QUADRATURE FORMULAS USING TSCHEBYSCHEFF ZEROS*

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By using the zeros of Tschebyscheff polynomials, quadrature formulas are developed to approximate $\int_{-1}^{1} f(x)\,dx$. A matrix representation is given which allows for simple extensions to interpolation, numerical differentiation and integration. Corresponding 'trapezoidal' and 'parabolic' rules are given. The various formulas are compared with the Newton-Cotes formulas and the errors are found to be smaller. Numerical examples are presented. Two-dimensional extensions are also given.

1. DESCRIPTION OF METHOD

Suppose we have a set of $n+1$ points (x_0, f_0), (x_1, f_1), ..., (x_n, f_n) and suppose we want to pass a polynomial $p_n(x)$ of degree n to be passed through these $n+1$ points. The usual solution to this problem is by the method of Lagrange. The polynomial $p_n(x)$ is given by

$$p_n(x) = f_0 l_0(x) + f_1 l_1(x) + \ldots + f_n l_n(x) \quad \ldots \quad (1)$$

where $l_i(x)$, $i = 0, 1, \ldots, n$, are polynomials of degree n or less determined by

$$l_i(x_j) = \delta_{ij} \quad \ldots \quad (2)$$

where δ_{ij} is the usual Kronecker delta.

Suppose we let

$$r_{n+1}(x) = (x-x_0)(x-x_1)\ldots(x-x_n) \quad \ldots \quad (3)$$

and define

$$s_i(x) = \frac{r_{n+1}(x)}{x-x_i}. \quad \ldots \quad (4)$$

From (4), we have

$$s_i(x_i) = r'_{n+1}(x_i) \quad \ldots \quad (5)$$

and since

$$l_i(x) = \frac{s_i(x)}{s_i(x_i)} \quad \ldots \quad (6)$$

it follows that

$$l_i(x) = \frac{s_i(x)}{r'_{n+1}(x_i)}. \quad \ldots \quad (7)$$

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Now, if we represent $l_i(x)$ as

$$l_i(x) = b_{0i}+b_{1i}x+b_{2i}x^2+\dots+b_{ni}x^n \qquad (8)$$

and introduce matrix notation, we can express $p_n(x)$ as

$$p_n(x) = XLF, \qquad (9)$$

where

$$L = \begin{bmatrix} b_{00} & b_{01} & \dots & b_{0n} \\ b_{10} & b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{n0} & b_{n1} & \dots & b_{nn} \end{bmatrix}$$

and

$$F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} \text{ and } X = [1, x, x^2, \dots, x^n].$$

For the above, $X\,L = [l_0(x), l_1(x), \dots, l_n(x)]$ and thus $X\,L\,F = p_n(x)$.

It follows that

$$p_n'(x) = [0, 1, 2x, \dots, nx^{n-1}]LF \qquad (10)$$

and

$$\int_0^x p_n(t)dt = \left[x, \frac{x^2}{2}, \dots, \frac{x^{n+1}}{n+1}\right] LF. \qquad (11)$$

It is well known that the error function

$$R_n(f; x) = f(x)-p_n(x)$$

satisfies

$$|R_n(f; x)| < \left\{ \max_{a \le t \le b} |f^{(n+1)}(t)| \right\} \frac{|x-x_0||x-x_1|\dots|x-x_n|}{(n+1)!}.$$

The part $|x-x_0|\,|x-x_1|\dots|x-x_n|$ is independent of the function and $\max_{a \le x \le b}$ $(x-x_0)(x-x_1)\dots(x-x_n)$ is minimized if $x_0, x_1, \dots, x_n$, are taken as the zeros of the Tschebyscheff polynomial $T_{n+1}^{(x)}$ and for this reason we take these zeros for our interpolation points.

2. Illustrative Example

Suppose we consider the zeros of the Tschebyscheff polynomial $T_3(x) = 4x^3 - 3x$, that is $0, \pm\frac{\sqrt{3}}{2}$.

