k}-(e_ie_k)_{,\,j}\}.$$

Denote the length of the vector λ_i by ϕ so that $e_i = \lambda_i/\phi$. Then the above equations can be written as

$$(-R_{ik}\lambda_j - R_{jk}\lambda_i) - (-R_{ij}\lambda_k - R_{jk}\lambda_i) = \frac{R}{n-1}\{(e_i e_j),_k - (e_i e_k),_j\}$$

$$= \frac{R}{(n-1)\phi^2}\left[\left\{\left(\lambda_{i,k} - \frac{\lambda_i \phi,_k}{\phi}\right)\lambda_j + \left(\lambda_{j,k} - \frac{\lambda_j \phi,_k}{\phi}\right)\lambda_i\right\}\right.$$

$$\left. - \left\{\left(\lambda_{i,j} - \frac{\lambda_i \phi,_j}{\phi}\right)\lambda_k + \left(\lambda_{k,j} - \frac{\lambda_k \phi,_j}{\phi}\right)\right\}\right].$$

Therefore, equating coefficients of λ_i, λ_j, λ_k on both the sides, we can write

$$R_{ij} = \frac{R}{(n-1)\phi^2}\left(\lambda_{i,j} - \frac{\lambda_i \phi,_j}{\phi}\right)$$

which must be symmetric in i, j. ($\lambda_{i,j}$ is symmetric, since λ_i is a gradient vector; so $\lambda_i\, \phi,_j$ must also be symmetric in i, j). Accordingly we get

$$\frac{R}{n-1}(e_i e_j),_k = -(R_{ik}\lambda_j + R_{jk}\lambda_i). \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Therefore (6) becomes

$$R_{ij,k} = 2R_{ij}\lambda_k + R_{ik}\lambda_j + R_{jk}\lambda_i. \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Hence by virtue of (1) we get the property (*c*) of the theorem.

Now suppose that R is a constant. Then from the properties (*c*) and (*b*) of the theorem, the properties (*d*) follow. Thus the space becomes a C_n admitting a non-null parallel vector field. Levine (1950) has shown that such a space can be expressed as a product space $K_1 \times K_{n-1}$, where K_m represents an m-dimensional space of constant curvature. Hence the property (*e*) of the theorem. This completes the proof of the theorem.

References

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