TEMPERATURE DISTRIBUTION IN COUETTE FLOW BETWEEN TWO PARALLEL FLAT PLATES

by G. N. Purohit, *Department of Mathematics, Rajasthan University, Jaipur*

(Communicated by P. L. Bhatnagar, F.N.I.)

(*Received August 17, 1966*)

In this paper we have studied the temperature distribution in a viscous incompressible fluid flowing between two parallel flatplates. The temperature of one of the plates is allowed to vary with time, while the other is kept at a constant temperature. The rate of heat generation per unit volume per unit time in the fluid has been taken as a function of time and the dissipation of energy due to friction has also been considered. Using Laplace transform technique it is found that the solution of the energy equation comes out in a form which exhibits the contributions of boundary conditions, dissipation due to friction and the rate of heat generation to the temperature distribution.

1. Introduction

The steady flow of a viscous incompressible fluid between two parallel flat plates caused by the motion of one of the plates, under constant pressure, is quite well known as plane Couette flow. Pai (1956) gives the velocity distribution and temperature distribution for this flow. However, he gives the solution of energy equation without considering the rate of heat generation per unit volume in the fluid (other than viscous dissipation). Recently, Bhatnagar and Tikekar (1965) gave temperature distribution in a channel bounded by two co-axial cylinders. They assumed the rate of heat generation per unit volume as a function of the time but did not include the effects of viscous dissipation.

In this paper, we have considered the distribution of temperature in a viscous incompressible fluid flowing between two parallel plates. The flow is caused by the motion of one of the plates. The moving plate is kept at a constant temperature, while the temperature of the stationary plate is allowed to vary with time. Effects of viscous dissipation have also been considered. The rate of heat generation per unit volume in the fluid is also taken as time-dependent. Using the Laplace transform technique the solution of the energy equation is obtained. The solution comes out in a form which exhibits the contribution of boundary conditions, dissipation due to friction and the rate of heat generation per unit volume in the fluid to the temperature distribution. Tables have been prepared to give the dimensionless temperature distribution for these effects for a particular case, i.e. for a periodic rate of heat generation, and periodic variation of the temperature of the lower plate.

2. Flow Distribution and Energy Equation

For the case of two-dimensional incompressible fluid flow with constant properties, the system of equations for the velocity distribution in steady flow along a xy-plane is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \text{(2.1)}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \qquad \text{(2.2)}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \qquad \text{(2.3)}$$

where ρ is the density of the fluid and ν is the coefficient of kinematic viscosity.

Now let us consider the flow between two parallel flat plates at a distance $2d$ apart, of which one is at rest and the other is moving with constant velocity U. For this flow we have

$$u=u(x,y), \qquad v=0, \qquad p=p(x,y).$$

The eqns. (2.1), (2.2) and (2.3) reduce to

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=0,\\ \frac{\partial p}{\partial x}&=\mu\frac{\partial^2 u}{\partial y^2}\\ \frac{\partial p}{\partial y}&=0\end{aligned}\right\}. \qquad \text{(2.4)}$$

For a plane Couette flow, $\frac{\partial p}{\partial x}=0$, the solution of eqn. (2.4) under the boundary conditions

$$u=0: y=-d; \quad u=U: y=d$$

is

$$u=\frac{U}{2}\left(1+\frac{y}{d}\right). \qquad \text{(2.5)}$$

The energy equation is

$$\rho C\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=\frac{\partial Q}{\partial t}+k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\phi \qquad \text{(2.6)}$$

where

$$\phi=\left[2\left\{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

is the energy dissipation function; $\frac{\partial Q}{\partial t}$ is the rate of heat generation per unit volume in the fluid; C and k are respectively the specific heat and the coefficient of heat conductivity of the fluid.

The velocity distribution is steady while the temperature distribution is unsteady. The temperature distribution does not influence the flow field of an incompressible fluid with constant properties. We have assumed a fluid having these properties.

3. Solution of Equations

If we assume that the temperature is independent of its axial position, then $\frac{\partial T}{\partial x} = 0$, eqn. (2.6) reduces to

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial Q}{\partial t} + k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y}\right)^2. \qquad \ldots \quad (3.1)$$

Introducing the non-dimensional quantities

$$\eta = \frac{y}{d}, \quad \bar{u} = \frac{u}{U}, \quad \tau = \frac{t\nu}{d^2}, \quad \theta = \frac{T - T_i}{T_i}, \quad f(\tau) = \frac{\nu Q}{kT_i}$$

in (3.1), we have

$$\frac{\partial^2 \theta}{\partial \eta^2} - \sigma \frac{\partial \theta}{\partial \tau} = -\frac{\partial f}{\partial \tau} - 4\beta\sigma \left(\frac{\partial \bar{u}}{\partial \eta}\right)^2, \qquad \ldots \quad (3.2)$$

where β is the non-dimensional constant $\frac{U^2}{4T_iC_p}$, σ is the Prandtl number, T_i is the initial temperature of the fluid and both the plates, and $f(\tau)$ is a function of time only such that $f(o) = 0$.

