NON-NULL GEOMETRODYNAMICS

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The Einstein-Maxwell field equations in presence of pressure and matter have been considered in the non-null case with special role attached to the complex conjugate null eigenvectors of the electromagnetic tensor field $k_{\mu\lambda}$. The method and technique used have been taken from previous papers (Misra 1964; Hlavatý 1963*b*). It has been concluded that the eigenvalues of $R_\mu{}^\lambda$ cannot be complex. The quantities μ, p, D and the complex conjugate null eigenvectors have been expressed as algebraic concomitants of the tensor $R_\mu{}^\lambda$.

1. Introduction

The operational space-time is a Reimannian V_4 characterized by a symmetric metric tensor $h_{\mu\lambda}$ of signature $+++-$. The rank of the matrix $((h_{\mu\lambda}))$ is 4, so that we can always construct the inverse tensor $h^{\lambda\mu}$ such that

$$h_{\mu\alpha}h^{\alpha\lambda} = \delta^\lambda_\mu . \qquad (1.1)$$

Throughout this paper we shall use $h_{\mu\lambda}$ ($h^{\mu\lambda}$) to lower (raise) the indices.

Let $R_{\omega\mu\lambda}{}^\nu$ and $R_{\mu\lambda} \overset{\text{def}}{=} R_{\alpha\mu\lambda}{}^\alpha$ be the Christoffel-Riemann curvature tensor and Ricci tensor respectively. Let $k_{\mu\lambda}$ $(= -k_{\lambda\mu})$ be the electromagnetic tensor field. The electromagnetic stress tensor $F_\mu{}^\lambda$ is given by

$$F_\mu{}^\lambda = k_{\mu\alpha}k^{\alpha\lambda} - \tfrac{1}{4}k_{\alpha\beta}k^{\beta\alpha}\delta^\lambda_\mu . \qquad (1.2a)$$

Let us put

$$\left.\begin{aligned}
&(a)\quad {}^{(o)}k_\mu{}^\lambda \overset{\text{def}}{=} \delta^\lambda_\mu \\
&(b)\quad {}^{(p)}k_\mu{}^\lambda \overset{\text{def}}{=} {}^{(p-1)}k_\mu{}^\alpha k_\alpha{}^\lambda, \quad p = 1, 2, 3, \ldots \\
&(c)\quad 4K \overset{\text{def}}{=} -{}^{(2)}k_\alpha{}^\alpha = k_{\alpha\beta}k^{\beta\alpha} \\
&(d)\quad k \overset{\text{def}}{=} \mathrm{Det}\left| k_\alpha{}^\beta \right|
\end{aligned}\right\} \qquad (1.3)$$

Then eqn. (1.2*a*) takes the form

$$F_\mu{}^\lambda = {}^{(2)}k_\mu{}^\lambda + K\delta^\lambda_\mu . \qquad (1.2b)$$

Note. Greek indices range from I to IV and Latin indices range from 1 to 4. Usual summation convention applies to the repeated indices except for x, y, z.

We assume that the combined Einstein-Maxwell field equations are given by (Hlavatý 1963*b*)

$$R_{\mu}^{\ \lambda}-\tfrac{1}{2}R\delta_{\mu}^{\lambda} = \mu u_{\mu}u^{\lambda}-F_{\mu}^{\ \lambda}+p\delta_{\mu}^{\lambda}, \qquad (1.4a)$$

$$\partial_{[\omega}k_{\mu\lambda]} = 0, \qquad (1.4b)$$

$$\nabla_{\alpha}k^{\alpha\beta} = \epsilon u^{\beta}. \qquad (1.4c)$$

Here ∇ is the operator of covariant derivation with respect to $h_{\mu\lambda}$. Also

$$R \overset{\text{def}}{=} R_{\alpha}^{\ \alpha}. \qquad (1.5)$$

The time-like unit velocity vector u_{μ} is tangential to the orbit of the charged invariant mass μ.

$$u_{\alpha}u^{\alpha} = -1, \qquad (1.6)$$

p is the pressure* and ϵ is the charge-density.

The electromagnetic tensor field $k_{\mu\lambda}$ is said to be of the

(*a*) first class if $kK \neq 0$,
(*b*) second class if $k = 0$, $K \neq 0$,
(*c*) third class if $k = 0 = K$, ${}^{(2)}k_{\alpha}^{\ \beta} \neq 0$.

It has been shown by Hlavatý (1958) that for the signature $+ + + -$ of $h_{\mu\lambda}$ we have only these three classes.

It has also been shown by Hlavatý (1958) that in the case the tensor $k_{\mu\lambda}$ belongs to the first two classes, it has four linearly independent null eigenvectors $\overset{i}{a}_{\mu}$. These vectors are such that $\overset{1}{a}_{\mu}$, $\overset{2}{a}_{\mu}$ are complex conjugate and $\overset{3}{a}_{\mu}$, $\overset{4}{a}_{\lambda}$ are real. Since these vectors are linearly independent, their inverse set $\underset{i}{a}^{\mu}$ exists such that

$$\overset{i}{a}_{\mu}\underset{j}{a}^{\mu} = \delta_{j}^{i}, \qquad (1.7a)$$

$$\overset{i}{a}_{\mu}\underset{i}{a}^{\lambda} = \delta_{\mu}^{\lambda}. \qquad (1.7b)$$

Hlavatý (1958) called the set $\overset{i}{a}_{\mu}$, $\underset{i}{a}^{\mu}$ a non-holonomic frame for the first two classes of $k_{\mu\lambda}$. The non-holonomic components $T_{i\ldots}^{\ j\ldots}$ of an arbitrary tensor $T_{\mu\ldots}^{\ \lambda\ldots}$ are defined by

$$T_{i\ldots}^{\ j\ldots} \overset{\text{def}}{=} T_{\mu\ldots}^{\ \lambda\ldots}\overset{j}{a}_{\lambda}\underset{i}{a}^{\mu}\cdots \qquad (1.8)$$

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* p must not be confused with the hydrodynamical pressure. According to Einstein (1950) p is the energetic presentation of dynamical relationship inside the matter.

