CONICALLY DIFFERENTIABLE FUNCTIONS

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Conical differentiability of an arc A at a point p in the projective plane has been discussed synthetically earlier (Lane and Singh 1964, 1965, 1966). The present paper deals with the conditions on a function $f(x)$ such that the arc A given by $y = f(x)$, $f(0) = 0$, $0 \leqslant x \leqslant 1$ is conically and strongly conically differentiable at the origin. It turns out that conical differentiability is closely connected with de la Vallée Poussin differentiability.

§ 1. The authors had discussed earlier conical differentiation from a synthetic point of view (Lane and Singh 1964, 1965, 1966). Here we examine conditions on a function $f(x)$ such that the arc A, given by $y = f(x)$, $f(0) = 0$, $0 \leqslant x \leqslant 1$, is conically differentiable at the origin. The definitions and notations used here are the same as those used in the papers referred to above. The prerequisites are summarized in our last-mentioned paper (Lane and Singh 1966, 1.2 and 1.3).

The topology of the space of conics in the projective plane coincides with the topology defined by regarding the conics $ax^2+bxy+cy^2+dx+ey+f = 0$ as points of a projective 5-space (a, b, c, d, e, f) as long as the double lines, double segments and point-conics are excluded. The conditions on $f(x)$ when these degenerate cases are involved are complicated.

§ 2. *Condition I.*—The arc A satisfies Condition I (i.e. the ordinary tangent exists (Lane and Singh 1964, 4.2)) at the origin p if and only if $\lim_{t\to 0} f(t)/t$ exists. Here, and in the following, we include $\pm\ \infty$ as possible values of the limit. If A satisfies Condition I it is well known that one can choose a coordinate system such that either

$$(\mathrm{I}_0)\ \lim_{t\to 0} f(t)/t = 0, \text{ or } (\mathrm{I}_\infty)\lim_{t\to 0} f(t)/t = +\ \infty.$$

We shall discuss case (I_0) only. Case (I_∞) can be dealt with by the same methods and it yields similar results, but with the roles of t and $f(t)$ reversed. Condition I′ in § 7 automatically excludes (I_∞).

§ 3. *Condition II.*—Assume (I_0). If the points p, P_1, P_2 are not collinear, and P_1, P_2 do not lie on the tangent $\mathfrak{T}$ of A at p, then the equation of the tangent conic through P_1, P_2 and the point $(t, f(t))$ of A near p is

$$\begin{vmatrix} x^2 & xy & y^2 & y \\ x_1^2 & x_1y_1 & y_1^2 & y_1 \\ x_2^2 & x_2y_2 & y_2^2 & y_2 \\ t^2 & tf(t) & f^2(t) & f(t) \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The arc A satisfies Condition II (i.e. the osculating conics exist (Lane and Singh 1964, 5.1 and 5.4)) if and only if eqn. (1) tends to a unique limit equation as $t \to 0$. Such a limit exists if and only if $\lim_{t\to 0} f(t)/t^2$ exists and is finite, or $\lim_{t\to 0} f(t)/t^2 = +\infty$. Now $\lim_{t\to 0} f(t)/t^2$ is finite, say k, if and only if (1) converges to

$$F(x, y, x_1, y_1, x_2, y_2) = \begin{vmatrix} xy & y^2 & y-kx^2 \\ x_1y_1 & y_1^2 & y_1-kx_1^2 \\ x_2y_2 & y_2^2 & y_2-kx_2^2 \end{vmatrix} = 0, \quad \ldots \quad \ldots \quad (2)$$

which represents a non-degenerate conic (thus p is of Type 1) if $k \neq 0$ (II_1), and the pair of lines $\mathfrak{T} \cup \mathcal{L}(P_1P_2)$ (p is of Type 3) if $k = 0$ (II_3). Also, $\lim_{t\to 0} |f(t)/t^2| = \infty$ if and only if (1) tends to

$$\begin{vmatrix} x^2 & xy & y^2 \\ x_1^2 & x_1y_1 & y_1^2 \\ x_2^2 & x_2y_2 & y_2^2 \end{vmatrix} = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

which represents the pair of lines $\mathcal{L}(p, P_1) \cup \mathcal{L}(p, P_2)$ (thus p is of Type 2 (Lane and Singh 1964, 5.6)). If p is of Type 2, we can assume $f(t) > 0$, and hence $\lim_{t\to 0} f(t)/t^2 = +\infty$. If A satisfies Condition II at p and there is a sequence of points $t \in A \cap \mathfrak{T}$ converging to p, then p is of Type 3. Hence we may also assume that $f(t) > 0$, and hence that $k > 0$, if p is of Type 1.