Introducing the non-dimensional variable in (2.5) we have

$$\frac{\partial \bar{u}}{\partial \eta} = \tfrac{1}{2}. \qquad \ldots \quad (3.3)$$

From (3.2) and (3.3), we have

$$\frac{\partial^2 \theta}{\partial \eta^2} - \sigma \frac{\partial \theta}{\partial \tau} = -\frac{\partial f}{\partial \tau} - \beta\sigma. \qquad \ldots \quad (3.4)$$

The upper plate is kept at constant temperature T_i for all times. The temperature of the lower plate is a prescribed time-dependent temperature defined by the function $T_i\ (1+g(\tau))$ such that $g(o) = 0$. The boundary and initial conditions are

$$\left.\begin{array}{lll} \theta = 0, & \eta = 1, & \tau \geqslant 0 \\ \theta = 0, & -1 < \eta < 1, & \tau = 0 \\ \theta = g(\tau), & \eta = -1, & \tau \geqslant 0 \end{array}\right\} \qquad \ldots \quad (3.5)$$

We have to solve eqn. (3.4) under the boundary conditions (3.5). Multiplying eqn. (3.4) by $e^{-s\tau}$ and then integrating the resulting equation with respect to τ in the limits 0 to ∞, we have

$$\frac{d^2\theta_s}{d\eta^2} - \sigma s\,\theta_s = -sf_s - \beta\sigma/s, \qquad (3.6)$$

where θ_s and f_s are the Laplace transforms of θ and f respectively defined by

$$\theta_s = \int_0^\infty \theta(\tau)e^{-s\tau}d\tau,\; f_s = \int_0^\infty f(\tau)e^{-s\tau}d\tau.$$

The boundary conditions (3.5) reduce to

$$\theta_s = 0,\; \eta = 1;\quad \theta_s = g_s,\; \eta = -1, \qquad (3.7)$$

where g_s is Laplace transform of $g(\tau)$.

The solution of eqn. (3.6) is

$$\theta_s = c_1 e^{\sqrt{\sigma s}\eta} + c_2 e^{-\sqrt{\sigma s}\eta} + \frac{f_s}{\sigma} + \frac{\beta}{s^2}. \qquad (3.8)$$

Evaluating the constants c_1 and c_2 from the boundary conditions (3.7), we have

$$\begin{aligned}\theta_s = {} & \frac{1}{\sigma}f_s\left[1 - \frac{\sinh\{(1+\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})} - \frac{\sinh\{(1-\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})}\right] \\ & + \frac{\beta}{s^2}\left[1 - \frac{\sinh\{(1+\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})} - \frac{\sinh\{(1-\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})}\right] \\ & + g_s\left[\frac{\sinh\{(1-\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})}\right]. \qquad (3.9)\end{aligned}$$

The temperature θ can be obtained by taking inverse the Laplace transform of (3.9), defined by

$$\theta = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \theta_s e^{s\tau}\,ds. \qquad (3.10)$$

The temperature θ can be expressed as a sum of the three terms

$$\theta = \theta_1 + \theta_2 + \theta_3,$$

where

$$\theta_1 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{1}{\sigma}f_s\left[1 - \frac{\sinh\{(1+\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})} - \frac{\sinh\{(1-\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})}\right]e^{s\tau}\,ds, \qquad (3.11)$$

$$\theta_2 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{\beta}{s^2}\left[1 - \frac{\sinh\{(1+\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})} - \frac{\sinh\{(1-\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})}\right]e^{s\tau}\,ds, \qquad (3.12)$$

$$\theta_3 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} g_s\left[\frac{\sinh\{(1-\eta)\sqrt{\sigma s}\}}{\sinh(2\sqrt{\sigma s})}\right]e^{s\tau}\,ds. \qquad (3.13)$$

Now θ_1 is the component corresponding to the rate of heat generation in the fluid (other than viscous dissipation), θ_2 due to viscous dissipation and θ_3 due to boundary conditions.