Hlavatý (1958) showed that the non-holonomic components h_{ij} of $h_{\mu\lambda}$ can be assumed in the form

$$((h_{ij})) = ((h^{ij})) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad \ldots \quad \ldots \quad (1.9)$$

Consequently, we have

$$h_{\mu\lambda} = 2\overset{1}{a}_{(\mu}\overset{2}{a}_{\lambda)} + 2\overset{3}{a}_{(\mu}\overset{4}{a}_{\lambda)}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.10a)$$

$$F_{\mu}^{\ \lambda} = D\sum_{x} \underset{x}{\omega}\overset{x}{a}_{\mu}\underset{x}{a}^{\lambda}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.10b)$$

where

$$-\underset{1}{\omega} = -\underset{2}{\omega} = \underset{3}{\omega} = \underset{4}{\omega} = 1, \qquad \ldots \quad \ldots \quad \ldots \quad (1.11a)$$

$$D = +\sqrt{K^2 - k} > 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.11b)$$

Rainich (1925) and later Misner and Wheeler (1957) solved the following problem confining themselves to the first two classes of $k_{\mu\lambda}$. Given $h_{\mu\lambda}$, to find a source free electromagnetic tensor field $k_{\mu\lambda}$, concomitant of $h_{\mu\lambda}$ which satisfies the following equations:

$$R_{\mu\lambda} - \tfrac{1}{2}Rh_{\mu\lambda} = -F_{\mu\lambda}, \qquad \ldots \quad \ldots \quad \ldots \quad (1.12a)$$

$$\partial_{[\omega}k_{\mu\lambda]} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.12b)$$

$$\nabla_{\alpha}k^{\alpha\beta} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.12c)$$

Later Hlavatý (1960*a*) devised a unified method which solves the Rainich problem for all the three classes, as well as the inverse problem. Hlavatý (1960*b*, 1963*b*) also solved the same problem when the field equations are given by (1.4), provided $k_{\mu\lambda}$ belongs to the first two classes. He gave necessary and sufficient conditions for the solution of the problem and the actual construction of all concomitants.*

In an earlier paper (Misra 1964) special role has been played by vectors $\overset{3}{a}_{\mu}$, $\overset{4}{a}_{\mu}$. The object of this paper is to study the equation (1.4*a*) with special reference to the complex conjugate null vectors $\overset{1}{a}_{\mu}$, $\overset{2}{a}_{\mu}$. We shall adopt the technique used earlier (Hlavatý 1963*b*; Mishra 1962; Misra 1964) and consider only the non-null case.

Before we proceed to the next section we mention a theorem proved by Hlavatý (1960*b*) in the non-null case and by Mishra (1962) in the null case.

Theorem: Necessary and sufficient condition that field equations be compatible with the continuity equation

$$\nabla_{\alpha}(\mu u^{\alpha}) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.13)$$

* For third class we refer to Mishra (1962).

is that

$$u^{\alpha}\partial_{\alpha}p = 0. \qquad (1.14)$$

2. Basic Preliminaries

If we substitute (1.10*b*) in (1.4*a*) we get

$$R_{\mu}{}^{\lambda}-\tfrac{1}{2}R\delta^{\lambda}_{\mu} = \mu u_{\mu}u^{\lambda}-D\sum_{x}\underset{x}{\omega}\overset{x}{a}_{\mu}\underset{x}{a}^{\lambda}+p\delta^{\lambda}_{\mu}. \qquad (2.1)$$

Before we come to the problem we shall prove some results in the form of Lemmas in this and in the following section. Some of the results have correspondence to the results obtained before (Hlavatý 1963*b*; Watanabe 1963).

Lemma (2.1). *Put*

$$T_{\mu}{}^{\lambda} = \mu u_{\mu}u^{\lambda}-2D\left(\overset{3}{a}_{\mu}\underset{3}{a}^{\lambda}+\overset{4}{a}_{\mu}\underset{4}{a}^{\lambda}\right), \qquad (2.2a)$$

$$\tilde{T}_{\mu}{}^{\lambda} = \mu u_{\mu}u^{\lambda}+2D\left(\overset{1}{a}_{\mu}\underset{1}{a}^{\lambda}+\overset{2}{a}_{\mu}\underset{2}{a}^{\lambda}\right). \qquad (2.2b)$$

Then we have

$$R_{\mu}{}^{\lambda}-\alpha\delta^{\lambda}_{\mu} = T_{\mu}{}^{\lambda}, \qquad (2.3a)$$

$$R_{\mu}{}^{\lambda}-\tilde{\alpha}\delta^{\lambda}_{\mu} = \tilde{T}_{\mu}{}^{\lambda}, \qquad (2.3b)$$

where

$$\alpha = \tfrac{1}{2}R+p+D, \qquad (2.4a)$$

$$\tilde{\alpha} = \tfrac{1}{2}R+p-D. \qquad (2.4b)$$

Proof: Proof follows easily from (1.10*b*), (1.11), (2.2) and (2.4).

Lemma (2.2). *We have*

$$\alpha+\tilde{\alpha} = R+2p, \qquad (2.5a)$$

$$\alpha-\tilde{\alpha} = 2D. \qquad (2.5b)$$

The proof follows from (2.4).

Lemma (2.3). *We have*

$$\mu = R+4p. \qquad (2.6)$$

Also

$$\mu = 2(\alpha+\tilde{\alpha})-R. \qquad (2.7)$$

Proof: The eqn. (2.6) has already been deduced by Hlavatý (1963*b*). The second eqn. (2.7) follows on eliminating p between (2.6) and (2.5*a*).

