ON THE TEMPERATURE DISTRIBUTION OF A VISCOUS LIQUID UNDER OSCILLATORY RATE OF HEAT ADDITION SUPERPOSED ON THE STEADY TEMPERATURE OF INCOMPRESSIBLE FLUID BETWEEN TWO CO-AXIAL CIRCULAR CYLINDERS

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In the present paper the temperature distribution in a channel bounded by two co-axial circular pipes is obtained when viscous incompressible fluid is flowing through it and the oscillatory rate of heat addition is superposed on the steady temperature. Solutions are also given in the two extreme cases of very small and very large frequencies. It is found that maximums of temperature distribution exist in the neighbourhood of the wall when frequencies are very large.

INTRODUCTION

Solutions for the temperature distribution in a circular pipe have been given by many authors, like Graetz, Nusselt, Goldstein; all these are cited in Goldstein's book (1938, § 266). Lal (1964) considered the temperature distribution in a channel bounded by two co-axial circular pipes when viscous incompressible fluid is flowing through it and the rate of heat addition is an exponential function of time. In the present paper the expression for the temperature distribution in a channel bounded by two co-axial circular pipes has been derived for a viscous incompressible fluid flowing through it by neglecting the dissipation due to friction and when an oscillatory rate of heat addition is superposed on the steady temperature. Here the solutions are also given in the two extreme cases of very small and very large frequencies.

1. EQUATION OF ENERGY AND ITS SOLUTION

The equation of energy (Pai 1956) in the present case is

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_V}\frac{\partial Q}{\partial t} + k'\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

where $k' = \dfrac{k}{\rho c_V}$ is a constant and the dissipation due to friction is neglected.

Now we assume that

$$\frac{1}{\rho c_V} \frac{\partial Q}{\partial t} = \sum_{n=1}^{\infty} a_n e^{int} \qquad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

and

$$T = T_0 + \sum_{n=1}^{\infty} T_n(r) e^{int}, \qquad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

where a_n and T_n are real and T_n is a function of r only.

Substituting eqns. (1.2) and (1.3) in (1.1) and comparing the terms of the same family, the differential equations for the coefficients are

$$\frac{d^2T_0}{dr^2} + \frac{1}{r} \frac{dT_0}{dr} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

and

$$\frac{d^2T_n}{dr^2} + \frac{1}{r} \frac{dT_n}{dr} - \frac{in}{k'} T_n + \frac{a_n}{k'} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

Integrating (1.4) we have

$$T_0 = A + B \log r. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

For (1.5) let $v = \frac{in}{k'} T_n - \frac{a_n}{k'}$, then

$$\frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{in}{k'} v = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (1.7)$$

The solution of (1.7) is (Watson 1922)

$$v = D_n' J_0(rpi^{3/2}) + E_n' Y_0(rpi^{3/2}), \qquad \ldots \quad \ldots \quad \ldots \quad (1.8)$$

where $p = (n/k')^{1/2}$ and J_0 and Y_0 are Bessel functions of the first and the second kinds of zero order. Hence

$$T_n(r) = \frac{a_n}{in} + D_n J_0(rpi^{3/2}) + E_n Y_0(rpi^{3/2}).$$

Before superposing the oscillatory flow, we must have the fully-developed steady motion. With this condition and with the following boundary conditions:

$$T_0 = T_1 \quad \text{when } r = r_1,$$

and

$$T_0 = T_2 \quad \text{when } r = r_2,$$

the constants A and B in (1.6) are determinate. Thus

$$T_0 = \frac{T_1 \log (r_2/r) + T_2 \log (r/r_1)}{\log (r_2/r_1)}.$$

So (1.3) gives

$$T = \frac{T_1 \log (r_2/r) + T_2 \log (r/r_1)}{\log (r_2/r_1)} + R \sum_{n=1}^{\infty} \left[\frac{a_n}{in} + D_n J_0(rpi^{3/2}) + E_n Y_0(rpi^{3/2}) \right] e^{int},$$

where R denotes the real part.

The boundary conditions are

$$T = T_1 e^{int} + T_1 \quad \text{when } r = r_1,$$

and

$$T = T_2 e^{int} + T_2 \quad \text{when } r = r_2.$$

With the help of the above conditions the constants D_n and E_n are determinate. Thus

$$T = \frac{T_1 \log (r_2/r) + T_2 \log (r/r_1)}{\log (r_2/r_1)}$$

$$+ R \sum_{n=1}^{\infty} \frac{a_n}{in} \left[1 - \frac{\{ Y_0(r_2 pi^{3/2}) - Y_0(r_1 pi^{3/2}) \} J_0(rpi^{3/2}) - \{ J_0(r_2 pi^{3/2}) - J_0(r_1 pi^{3/2}) \} Y_0(rpi^{3/2})}{J_0(r_1 pi^{3/2}) Y_0(r_2 pi^{3/2}) - J_0(r_2 pi^{3/2}) Y_0(r_1 pi^{3/2})} \right] e^{int}$$

$$+ R \sum_{n=1}^{\infty} \left[\frac{\{ T_1 Y_0(r_2 pi^{3/2}) - T_2 Y_0(r_1 pi^{3/2}) \} J_0(rpi^{3/2}) - \{ T_1 J_0(r_2 pi^{3/2}) - T_2 J_0(r_1 pi^{3/2}) \} Y_0(rpi^{3/2})}{J_0(r_1 pi^{3/2}) Y_0(r_2 pi^{3/2}) - J_0(r_2 pi^{3/2}) Y_0(r_1 pi^{3/2})} \right] e^{int}. \quad \ldots (1.9)$$