4. Particular Case

The function $f(\tau)$ and $g(\tau)$ are independent of each other, but they can be related also. For a particular example let us assume them to be periodic function of time such that

$$f(\tau) = g(\tau) = \sin(\tau)$$

$$\therefore \quad f_s = g_s = \frac{1}{1+s^2}. \quad \cdots \quad (4.1)$$

Substituting the value of f_s and g_s in (3.11) and (3.13) respectively and then evaluating the integrals, we obtain

$$\theta_1 = \frac{4}{\sigma\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left\{ \frac{16\sigma^2 \sin\tau + 4\sigma n^2\pi^2 \cos\tau - 4\sigma n^2\pi^2 e^{-n^2\pi^2\tau/4\sigma}}{n^4\pi^4 + 16\sigma^2} \right\} \times \left\{ \sin\frac{n\pi}{2} \cos\frac{n\pi\eta}{2} \right\}, \quad \cdots \quad (4.2)$$

$$\theta_3 = \frac{1-\eta}{2} \sin\tau + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left\{ \frac{16\sigma^2 \sin\tau + 4\sigma n^2\pi^2 \cos\tau - 4\sigma n^2\pi^2 e^{-n^2\pi^2\tau/4\sigma}}{n^4\pi^4 + 16\sigma^2} \right\} \times \sin\left\{ \frac{n\pi}{2}(1-\eta) \right\}. \quad \cdots \quad (4.3)$$

These integrals are evaluated with the help of convolution theorem.

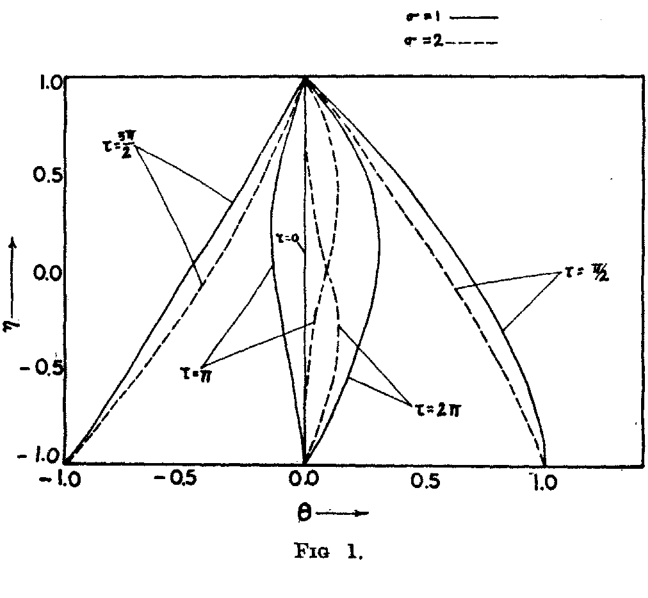

Fig 1.

The temperature θ_2 is independent of plate temperature and of other heat generation. It is given by

$$\theta_2 = \frac{16\beta\sigma}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} [1-\exp(-n^2\pi^2\tau/4\sigma)] \sin\frac{n\pi}{2} \cos\frac{n\pi\eta}{2} \quad \ldots \quad \ldots \quad (4.4)$$

Numerical work has been done to tabulate θ_1, θ_2 and θ_3, for Prandtl numbers $\sigma = 1$ and 2 for $\tau = 0$, $\pi/2$, π, $3\pi/2$ and 2π in Tables I, II and III respectively. In its course the temperature due to viscous dissipation $\theta_2/\beta\sigma$ goes on constantly increasing with time and becomes steady after some

TABLE I

For θ_1

	$\sigma = 1$				$\sigma = 2$			
η \ τ	$\pi/2$	π	$3\pi/2$	2π	$\pi/2$	π	$3\pi/2$	2π
−1·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
−0·75	0·0660	−0·0833	−0·0696	0·1872	0·0810	−0·1393	−0·1088	0·1366
−0·50	0·1211	−0·1410	−0·1276	0·3269	0·1480	−0·2384	−0·1802	0·2335
−0·25	0·1572	−0·1674	−0·1656	0·4022	0·1911	−0·2868	−0·2333	0·2803
0·00	0·1696	−0·1637	−0·1787	0·4241	0·2059	−0·2993	−0·2515	0·2924
0·25	0·1572	−0·1674	−0·1656	0·4022	0·1911	−0·2868	−0·2333	0·2803
0·50	0·1211	−0·1410	−0·1276	0·3269	0·1480	−0·2384	−0·1802	0·2335
0·75	0·0660	−0·0833	−0·0696	0·1872	0·0810	−0·1393	−0·1088	0·1366
1·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

Note : The value of θ_1 for $\tau = 0$ is zero for all values of η.