Denote by r and $\tilde{r}$ the ranks of the matrices $((T_{\mu}{}^{\lambda}))$ and $((\tilde{T}_{\mu}{}^{\lambda}))$ respectively. It is obvious from (2.2) that both r and $\tilde{r}$ are less than 4. Thus there exists a vector v^{λ} such that in the case (2.2*a*),

$$T_{\mu}{}^{\lambda}v^{\mu} = 0,$$

which in consequence of (2.3*a*) gives

$$(R_{\mu}{}^{\lambda}-\alpha\delta^{\lambda}_{\mu})v^{\mu} = 0 \qquad (2.8a)$$

or

$$R_{\mu}{}^{\lambda}v^{\mu} = \alpha v^{\lambda}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.8b)$$

showing that α is an eigenvalue of $R_{\mu}{}^{\lambda}$, the corresponding eigenvector being v^{λ}. In a similar manner we can show that $\tilde{\alpha}$ is also an eigenvalue of $R_{\mu}{}^{\lambda}$.

Denote by $\underset{x}{v}{}^{\lambda}$ the eigenvectors of $R_{\mu}{}^{\lambda}$ and the corresponding eigenvalues by $\underset{x}{\rho}$. Also denote by $\underset{x}{\tau}$ and $\underset{x}{\tilde{\tau}}$ the eigenvalues of $T_{\mu}{}^{\lambda}$ and $\tilde{T}_{\mu}{}^{\lambda}$ respectively. In the following lemma we shall derive relations among $\underset{x}{\rho}$, $\underset{x}{\tau}$ and $\underset{x}{\tilde{\tau}}$.

Lemma (2.4). *$\underset{x}{\rho}$, $\underset{x}{\tau}$ and $\underset{x}{\tilde{\tau}}$ are related by*

$$\underset{x}{\tilde{\tau}} = \underset{x}{\rho} - \tilde{\alpha} = \underset{x}{\tau} + 2D. \qquad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

The eigenvectors of $R_{\mu}{}^{\lambda}$, $T_{\mu}{}^{\lambda}$ and $\tilde{T}_{\mu}{}^{\lambda}$ are the same.

Proof: It has been proved by Hlavatý (1963*b*) that the eigenvectors of $R_{\mu}{}^{\lambda}$ and $T_{\mu}{}^{\lambda}$ are the same. Corresponding proof for $\tilde{T}_{\mu}{}^{\lambda}$ follows exactly on the same lines.

Since $\underset{x}{v}{}^{\lambda}$ are eigenvectors of $R_{\mu}{}^{\lambda}$ we have

$$R_{\mu}{}^{\lambda}\underset{x}{v}{}^{\mu} = \underset{x}{\rho}\,\underset{x}{v}{}^{\lambda}.$$

Also from (2.3*b*) we have

$$\tilde{T}_{\mu}{}^{\lambda}\underset{x}{v}{}^{\mu} = (R_{\mu}{}^{\lambda} - \tilde{\alpha}\delta_{\mu}^{\lambda})\underset{x}{v}{}^{\mu} = (\underset{x}{\rho} - \tilde{\alpha})\underset{x}{v}{}^{\lambda} = \underset{x}{\tilde{\tau}}\,\underset{x}{v}{}^{\lambda}.$$

Hence

$$\underset{x}{\tilde{\tau}} = \underset{x}{\rho} - \tilde{\alpha}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

The remaining part follows in a similar manner with the help of (2.5*b*).

Note (2.1). It is now obvious from the above that $R_{\mu}{}^{\lambda}$ has at least two real eigenvalues, viz. α and $\tilde{\alpha}$.

Lemma (2.5). *Put*

$$T \overset{\text{def}}{=} T_{\alpha}{}^{\alpha}, \quad \tilde{T} \overset{\text{def}}{=} \tilde{T}_{\alpha}{}^{\alpha}. \qquad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

Then $D = 0$ implies $T = \tilde{T}$ and $\alpha = \tilde{\alpha}$.

Proof: From (2.3) and (2.5*b*) we have

$$T_{\mu}{}^{\lambda} - \tilde{T}_{\mu}{}^{\lambda} = -2D\delta_{\mu}^{\lambda} = (\tilde{\alpha} - \alpha)\delta_{\mu}^{\lambda}.$$

Contracting this equation we get the results.

Hlavatý (1963*b*) has shown that the eigenvalue α is exactly of multiplicity $m = 4 - r$. In a similar manner we can show that $\tilde{\alpha}$ is exactly of multiplicity $\tilde{m} = 4 - \tilde{r}$.

Again, it has been shown (Hlavatý 1963*b*) that a necessary and sufficient condition for $D = 0$, $\mu \neq 0$ is $r = 1$. Similarly we can show that

Lemma (2.6). *For $\mu \neq 0$, $D = 0$ implies $\tilde{r} = 1$ and conversely.*

Lemma (2.7). *$r = 1$ implies $\tilde{r} = 1$ and conversely.*

Lemma (2.8). *$r = 1$ $(= \tilde{r})$ implies $\alpha = \tilde{\alpha}$. Otherwise $\alpha \neq \tilde{\alpha}$.*

The proofs are easy.

Our basic equations under consideration are

$$R_{\mu}{}^{\lambda} - \tilde{\alpha}\delta_{\mu}^{\lambda} = \mu u_{\mu} u^{\lambda} + 2D\left(\overset{1}{a}_{\mu}\underset{1}{a}^{\lambda} + \overset{2}{a}_{\mu}\underset{2}{a}^{\lambda}\right), \qquad (2.12a)$$

$$\tilde{\alpha} = \tfrac{1}{2}R + p - D, \qquad (2.12b)$$

$$u^{\alpha}\partial_{\alpha} p = 0. \qquad (2.12c)$$

Agreement (2.1). *We assume that* (2.12*a*) *admits at least one set of solutions*

$$\mu,\ u_{\mu},\ D,\ p,\ \overset{1}{a}_{\mu},\ \overset{2}{a}_{\mu} \qquad (2.13)$$

expressed as algebraic concomitants of the known tensor $R_{\mu}{}^{\lambda}$ *.