Altogether, if (I_0) holds and $f(t) \geqslant 0$, when p is of Type 1 or 2,

$$\left.\begin{matrix} (\mathrm{II}_1) \\ (\mathrm{II}_2) \\ (\mathrm{II}_3) \end{matrix}\right\} \textit{Condition II holds with } p \textit{ of Type} \begin{cases} 1 \\ 2 \\ 3 \end{cases} \longleftrightarrow \lim_{t\to 0} f(t)/t^2 = \begin{cases} k > 0 \\ +\infty \\ 0. \end{cases}$$

§ 4. *Condition III.*—Suppose that (I_0) and (II_1) hold. From (2), the osculating conic through $P \notin \mathfrak{T}$ and a point of A near p has the equation

$$F(x, y, x_1, y_1, t, f(t)) = 0. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The arc A satisfies Condition III at p (i.e. the superosculating conics at p exist) if and only if (4) converges as t tends to 0, and hence if and only if either

$\lim_{t\to 0} (f(t)-kt^2)/t^3$ exists and is finite, say, $h(\mathrm{III}_a)$, or $\lim_{t\to 0} |(f(t)-kt^2)/t^3| = \infty$ (III_b).

Now, (4) tends to the equation to a unique non-degenerate conic (thus, p is of Type $1a$ (Lane and Singh 1964, 6.4)) if and only if (III_a) holds, in which case (4) tends to

$$(y_1 - kx_1^2 - lx_1y_1)y^2 - y_1^2(y - kx^2 - lxy) = 0, \text{ where } h = kl. \quad .. \quad (5)$$

Next, (4) tends to the pair of lines $\mathfrak{T} \cup \mathcal{L}(p, P_1)$(thus p is of Type $1b$ (Lane and Singh 1964, 6.4)) if and only if (III_b) holds. We observe that (III_b) implies $f(t) \neq kt^2$ if t is small, $t \neq 0$. If (III_b) holds, then (4) converges to

$$x_1y_1y^2 - y_1^2xy = 0. \quad .. \quad .. \quad .. \quad .. \quad (6)$$

Since p is of Type 1, every osculating conic has an equation of the form (2), i.e. of the form $axy+by^2+y-kx^2 = 0$, $k > 0$. In particular, if p is of Type $1b$ then $y-kx^2 = 0$ represents a non-degenerate osculating conic of A. A point (x_0, y_0) lies inside [outside] this conic according as $y_0 - kx_0^2$ is positive [negative]. Hence the point $(t, f(t))$ lies inside [outside] *every* non-degenerate osculating conic (thus p is of Type $1b_*$ [Type $1b^*$]) if and only if $f(t)-kt^2$ is positive [negative] (Lane and Singh 1964, 6.5 Corollary). Altogether, if (I_0) and (II_1) hold, we may assume

$$\left.\begin{array}{l}(\mathrm{III}_a)\\(\mathrm{III}_{b*})\\(\mathrm{III}_b{}^*)\end{array}\right\} \begin{array}{c}\textit{Condition III holds}\\ \textit{with } p \textit{ of Type}\end{array} \left\{\begin{array}{l}1a\\1b_* \leftarrow\rightarrow \lim_{t\to 0}(f(t)-kt^2)/t^3 = \\1b^*\end{array}\right. \left\{\begin{array}{l}h\\+\infty\\-\infty.\end{array}\right.$$