TABLE II

For $\theta_2/\beta\sigma$

	$\sigma = 1$				$\sigma = 2$			
η \ τ	$\pi/2$	π	$3\pi/2$	2π	$\pi/2$	π	$3\pi/2$	2π
−1·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
−0·75	0·2148	0·2188	0·2188	0·2188	0·1904	0·2138	0·2182	0·2188
−0·50	0·3679	0·3753	0·3754	0·3755	0·3230	0·3680	0·3742	0·3754
−0·25	0·4580	0·4677	0·4678	0·4679	0·3992	0·4580	0·4663	0·4676
0·00	0·4822	0·5008	0·5009	0·5010	0·4268	0·4904	0·4992	0·5008
0·25	0·4580	0·4677	0·4678	0·4679	0·3992	0·4580	0·4663	0·4676
0·50	0·3679	0·3753	0·3754	0·3755	0·3230	0·3680	0·3742	0·3754
0·75	0·2148	0·2188	0·2188	0·2188	0·1904	0·2138	0·2182	0·2188
1·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

Note: The value of $\frac{\theta_2}{\beta\sigma}$ for $\tau = 0$ is zero for all values of η.

time. The temperature θ_1 due to heat generation is periodic. The temperature θ_3 goes on decreasing or increasing with the increase in η according as θ_3 is positive or negative at $\eta = -1$. The temperature θ_3 is also periodic.

TABLE III

For θ_3

η \ τ	$\sigma = 1$: $\pi/2$	π	$3\pi/2$	2π	$\sigma = 2$: $\pi/2$	π	$3\pi/2$	2π
−1·00	1·0000	0·0000	−1·0000	0·0000	1·0000	0·0000	−1·0000	−0·0000
−0·75	0·8397	0·1161	−0·8378	−0·1161	0·8028	0·0955	−0·7751	−0·0928
−0·50	0·7063	0·1954	−0·6830	−0·1954	0·6145	0·1765	−0·5824	−0·1716
−0·25	0·5435	0·2237	−0·5399	−0·2237	0·4427	0·2430	−0·4006	−0·2365
0·00	0·4152	0·2121	−0·4107	−0·2121	0·2941	0·2993	−0·2485	−0·2924
0·25	0·2986	0·1786	−0·2945	−0·1786	0·1751	0·3306	−0·1328	−0·3241
0·50	0·1927	0·1316	−0·1291	−0·1316	0·0895	0·3003	−0·0572	−0·2954
0·75	0·0943	0·0711	−0·0924	−0·0710	0·0352	0·1831	−0·0073	−0·1814
1·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

Note: The value of θ_3 for $\tau = 0$ is zero for all values of η.

TABLE IV

η \ τ	$\sigma = 1$: $\pi/2$	π	$3\pi/2$	2π	$\sigma = 2$: $\pi/2$	π	$3\pi/2$	2π
−1·00	1·0000	0·0000	−1·0000	0·0000	1·0000	0·0000	−1·0000	0·0000
−0·75	0·9486	−0·0274	−0·8636	0·1149	0·9219	0·0010	−0·8403	0·0876
−0·50	0·9110	−0·0665	−0·7355	0·2066	0·8271	0·0117	−0·6878	0·1370
−0·25	0·7933	−0·0851	−0·6119	0·2721	0·7136	0·0478	−0·5407	0·1373
0·00	0·6812	−0·1120	−0·4892	0·3122	0·5854	0·0916	−0·4002	0·1002
0·25	0·5474	−0·1302	−0·3665	0·3172	0·4460	0·1354	−0·2729	0·0497
0·50	0·3874	−0·1203	−0·1816	0·2704	0·3021	0·1355	−0·1626	0·0132
0·75	0·2032	−0·0724	−0·1182	0·1600	0·1543	0·0866	−0·0725	0·0000
1·00	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

Note: The value of θ for $\tau = 0$ is zero for all values of η.

For a particular example of a flowing fluid having $u = 4{\cdot}5$ ft/sec, $T_i = 104°$ F and $C = 0{\cdot}998$ Btu./lb °F the constant $\beta\sigma$ approximately equals to 0·2. Table IV gives the temperature $\theta\,(=\theta_1+\theta_2+\theta_3)$ for this particular value of $\beta\sigma$. It has been presented graphically also (Fig. 1).

Acknowledgements

The author wishes to record his deep sense of gratitude to Dr. M. C. Gupta and Dr. P. D. Verma for their kind guidance and constant help in the preparation of this paper and to Dr. G. C. Patni for his many valuable suggestions and encouragement. The author is also thankful to the Ministry of Education, Government of India, for having provided a research scholarship.

References

Bhatnagar, P. L., and Tikekar, V. G. (1965). A note on the temperature distribution in a channel bounded by two coaxial cylinders. 31st Conf. Indian math. Soc.

Pai, S. I. (1956). Viscous Flow Theory I—Laminar Flow. D. Van Nostrand Company Inc., New York.