3. Ranks r and $\tilde{r}$

Our consideration of the problem is according to the following scheme:

$$\left.\begin{matrix} r = 1 \\ \tilde{r} = 1 \end{matrix}\right\},\quad \left.\begin{matrix} r = 2 \\ \tilde{r} = 2 \end{matrix}\right\},\quad \left.\begin{matrix} r = 2,\,3 \\ \tilde{r} = 3,\,2 \end{matrix}\right\},\quad \left.\begin{matrix} r = 3 \\ \tilde{r} = 3 \end{matrix}\right\}.$$

By Lemma (2.6) the first case is excluded as it amounts to the absence of the electromagnetic field in the non-null case. In the following Lemma we shall show that the requirement $r = 2 = \tilde{r}$ is also not compatible when $\mu \neq 0$.

Lemma (3.1). *When $\mu \neq 0$ the requirement $r = 2 = \tilde{r}$ is incompatible.*

Proof: From (2.2*a*) when $\mu \neq 0$ and $r = 2$, u_{μ} is a linear combination of $\overset{3}{a}_{\mu}, \overset{4}{a}_{\mu}$. Similarly from (2.2*b*) with $\mu \neq 0$, $\tilde{r} = 2$, u_{μ} is a linear combination of $\overset{1}{a}_{\mu}$ and $\overset{2}{a}_{\mu}$. But the bivectors $\overset{1}{a}_{[\mu}\overset{2}{a}_{\lambda]}$ and $\overset{3}{a}_{[\mu}\overset{4}{a}_{\lambda]}$ are totally perpendicular (Hlavatý 1958, p. 16) which in this case yields $u_{\mu}u^{\mu} = 0$. This contradicts the assumption that u_{α} is a time-like unit vector. Hence the Lemma.

Lemma (3.2). *The requirement $r = 2 = \tilde{r}$ yields $\mu = 0$, and conversely.*

The proof follows from the fact that $\overset{i}{a}_{\mu}$ are linearly independent.

Thus we are left with the following (Mishra and Upadhyaya 1963) cases:

$$\left.\begin{matrix} r = 3 \\ \tilde{r} = 3 \end{matrix}\right\},\quad \left.\begin{matrix} r = 2,\,3 \\ \tilde{r} = 3,\,2 \end{matrix}\right\}.$$

In this paper we shall consider only the case $r = 3 = \tilde{r}$. The remaining cases will be considered in a separate paper.

* That the equation (2.12*a*) or (1.4*a*) does not always have a solution has been shown in Hlavatý (1963*c*). Hence the necessity of the agreement.

4. $r = 3 = \tilde{r}$

When the requirement $r = 3 = \tilde{r}$ is imposed, the set of vectors u_μ, $\overset{1}{a}_\mu$ and $\overset{2}{a}_\mu$ are linearly independent. In that case we define a set of three mutually perpendicular unit vectors as follows:

$$\left.\begin{aligned} \overset{x}{v}_\lambda \overset{y}{v}{}^\lambda &= 0 \qquad xy = 12,\, 14,\, 24,\, 13,\, 34 \\ &= 1 \qquad xy = 11,\, 33 \\ &= -1 \quad\; xy = 44 \end{aligned}\right\} \qquad (4.1)$$

We can express the vectors $\overset{1}{a}_\mu$, $\overset{2}{a}_\mu$ and u_μ as a linear combination of these vectors

$$\overset{1}{a}_\mu = A\overset{1}{v}_\mu + B\overset{3}{v}_\mu + \epsilon\sqrt{A^2+B^2}\,\overset{4}{v}_\mu, \qquad (4.2a)$$

$$\overset{2}{a}_\mu = \bar{A}\overset{1}{v}_\mu + \bar{B}\overset{3}{v}_\mu + \epsilon'\sqrt{\bar{A}^2+\bar{B}^2}\,\overset{4}{v}_\mu, \qquad (4.2b)$$

$$u_\mu = x\overset{1}{v}_\mu + y\overset{3}{v}_\mu + z\overset{4}{v}_\mu, \quad \epsilon,\, \epsilon' = \pm 1. \qquad (4.2c)$$

Here the quantities A, B are complex and a bar over a quantity denotes its complex conjugate; x, y and z are real. These satisfy the following relations:

$$A\bar{A} + B\bar{B} - \epsilon\epsilon'\sqrt{A^2+B^2}\sqrt{\bar{A}^2+\bar{B}^2} = 1, \qquad (4.3a)$$

$$x^2 + y^2 - z^2 = -1. \qquad (4.3b)$$

Lemma (4.1). *The eigenvalues of* $R_\mu^{\ \lambda} - \tilde{\alpha}\delta_\mu^\lambda$ *are given by*

$$\lambda\Big[\lambda^3 + \lambda^2\big(\underset{1}{\tau} + \underset{3}{\tau} - \underset{4}{\tau}\big) + \lambda\big(\underset{1}{\tau}\underset{4}{\tau} + \underset{3}{\tau}\underset{4}{\tau} + \underset{1}{\tau}^2 - \underset{1}{\tau}\underset{3}{\tau} - \tau''^2 - \tau'''^2\big)$$

$$+ \big(\underset{1}{\tau}\tau'''^2 + \underset{4}{\tau}\tau'^2 + 2\tau'\tau'\tau'' + \underset{3}{\tau}\tau''^2 - \underset{1}{\tau}\underset{3}{\tau}\underset{4}{\tau}\big)\Big] = 0, \qquad (4.4)$$