Now,

$$r_3(x) = x\left(x-\frac{\sqrt{3}}{2}\right)\left(x+\frac{\sqrt{3}}{2}\right),$$

$$r_3'(x) = 3x^2-\tfrac{3}{4},$$

and

$$r_3'(0) = -\frac{3}{4},\ r_3'\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2},\ r_3'\left(-\frac{\sqrt{3}}{2}\right) = \frac{3}{2}.$$

Thus,

$$s_0(x) = x^2 - \frac{3}{4},\ s_1(x) = x^2 + \frac{\sqrt{3}}{2}x,\ s_2(x) = x^2 - \frac{\sqrt{3}}{2}x,$$

and

$$l_0(x) = 1 - \frac{4}{3}x^2,\ l_1(x) = \frac{1}{\sqrt{3}}x + \frac{2}{3}x^2,\ l_2(x) = -\frac{1}{\sqrt{3}}x + \frac{2}{3}x^2,$$

and thus

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{4}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

from which it follows that (putting F in a more convenient form)

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} p_3^{(x)}\,dx = \int_{-1}^{1} [1, x, x^2] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{4}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} f_0 \\ f_{-1} \\ f_1 \end{bmatrix} dx$$

$$= [2, 0, \tfrac{2}{3}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{4}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} f_0 \\ f_{-1} \\ f_1 \end{bmatrix}$$

$$= \frac{1}{9}\{4f_{-1} + 10f_0 + 4f_1\}.$$

This formula can also be expressed as

$$\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[4f\left(-\frac{\sqrt{3}}{2}\right) + 10f(0) + 4f\left(\frac{\sqrt{3}}{2}\right)\right] + E(f), \quad \text{.. (12)}$$

where the error term is found by the same technique as in Davis (1963). By using the Peano Kernel, $E(f)$ is given by

$$E(f) = \int_{-1}^{+1} f^{(4)}(t)K(t)\,dt,$$

where

$$K(t) = \frac{1}{6}\left\{\frac{t^4}{4} - \frac{5}{9}t^3 + \frac{1}{6}(9-4\sqrt{3})t^2 + \frac{1}{12}(3-2\sqrt{3})\right\}, \quad 0 \le t < \frac{\sqrt{3}}{2}$$

$$= \frac{1}{24}(1-t)^4, \quad \frac{\sqrt{3}}{2} \le t \le 1$$

and $$K(t) = K(-t).$$

In the appendix, we have listed the formulas corresponding to

$\int_{-1}^{1} p_n(x)\,dx$ for $n = 2$, 4 and 5, and the associated L matrix.

3. One-dimensional Extensions

Corresponding to the above expressions for $\int_{-1}^{+1} f(x)\,dx$, we can write

$$\int_a^b f(x)\,dx \simeq \frac{h}{18}\left[4f\left(a+\frac{h}{2}-\frac{\sqrt{3}}{4}h\right)+10f\left(a+\frac{h}{2}\right)+4f\left(a+\frac{h}{2}+\frac{\sqrt{3}}{4}h\right)\right] \tag{13}$$

where $h = b-a$. We can also give another extension, a 'parabolic' rule,

$$\int_a^b f(x)\,dx \simeq \frac{h}{18}\left[4\sum_{i=1}^{n} f\left(a+(2i-1)\frac{h}{2}-\frac{\sqrt{3}}{4}h\right)+10\sum_{i=1}^{n} f\left(a+(2i-1)\frac{h}{2}\right)\right.$$
$$\left.+4\sum_{i=1}^{n}\left(a+(2i-1)\frac{h}{2}+\frac{\sqrt{3}}{4}h\right)\right] \quad \ldots \tag{14}$$

where

$$h = \frac{b-a}{n}.$$

In the appendix, we have listed formulas similar to (13) corresponding to $n = 2$, 4, and 5. In addition, we have listed a corresponding 'trapezoidal' rule.