When $T_1 = T_2$, we have, as $r_1 \to 0$,

$$T = T_1 + R \sum_{n=1}^{\infty} \frac{a_n}{in} \left[1 - \frac{J_0(rpi^{3/2})}{J_0(r_2 pi^{3/2})} \right] e^{int} + R \sum_{n=1}^{\infty} T_1 \frac{J_0(rpi^{3/2})}{J_0(r_2 pi^{3/2})} e^{int},$$

and this gives the temperature distribution in a circular pipe.

2. Temperature Distribution for Small Frequencies

For small frequencies, that is n is small, we have

$$J_0(rpi^{3/2}) \simeq 1 + \frac{ir^2 p^2}{4}$$

and

$$Y_0(rpi^{3/2}) \simeq \frac{2}{\pi} \left[\left(\gamma + \log \frac{rpi^{3/2}}{2} \right) \left(1 + \frac{ir^2 p^2}{4} \right) - \frac{ir^2 p^2}{4} \right],$$

where γ is the Euler's constant.

Hence

$$[Y_0(r_2 pi^{3/2}) - Y_0(r_1 pi^{3/2})] J_0(rpi^{3/2}) - [J_0(r_2 pi^{3/2}) - J_0(r_1 pi^{3/2})] Y_0(rpi^{3/2})$$
$$= \frac{2}{\pi} \left[\log (r_2/r_1) + \frac{ip^2}{4} \left\{ r_2^2 \log (r_2/r) - r_1^2 \log (r_1/r) + r^2 \log (r_2/r_1) - (r_2^2 - r_1^2) \right\} \right]$$

and

$$J_0(r_1 pi^{3/2}) Y_0(r_2 pi^{3/2}) - J_0(r_2 pi^{3/2}) Y_0(r_1 pi^{3/2})$$
$$= \frac{2}{\pi} \left[\log (r_2/r_1) + \frac{ip^2}{4} \left\{ r_2^2 \log (r_2/r_1) + r_1^2 \log (r_2/r_1) - (r_2^2 - r_1^2) \right\} \right].$$

Substituting these values in (1.9) and putting $T_1 = T_2$, we get

$$T = T_1 + \sum_{n=1}^{\infty} \frac{a_n}{4k' \log (r_2/r_1)} \left[r_1^2 \log (r_2/r) + r_2^2 \log (r/r_1) - r^2 \log (r_2/r_1) \right] \cos nt$$
$$+ \sum_{n=1}^{\infty} \frac{T_1}{4k' \log (r_2/r_1)} \left[4k' \log (r_2/r_1) \cos nt + n(r_1^2 \log (r_2/r) \right.$$
$$\left. + r_2^2 \log (r/r_1) - r^2 \log (r_2/r_1)) \sin nt \right], \qquad (2.1)$$

when both the walls of the pipe have the same temperature.

3. Temperature Distribution for Large Frequencies

When n is large, taking asymptotic expansions of Bessel functions, $rp > 10$ and $-\frac{\pi}{2} \leqslant$ phase $(rpi^{3/2}) \leqslant \frac{\pi}{2}$, we have

$$J_0(rpi^{3/2}) \simeq \sqrt{\frac{2}{\pi rpi^{3/2}}} \cdot \cos \left(rpi^{3/2} - \frac{\pi}{4} \right)$$

and

$$Y_0(rpi^{3/2}) \simeq \sqrt{\frac{2}{\pi rpi^{3/2}}} \cdot \sin \left(rpi^{3/2} - \frac{\pi}{4} \right).$$

Substituting these values in (1.9) and again putting $T_1 = T_2$, we obtain

$$T = T_1 + \sum_{n=1}^{\infty} \frac{a_n}{n} \left[\sin nt + \sqrt{\frac{r_1}{r}} \exp \left\{ -(r_1 - r) \frac{p}{\sqrt{2}} \right\} . \sin \left\{ (r_1 - r) \frac{p}{\sqrt{2}} - nt \right\} \right.$$
$$\left. + \sqrt{\frac{r_2}{r}} . \exp \left\{ -(r_2 - r) \frac{p}{\sqrt{2}} \right\} . \sin \left\{ (r_2 - r) \frac{p}{\sqrt{2}} - nt \right\} \right]$$
$$+ T_1 \sum_{n=1}^{\infty} \left[\sqrt{\frac{r_1}{r}} . \exp \left\{ -(r_1 - r) \frac{p}{\sqrt{2}} \right\} . \cos \left\{ (r_1 - r) \frac{p}{\sqrt{2}} - nt \right\} \right.$$
$$\left. + \sqrt{\frac{r_2}{r}} . \exp \left\{ -(r_2 - r) \frac{p}{\sqrt{2}} \right\} . \cos \left\{ (r_2 - r) \frac{p}{\sqrt{2}} - nt \right\} \right]. \qquad (3.1)$$

Hence when n is large, it is found that maximums of temperature distribution exist in the neighbourhood of the wall.

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