where τ's *have been defined by* (4.6) *below.*

Proof: Substituting the values of $\overset{1}{a}_\mu$, $\overset{2}{a}_\mu$ and u from (4.2) in the right-hand side of (2.12a), we have

$$R_\mu^{\ \lambda} - \tilde{\alpha}\delta_\mu^\lambda = \underset{1}{\tau}\overset{1}{v}_\mu\overset{1}{v}{}^\lambda + \underset{3}{\tau}\overset{3}{v}_\mu\overset{3}{v}{}^\lambda + \underset{4}{\tau}\overset{4}{v}_\mu\overset{4}{v}{}^\lambda + \tau'\big(\overset{1}{v}_\mu\overset{3}{v}{}^\lambda + \overset{3}{v}_\mu\overset{1}{v}{}^\lambda\big)$$

$$+ \tau''\big(\overset{1}{v}_\mu\overset{4}{v}{}^\lambda + \overset{4}{v}_\mu\overset{1}{v}{}^\lambda\big) + \tau'''\big(\overset{3}{v}_\mu\overset{4}{v}{}^\lambda + \overset{4}{v}_\mu\overset{3}{v}{}^\lambda\big), \qquad (4.5a)$$

where

$$\left.\begin{aligned}\underset{1}{\tau} &= \mu x^2+4DA\bar{A}\\ \underset{3}{\tau} &= \mu y^2+4DB\bar{B}\\ \underset{4}{\tau} &= \mu z^2+4D\epsilon\epsilon'\sqrt{A^2+B^2}\sqrt{\bar{A}^2+\bar{B}^2}\\ \tau' &= \mu xy+2D(A\bar{B}+\bar{A}B)\\ \tau'' &= \mu xz+2D\left(A\epsilon'\sqrt{\bar{A}^2+\bar{B}^2}+\bar{A}\epsilon\sqrt{A^2+B^2}\right)\\ \tau''' &= \mu yz+2D\left(B\epsilon'\sqrt{\bar{A}^2+\bar{B}^2}+\bar{B}\epsilon\sqrt{A^2+B^2}\right)\end{aligned}\right\}. \quad .. \quad (4.6)$$

If a set of mutually orthogonal unit vectors $\overset{i}{v}_\mu$ with their inverse $\underset{i}{v}^\mu$ (with $\overset{4}{v}_\mu$ or $\underset{4}{v}^\lambda$ time-like) form a non-holonomic frame then

$$\overset{1}{v}_\mu = \underset{1}{v}_\mu, \quad \overset{2}{v}_\mu = \underset{2}{v}_\mu, \quad \overset{3}{v}_\mu = \underset{3}{v}_\mu, \quad \overset{4}{v}_\mu = -\underset{4}{v}_\mu .$$

Equation (4.5*a*) is equivalent to

$$R_\mu{}^\lambda - \tilde{\alpha}\delta^\lambda_\mu = \underset{1}{\tau}\overset{1}{v}_\mu\underset{1}{v}^\lambda+\underset{3}{\tau}\overset{3}{v}_\mu\underset{3}{v}^\lambda - \underset{4}{\tau}\overset{4}{v}_\mu\underset{4}{v}^\lambda+\tau'\left(\overset{1}{v}_\mu\underset{3}{v}^\lambda+\overset{3}{v}_\mu\underset{1}{v}^\lambda\right)+\tau''\left(\overset{4}{v}_\mu\underset{1}{v}^\lambda-\overset{1}{v}_\mu\underset{4}{v}^\lambda\right)$$
$$+\tau'''\left(\overset{4}{v}_\mu\underset{1}{v}^\lambda-\overset{1}{v}_\mu\underset{4}{v}^\lambda\right). \quad .. \quad .. \quad .. \quad .. \quad .. \quad (4.5b)$$

The eigenvalues of the right-hand side of (4.5*b*) are given by

$$\begin{vmatrix} \underset{1}{\tau}-\lambda & 0 & \tau' & \tau'' \\ 0 & -\lambda & 0 & 0 \\ \tau' & 0 & \underset{3}{\tau}-\lambda & \tau''' \\ -\tau'' & 0 & -\tau''' & -\left(\lambda+\underset{4}{\tau}\right) \end{vmatrix} = 0$$

which on simplification yields (4.4).

Note (4.1). It is obvious from (4.4) that we have only one root $\lambda = 0$ unless the expression in the last parenthesis vanishes. That the last parenthesis does not vanish may be seen from eqn. (5.5).

Note (4.2). We have already seen that α and $\tilde{\alpha}$ are two real eigenvalues of $R_\mu{}^\lambda$. If the remaining two eigenvalues be β and γ then the eigenvalues of $R_\mu{}^\lambda - \tilde{\alpha}\delta^\lambda_\mu$ are $\alpha-\tilde{\alpha}$, 0, $\beta-\tilde{\alpha}$, $\gamma-\tilde{\alpha}$.

In the case $r = 3 = \tilde{r}$ the eigenvalues α and $\tilde{\alpha}$ are the single real eigenvalues. Consequently the following essentially different characteristics are possible

$$[11\ 11]\ [11\ (11)]\ [11\ 2]. \quad .. \quad .. \quad .. \quad (4.7)$$

We have written the above characteristics with the understanding that the eigenvalues correspond to the symbols in the square bracket in the order α, $\tilde{\alpha}$, β, γ respectively. Consequently we have the following matrices for non-holonomic components $R_i^{\ j}$ of $R_\mu^{\ \lambda}$ corresponding to the above characteristics.

$$R_i^j = \begin{pmatrix} \alpha & . & . & . \\ . & \tilde{\alpha} & . & . \\ . & . & \beta & . \\ . & . & . & \gamma \end{pmatrix} \quad \alpha, \tilde{\alpha}, \beta, \gamma \neq, \qquad \ldots \qquad \ldots (4.8a)$$

$$R_i^j = \begin{pmatrix} \alpha & . & . & . \\ . & \tilde{\alpha} & . & . \\ . & . & \beta & . \\ . & . & . & \beta \end{pmatrix} \quad \alpha, \tilde{\alpha}, \beta \neq, \qquad \ldots \qquad \ldots (4.8b)$$

$$R_i^j = \begin{pmatrix} \alpha & . & . & . \\ . & \tilde{\alpha} & . & . \\ . & . & \beta & 1 \\ . & . & . & \beta \end{pmatrix} \quad \alpha, \tilde{\alpha}, \beta \neq. \qquad \ldots \qquad \ldots (4.8c)$$

We shall consider these characteristics one by one.