4. Two-dimensional Extensions

When we consider the above formulas extended to two dimensions, we have

$$\int_{-1}^{1}\int_{-1}^{1} f(x, y)dx\,dy \simeq \frac{1}{81}\left\{16\left[f\left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)+f\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)\right.\right.$$
$$\left.+f\left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)+f\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)\right]+40\left[f\left(-\frac{\sqrt{3}}{2}, 0\right)\right.$$
$$\left.\left.+f\left(0, -\frac{\sqrt{3}}{2}\right)+f\left(0, \frac{\sqrt{3}}{2}\right)+f\left(\frac{\sqrt{3}}{2}, 0\right)\right]+100f(0, 0)\right\}. \quad \ldots \tag{15}$$

In a more general form, we have

$$\int_{x_0}^{x_n}\int_{y_0}^{y_m} f(x, y)dy\,dx \simeq \frac{hk}{324}\left\{100\sum_{j=1}^{m}\sum_{i=1}^{n} f\left(x_0+(2i-1)\frac{h}{2}, y_0+(2j-1)\frac{k}{2}\right)\right.$$
$$+16\sum_{j=1}^{m}\sum_{i=1}^{n} f\left(x_0+(2i-1)\frac{h}{2}\pm\frac{h\sqrt{3}}{4}, y_0+(2j-1)\frac{k}{2}\pm\frac{k\sqrt{3}}{4}\right)$$

$$+40\sum_{j=1}^{m}\sum_{i=1}^{n}f\left(x_0+(2i-1)\frac{h}{2},\, y_0+(2j-1)\frac{k}{2}\pm\frac{k\sqrt{3}}{4}\right)$$

$$+40\sum_{j=1}^{m}\sum_{i=1}^{n}f\left(x_0+(2i-1)\frac{h}{2}\pm\frac{h\sqrt{3}}{4},\, y_0+(2j-1)\frac{k}{2}\right)\Bigg\} \quad \ldots \quad (16)$$

where all possible sign combinations are taken in each double summation above and where $x_n = x_0+nh$, $y_m = y_0+mk$.

In the appendix, we have listed the two formulas similar to the above, only for the case $n = 2$.

If the region R we want to integrate over by use of the above or similar formulas is not rectangular, we construct a rectangular region S to enclose R and define a new function $F(x, y)$ at the desired points to be zero if outside R and inside S and to be the functional value if inside R or on boundary of R. We then apply the quadrature formulas to the function $F(x, y)$ over S to obtain approximate value of $\int_R\int f(x, y)dx\, dy$. Convergence cannot be guaranteed in this case.

5. Numerical Examples

The formulas found were checked extensively on a G.E. 235 computer. These formulas were compared with the Newton-Cotes rules, the parabolic rule and the trapezoidal rule (Hildebrand 1956). We found a number of interesting results in the comparison of these formulas. To set forth an example, we consider $\int_a^b \frac{dx}{1+x^4}$. First we let $a = 0$, $b = 3$, and compare the usual trapezoidal rule with the 'trapezoidal' rule used by us $\left(\int_0^3 \frac{dx}{x^4+1} = 1{\cdot}09844\right)$.

Number of points used	Trapezoidal rule	'Trapezoidal' rule used by us
2	1·51829	1·48022
4	1·06492	1·04097
6	1·09977	1·07869
8	1·09810	1·10037
10	1·09830	1·09942
12	1·09834	1·09829
14	1·09837	1·09832
16	1·09839	1·09839
18	1·09840	1·09841
20	1·09841	1·09841
22	1·09841	1·09842
24	1·09842	1·09842
26	1·09842	1·09842

Next, we consider $a = 0$, $b = 5$, and compare the usual parabolic rule with the 'parabolic' rule used by us $\left(\int_a^b \frac{dx}{x^4+1} = 1{\cdot}10806\right)$.

Number of points used	Parabolic rule	'Parabolic' rule used by us
3	0·91787	1·16898
9	1·13854	1·11559
15	1·10669	1·11278
21	1·10809	1·10744
27	1·10806	1·10796
33	1·10806	1·10808
39	1·10806	1·10806

Now, for various ranges of integration, we consider $\int_a^b \frac{dx}{x^4+1}$ and compare the Newton-Cotes five-point rule with our five-point rule:

a	b	N-C five-point rule	Our five-point rule	Exact
0	1	0·866425	0·866912	0·866973
0	2	1·08462	1·06753	1·07013
0	3	1·15250	1·11836	1·09844
0	4	1·07215	1·13833	1·10552
0	5	0·931680	1·08111	1·10806
0	6	0·833862	1·00127	1·10918
0	7	0·793924	0·948063	1·10975
0	8	0·79603	0·931671	1·11007
0	9	0·824737	0·945179	1·11026
0	10	0·869859	0·979527	1·11039

6. Conclusion

The above examples illustrate the type of results we found for various functions. Concerning the trapezoidal and parabolic rules in general, our rules were better at the start. As the number of points increased, our rules fell behind only in certain cases to catch up once again as the results converge.