5. The Characteristic [11 11]

As already mentioned this characteristic corresponds to the matrix (4.8a).

Theorem (5.1)*: Necessary and sufficient conditions for the field equations to admit a solution in the case of the characteristic* [11 11] *are*

$$\underset{1}{\tau}+\underset{4}{\tau} > \underset{3}{\tau}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (5.1a)$$

$$\beta+\gamma < 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (5.1b)$$

$$\underset{1}{\tau} > 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (5.1c)$$

$$\left(y\overset{3}{v}{}^{\mu}+z\overset{4}{v}{}^{\mu}\right)\partial_\mu(\beta+\gamma) = 0. \qquad \ldots \quad \ldots \quad \ldots (5.1d)$$

When these conditions are satisfied we have

$$\mu = \underset{1}{\tau}+\underset{4}{\tau}-\underset{3}{\tau}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (5.2a)$$

$$2p = -(\beta+\gamma), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (5.2b)$$

$$2D = \underset{1}{\tau}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (5.2c)$$

The vectors $\overset{1}{a}_\mu$, $\overset{2}{a}_\mu$, u_μ *are given by equation* (4.2) *with* A, x, *etc., given by equations* (5.9) *below.* .. .. (5.3)

Proof: Since v_μ are the unit orthogonal eigenvectors of $R_\mu^{\ \lambda}$ and α, $\tilde{\alpha}$, β, γ the eigenvalues, eqn. (4.5a) may be written as

$$R_\mu^{\ \lambda}-\tilde{\alpha}\delta_\mu^\lambda = (\alpha-\tilde{\alpha})\overset{1}{v}_\mu\overset{1}{v}{}^\lambda+(\beta-\tilde{\alpha})\overset{3}{v}_\mu\overset{3}{v}{}^\lambda+(\tilde{\alpha}-\gamma)\overset{4}{v}_\mu\overset{4}{v}{}^\lambda. \qquad \ldots \quad \ldots (5.4)$$

Comparing this eqn. with (4.5*a*) we have

$$\alpha-\tilde{\alpha} = \tau_1, \qquad (5.5a)$$

$$\beta-\tilde{\alpha} = \tau_3, \qquad (5.5b)$$

$$\tilde{\alpha}-\gamma = \tau_4, \qquad (5.5c)$$

$$\tau' = 0 = \tau'' = \tau'''. \qquad (5.5d)$$

When these equations are solved with (4.6) we get

$$4DA\bar{A} = \frac{\tau_1\left(\tau_4-\tau_3+4D\right)}{\tau_1-\tau_3+\tau_4+2D}, \qquad (5.6a)$$

$$4DB\bar{B} = \frac{\tau_3\left(\tau_4-\tau_1+4D\right)}{\tau_3-\tau_1+\tau_4+2D}, \qquad (5.6b)$$

$$4D\epsilon\epsilon'\sqrt{A^2+B^2}\sqrt{\bar{A}^2+\bar{B}^2} = \frac{\tau_4\left(\tau_1+\tau_3-4D\right)}{\tau_1+\tau_3+\tau_4-2D}. \qquad (5.6c)$$

These combined with (4.3*a*) give an equation exclusively in *D* on solving which we have

$$2D = \tau_1. \qquad (5.7)$$

From (4.6*a*, *b*, *c*) and (4.3*b*) we get

$$4D-\mu = \tau_1-\tau_4+\tau_3 \qquad (5.8)$$

which in consequence of (5.7) gives (5.2*a*). The relation (5.2*b*) is easily obtained from (5.4*a*, *b*, *c*), (2.5*a*) and (2.7).

To obtain (5.3*a*, *b*, *c*) we solve (5.6) together with (5.7) and (4.6). We get

$$x = 0, \qquad (5.9a)$$

$$y^2 = \frac{\tau_3\left(\tau_3-\tau_1\right)}{\left(\tau_1+\tau_4-\tau_3\right)\left(\tau_3+\tau_4\right)}, \qquad (5.9b)$$

$$z^2 = \frac{\tau_4\left(\tau_1+\tau_4\right)}{\left(\tau_1+\tau_4-\tau_3\right)\left(\tau_3+\tau_4\right)}, \qquad (5.9c)$$

$$\bar{A} = \frac{1}{2A}, \qquad (5.9d)$$

$$B^2 = \frac{-A^2\tau_3\left(\tau_1+\tau_4\right)}{\tau_1\left(\tau_3+\tau_4\right)}, \qquad (5.9e)$$

$$\bar{B}^2 = -\frac{\tau_3\left(\tau_1+\tau_4\right)}{4A^2\tau_1\left(\tau_3+\tau_4\right)}. \qquad (5.9f)$$

On substituting these values in (4.2) we get $\overset{1}{a}_\mu$, $\overset{2}{a}_\mu$, u_μ explicitly.