References

Davis, Philip J. (1963). Interpolation and Approximation. Blaisdell Publishing Company, Waltham, Mass.

Hildebrand, F. B. (1956). Introduction to Numerical Analysis. McGraw-Hill Book Company Inc., New York.

Appendix

We now list the formulas and matrices called for in the above discussion. The order is as called for in the paper.

$$\int_{-1}^{1} f(x)dx \simeq f\left(-\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}}\right)$$

$$L_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The error is given by $E(f) = \int_{-1}^{1} f_{(t)}^{(2)} K_1(t)\, dt$,

$$\text{where } K_1(t) = \begin{cases} \frac{(1+t)^2}{2} & -1 \le t < -\frac{1}{\sqrt{2}} \\ \frac{1+t^2}{2} - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \le t < \frac{1}{\sqrt{2}} \\ \frac{(1-t)^2}{2} & \frac{1}{\sqrt{2}} \le t \le 1 \end{cases}$$

$$\int_{-1}^{1} f(x)dx \simeq \int_{-1}^{1} P_4(x)dx = \left(\tfrac{1}{2}-\frac{\sqrt{2}}{6}\right)f\left(-\cos\frac{\pi}{8}\right)+\left(\tfrac{1}{2}+\frac{\sqrt{2}}{6}\right)f\left(-\cos\frac{3\pi}{8}\right)$$

$$+\left(\tfrac{1}{2}+\frac{\sqrt{2}}{6}\right)f\left(\cos\frac{3\pi}{8}\right)+\left(\tfrac{1}{2}-\frac{\sqrt{2}}{6}\right)f\left(\cos\frac{\pi}{8}\right)$$

$$L_3 = \begin{bmatrix} \frac{-2+\sqrt{2}}{4\sqrt{2}} & \frac{-2+\sqrt{2}}{4\sqrt{2}} & \frac{2+\sqrt{2}}{4\sqrt{2}} & \frac{2+\sqrt{2}}{4\sqrt{2}} \\ \frac{1-\sqrt{2}}{2\sqrt{2+\sqrt{2}}} & \frac{\sqrt{2}-1}{2\sqrt{2+\sqrt{2}}} & \frac{\sqrt{2}+1}{2\sqrt{2-\sqrt{2}}} & \frac{-1-\sqrt{2}}{2\sqrt{2-\sqrt{2}}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{2+\sqrt{2}}} & \frac{-\sqrt{2}}{\sqrt{2+\sqrt{2}}} & \frac{-\sqrt{2}}{\sqrt{2-\sqrt{2}}} & \frac{\sqrt{2}}{\sqrt{2-\sqrt{2}}} \end{bmatrix}$$

or

$$\int_{1}^{1} f(x)dx \simeq \int_{1}^{1} p_4(x)dx = 0{\cdot}264298\, f(-0{\cdot}923880)+0{\cdot}735702\, f(-0{\cdot}382684)$$

$$+0{\cdot}735702\, f(0{\cdot}382684)+0{\cdot}264298\, f(0{\cdot}923880)$$

$$L_3 = \begin{bmatrix} -0{\cdot}103553 & -0{\cdot}103553 & 0{\cdot}603553 & 0{\cdot}603553 \\ -0{\cdot}112085 & 0{\cdot}112085 & 1{\cdot}57716 & -1{\cdot}57716 \\ 0{\cdot}707107 & 0{\cdot}707107 & -0{\cdot}707107 & -0{\cdot}707107 \\ 0{\cdot}765367 & -0{\cdot}765367 & -1{\cdot}84776 & 1{\cdot}84776 \end{bmatrix}$$

$$\int_{-1}^{1} f(x)dx \simeq \int_{-1}^{1} p_5(x)dx = 0{\cdot}167780\ f(-0{\cdot}951057)+0{\cdot}525552\ f(-0{\cdot}587786)$$
$$+0{\cdot}613336\ f(0)+0{\cdot}525552\ f(0{\cdot}587786)+0{\cdot}167780\ f(0{\cdot}951057)$$

$$L_4 = \begin{bmatrix} 1{\cdot}00000 & 0 & 0 & 0 & 0 \\ 0 & 0{\cdot}324920 & -0{\cdot}324920 & -1{\cdot}37638 & 1{\cdot}37638 \\ -4{\cdot}00000 & -0{\cdot}341641 & -0{\cdot}341641 & 2{\cdot}34164 & 2{\cdot}34164 \\ 0 & -0{\cdot}940456 & 0{\cdot}940456 & 1{\cdot}52169 & -1{\cdot}52169 \\ 3{\cdot}20000 & 0{\cdot}988854 & 0{\cdot}988854 & -2{\cdot}58885 & -2{\cdot}58885 \end{bmatrix}$$

$$\int_{a}^{b} f(x)dx \simeq \frac{h}{2}\left[f\left(a+\frac{h}{2}-\frac{h}{2\sqrt{2}}\right)+f\left(a+\frac{h}{2}+\frac{h}{2\sqrt{2}}\right)\right]$$

where $h = b-a$.

$$\int_{a}^{b} f(x)dx \simeq \int_{a}^{b} p_4(x)dx = \frac{h}{2}\left[0{\cdot}264298\ f\left(a+\frac{h}{2}-0{\cdot}461940\ h\right)\right.$$
$$+0{\cdot}735702\ f\left(a+\frac{h}{2}-0{\cdot}191342\ h\right)$$
$$+0{\cdot}735702\ f\left(a+\frac{h}{2}+0{\cdot}191342\ h\right)$$
$$\left.+0{\cdot}264298\ f\left(a+\frac{h}{2}+0{\cdot}461940\ h\right)\right]$$

where $h = b-a$.

$$\int_{a}^{b} f(x)dx \simeq \int_{a}^{b} p_5(x)dx = \frac{h}{2}\left[0{\cdot}167780\ f\left(a+\frac{h}{2}-0{\cdot}475529\ h\right)\right.$$
$$+0{\cdot}525552\ f\left(a+\frac{h}{2}-0{\cdot}293893\ h\right)$$
$$+0{\cdot}613336 f\left(a+\frac{h}{2}\right)$$
$$+0{\cdot}525552 f\left(a+\frac{h}{2}+0{\cdot}293893\ h\right)$$
$$\left.+0{\cdot}167780 f\left(a+\frac{h}{2}+0{\cdot}475529\ h\right)\right]$$

where $h = b-a$.

$$\int_{a}^{b} f(x)dx \simeq \frac{h}{2}\left[\sum_{i=1}^{n} f\left[a+(2i-1)\frac{h}{2}-\frac{h}{2\sqrt{2}}\right]\right]+\sum_{i=1}^{n} f\left[a+(2i-1)\frac{h}{2}+\frac{h}{2\sqrt{2}}\right]$$

where $h = \dfrac{b-a}{n}$.

$$\int_{-1}^{1}\int_{-1}^{1} f(x, y)dx\,dy \simeq f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$+f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\int_{x_0}^{x_n}\int_{y_0}^{y_m} f(x, y)dy\,dx \simeq \frac{hk}{4}\left[\sum_{j=1}^{m}\sum_{i=1}^{n} f\left(x_0+(2i-1)\frac{h}{2}\pm\frac{h}{2\sqrt{2}}, y_0+(2j-1)\frac{k}{2}\pm\frac{k}{2\sqrt{2}}\right)\right]$$

where sum is as indicated in paper, $x_n = x_0+nh$, $y_m = y_0+mk$.

The methods can be extended to Hermite interpolation using Tschebyscheff zeros. This will be taken up in a subsequent paper.