From physical requirements

$$\mu > 0, \quad p > 0, \quad D > 0$$

we have

$$\underset{1}{\tau} - \underset{3}{\tau} + \underset{4}{\tau} > 0, \qquad \ldots \quad (5.10a)$$

$$\beta + \gamma < 0, \qquad \ldots \quad (5.10b)$$

$$\underset{1}{\tau} > 0. \qquad \ldots \quad (5.10c)$$

These give the conditions (5.1*a*, *b*, *c*). The condition (5.1*d*) is obtained from (5.9), (5.2) and (2.12*c*).

Theorem (5.2): *The eigenvalues of* $R_\mu{}^\lambda$ *for the characteristic* [11 11] *are always real.*

Proof: Assume that the eigenvalues of $R_\mu{}^\lambda$ in the present case are complex. Since α and $\tilde{\alpha}$ are real (Note (2.1)) the only possibility left is that β and γ be complex conjugate. Then the right-hand side of eqn. (5.5*b*) is real while the left-hand side is complex. This requires that the imaginary part of the left-hand side must vanish. Similar argument holds for γ in eqn. (5.5*c*).

Note (5.2). The above theorem renders the consideration of complex eigenvalues in the papers of Watanabe (1963) and Hlavatý (1963*a*) unnecessary.

Note (5.3). We could have obtained (5.7) directly from (5.5*c*) and (2.5*b*).

Theorem (5.3): *The characteristic* [11 (11)] *is not admissible when* $\mu \neq 0$.

Proof: A proof similar to the one given by Misra (1964, p. 1580) can be given to prove the theorem.

We shall consider the remaining characteristic in the next section.

6. The Characteristic [11 2]

In this section we shall denote by β the double eigenvalues of $R_\mu{}^\lambda$. Of course, β of this section is not the same as that of the previous section.

The matrix (4.8*c*) describing the characteristic [11 2] has only three eigenvectors. Also the requirement

$$R_{ij} = R_i{}^k h_{kj} = R_j{}^k h_{ki} = R_{ji} \qquad \ldots \quad (6.1)$$

leads to the matrix

$$((h_{ij})) = ((h^{ij})) = \begin{pmatrix} h_{11} & 0 & 0 & 0 \\ 0 & h_{22} & 0 & 0 \\ 0 & 0 & 0 & h_{34} \\ 0 & 0 & h_{43} & 0 \end{pmatrix}^{*} \qquad \ldots \quad (6.2)$$

* We have taken $h_{44} = 0$ though it is not necessary (Hlavatý 1963*b*; Misra 1964).

so that we can define two unit vectors $\overset{1}{v}_\mu$, $\overset{2}{v}_\mu$ orthogonal to each other and two null vectors $\overset{3}{v}_\mu$, $\overset{4}{v}_\mu$ such that

$$\begin{aligned} \overset{x}{v}_\mu \overset{y}{v}^\mu &= 1, \quad xy = 11, 22, 34, 43, \\ &= 0, \quad \text{in all remaining values of } xy. \end{aligned} \qquad (6.3)$$

We see that $\overset{1}{v}_\mu$, $\overset{2}{v}_\mu$ are both space-like vectors.

In order to determine the eigenvalues and other quantities we express $\overset{1}{a}_\mu$, $\overset{2}{a}_\mu$ and u_μ as linear combinations of these linearly independent vectors.

$$u_\mu = x\overset{1}{v}_\mu + y\overset{3}{v}_\mu + z\overset{4}{v}_\mu, \qquad (6.4a)$$

$$\overset{1}{a}_\mu = A\overset{1}{v}_\mu + B\overset{3}{v}_\mu + C\overset{4}{v}_\mu, \qquad (6.4b)$$

$$\overset{2}{a}_\mu = \bar{A}\overset{1}{v}_\mu + \bar{B}\overset{3}{v}_\mu + \bar{C}\overset{4}{v}_\mu. \qquad (6.4c)$$

Here again a quantity with a bar denotes the complex conjugate of the unbarred one. Of course, the x, A ... in (6.4) are different from the quantities denoted by the same letters in eqn. (4.2).

Remark. It is obvious from (6.2), (6.3) that

$$\overset{1}{v}_\mu = \underset{1}{v}_\mu, \quad \overset{2}{v}_\mu = \underset{2}{v}_\mu, \quad \overset{3}{v}_\mu = \underset{4}{v}_\mu, \quad \overset{4}{v}_\mu = \underset{3}{v}_\mu.$$

Lemma (6.1). *The quantities x, A ..., etc., satisfy the following relations:*

$$1 + x^2 + 2yz = 0, \qquad (6.5a)$$

$$A^2 + 2BC = 0, \qquad (6.5b)$$

$$\bar{A}^2 + 2\bar{B}\bar{C} = 0, \qquad (6.5c)$$

$$A\bar{A} + B\bar{C} + \bar{B}C = 1. \qquad (6.5d)$$

Proof: The proofs based on (1.9), (6.3) and (6.4) are easy.

When $\overset{1}{a}_\mu$, $\overset{2}{a}_\mu$ and u_μ are substituted from (6.4) into (2.12a) we get

$$\begin{aligned} R_\mu{}^\lambda - \tilde{\alpha}\delta_\mu^\lambda = \underset{1}{\tau}\overset{1}{v}_\mu \overset{1}{v}^\lambda + \underset{3}{\tau}\overset{3}{v}_\mu \overset{3}{v}^\lambda + \underset{4}{\tau}\overset{4}{v}_\mu \overset{4}{v}^\lambda + \tau'\left(\overset{1}{v}_\mu \overset{4}{v}^\lambda + \overset{4}{v}_\mu \overset{1}{v}^\lambda\right) \\ + \tau''\left(\overset{1}{v}_\mu \overset{3}{v}^\lambda + \overset{3}{v}_\mu \overset{1}{v}^\lambda\right) + \tau\left(\overset{3}{v}_\mu \overset{4}{v}^\lambda + \overset{4}{v}_\mu \overset{3}{v}^\lambda\right), \end{aligned} \qquad (6.6a)$$

where τ's are now given by

$$\underset{1}{\tau} = \mu x^2 + 4DA\bar{A}, \qquad (6.7a)$$

$$\underset{3}{\tau} = \mu y^2 + 4DB\bar{B}, \qquad (6.7b)$$

$$\underset{4}{\tau} = \mu z^2 + 4DC\bar{C}, \qquad (6.7c)$$

$$\tau = \mu yz + 2D(B\bar{C} + \bar{B}C), \qquad (6.7d)$$

$$\tau' = \mu xz + 2D(A\bar{C} + \bar{A}C), \qquad (6.7e)$$

$$\tau'' = \mu xy + 2D(A\bar{B} + \bar{A}B). \qquad (6.7f)$$

Theorem (6.1)*: In the case of the characteristic* [11 2], *i.e. when the non-holonomic components $R_i^{\ j}$ are given by* (4.8c), *necessary and sufficient conditions for the field equations to admit a solution are*

$$\underset{1}{\tau} > 0, \qquad (6.8a)$$

$$\beta < 0, \qquad (6.8b)$$

$$\underset{1}{\tau} > 2\tau, \qquad (6.8c)$$

$$\left(y\overset{3}{v}{}^{\mu}+z\overset{4}{v}{}^{\mu}\right)\partial_{\mu}\beta = 0. \qquad (6.8d)$$

When these requirements are satisfied, we have

$$\mu = \underset{1}{\tau}-2\tau, \qquad (6.9a)$$

$$2D = \underset{1}{\tau}, \qquad (6.9b)$$

$$p = -\beta. \qquad (6.9c)$$

$\overset{1}{a}_{\mu}$, $\overset{2}{a}_{\mu}$, u_{μ} *are given by* (6.4) *with* A, x, . . . , *etc., given by* (6.13), (6.14) *below.* (6.9d)

Proof: Since $\overset{i}{v}_{\mu}$ are the eigenvectors of $R_{\mu}^{\ \lambda}$ we can also write (6.6a) as

$$R_{\mu}^{\ \lambda}-\tilde{\alpha}\delta_{\mu}^{\lambda} = (\beta-\tilde{\alpha})\left(\overset{3}{v}_{\mu}\overset{4}{v}{}^{\lambda}+\overset{4}{v}_{\mu}\overset{3}{v}{}^{\lambda}\right)+\overset{3}{v}_{\mu}\overset{3}{v}{}^{\lambda}+(\alpha-\tilde{\alpha})\overset{1}{v}_{\mu}\overset{1}{v}{}^{\lambda}. \qquad (6.6b)$$

Comparing (6.6*a*, *b*) we get

$$\alpha-\tilde{\alpha} = \underset{1}{\tau}, \qquad (6.10a)$$

$$1 = \underset{3}{\tau}, \qquad (6.10b)$$

$$\beta-\tilde{\alpha} = \tau, \qquad (6.10c)$$

$$\tau' = 0 = \tau'' = \underset{4}{\tau}. \qquad (6.10d)$$

From (6.10) and (6.5*a*) we get

$$\mu+\underset{1}{\tau}+2\tau = 4D. \qquad (6.11a)$$

Again (6.10), (6.5*b*, *c*, *d*) give

$$2D = \underset{1}{\tau}. \qquad (6.12)$$

Hence from (6.11*a*) and (6.12) we get

$$\mu = \underset{1}{\tau}-2\tau. \qquad (6.11b)$$

With the help of (6.5), (6.7) and (6.10) we have

$$x = 0, \qquad \ldots \quad (6.13a)$$

$$y^2 = \frac{\tau - 2\underset{1}{\tau}}{4\tau\left(\tau - \underset{1}{\tau}\right)}, \qquad \ldots \quad (6.13b)$$

$$z^2 = \frac{\tau\left(\tau - \underset{1}{\tau}\right)}{\tau - 2\underset{1}{\tau}}, \qquad \ldots \quad (6.13c)$$

$$\bar{A} = \frac{1}{2A}, \qquad \ldots \quad (6.14a)$$

$$B^2 = A^2 \frac{\underset{1}{\tau}}{4\tau\left(\tau - \underset{1}{\tau}\right)}, \qquad \ldots \quad (6.14b)$$

$$\bar{B}^2 = \frac{\underset{1}{\tau}}{16A^2\tau\left(\tau - \underset{1}{\tau}\right)}, \qquad \ldots \quad (6.14c)$$

$$C^2 = \frac{A^2\tau\left(\tau - \underset{1}{\tau}\right)}{\underset{1}{\tau}}, \qquad \ldots \quad (6.14d)$$

$$\bar{C}^2 = \frac{\tau\left(\tau - \underset{1}{\tau}\right)}{4A^2\underset{1}{\tau}}. \qquad \ldots \quad (6.14e)$$

When these values are substituted in (6.4) we get (6.4) explicitly. The eqn. (6.9*c*) is obtained with the help of (2.6), (2.7) and (6.10*a*, *b*)

Again, the physical requirements

$$\mu > 0, \quad p > 0, \quad D > 0$$

lead to (6.8*a*, *b*). The eqn. (6.9*c*) is obtained from (6.13*a*, *b*, *c*) and (2.12*c*).

Note (6.1). We could have obtained (6.12) directly from (6.10*c*) and (2.5*b*).

Note (6.2). The values of x, A . . . , etc., in (6.13), (6.14) should be taken with the sign in such a way as to satisfy (6.5).

Thus we have seen that in the case $r = 3 = \tilde{r}$ the quantities $\overset{1}{a}_\mu$, $\overset{2}{a}_\mu$, u_μ, p, D and μ have been expressed as algebraic concomitants of the known tensor $R_\mu{}^\lambda$.

We shall consider the remaining cases in another paper.

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