§ 5. *Condition IV.*—Let p be of Type $1a$; thus we assume (I_0), (II_1) and (III_a). Then, by (5), the superosculating conic through p and a point of A near p has the equation

$$(f(t)-kt^2-ltf(t))y^2 - f^2(t)(y-kx^2-lxy) = 0. \quad .. \quad .. \quad (7)$$

If A satisfies Condition IV at p (i.e. the superosculating conic at p through the point t converges as $t \to p$ (Lane and Singh 1964, 7.1)), then (7) will converge as $t \to 0$. (However, the convergence of (7) does not by itself imply Condition IV.) Now, (7) will converge if and only if either $\lim_{t\to 0}(f(t)-kt^2-ht^3)/t^4$ exists (IV) and is finite, say n, $(\mathrm{IV}_{(i)})$, or $\lim_{t\to 0} |f(t)-kt^2-ht^3|/t^4 = \infty$ $(\mathrm{IV}_{(ii,\, iii)})$.

The arc A satisfies Condition IV with p of Type $1a(i)$ (thus the ultra-osculating conic is non-degenerate (Lane and Singh 1964, 7.3)) if and only if $(\mathrm{IV}_{(i)})$ holds. In this case, the equation of the ultra-osculating conic is

$$my^2+lxy+kx^2-y = 0,$$

where

$$m = (nk-h^2)/k^3. \quad .. \quad .. \quad .. \quad .. \quad (8)$$

If p is of Type $1a(ii)$ or $1a(iii)$, then $\mathrm{IV}_{(ii,\, iii)}$ holds. Conversely, if $\mathrm{IV}_{(ii,\, iii)}$ holds and (7) represents a small ellipse [a hyperbola] whenever t is small, then A satisfies Condition IV at p and p is of Type $1a(ii)$ [Type $1a(iii)$] (thus the ultra-osculating conic is the point p [the double line on $\mathfrak{T}$] (Lane and Singh 1964, 7.1). For any fixed $x > 0$, it can readily be verified that for sufficiently small values of t the discriminant of (7) considered as a quadratic function of y is negative [positive], and hence the corresponding values of y are imaginary [real], if the limit IV is $+\infty$ ($\mathrm{IV}_{(ii)}$)[$-\infty$ ($\mathrm{IV}_{(iii)}$)].

Altogether, if (I_0), (II_1) and (III_a) hold, then

$$\left.\begin{matrix}(\mathrm{IV}_{(i)})\\ (\mathrm{IV}_{(ii)})\\ (\mathrm{IV}_{(iii)})\end{matrix}\right\} \begin{matrix}\textit{Condition IV}\\ \textit{with } p \textit{ of Type}\end{matrix} \left\{\begin{matrix}1a(i)\\ 1a(ii)\\ 1a(iii)\end{matrix}\right. \longleftrightarrow \lim_{t\to 0}\,(f(t)-kt^2-ht^3)/t^4 = \left\{\begin{matrix}n\\ +\infty\\ -\infty.\end{matrix}\right.$$

Remark.—p is of Type $1a(i)_*$ or $1a(i)^*$ according as $(f(t)-kt^2-ltf(t))/f^2(t)$ tends to m from above or from below (Lane and Singh 1966, 2.3).

§ 6. *Divided differences.*—Each of the following functions is symmetric with respect to its variables. Put

$[s] \equiv f(s)$, $[ts] \equiv ([t]-[s])/(t-s)$, $[uts] \equiv ([ut]-[ts])/(u-s)$, and so on,

$\phi_1(s) \equiv s[s]$, $\phi_2 \equiv (\phi_1(t)-\phi_1(s))/(t-s) \equiv t[ts]+[s]$,

$\phi_3 \equiv (\phi_2(u,t)-\phi_2(t,s))/(u-s) \equiv u[uts]+[ts]$,

$\phi_4 \equiv (\phi_2(v,u,t)-\phi_3(u,t,s))/(v-s) \equiv v[vuts]+[uts]$,

$\phi_5 \equiv (\phi_4(w,v,u,t)-\phi_4(v,u,t,s))/(w-s) \equiv w[wvuts]+[vuts]$,

$\psi_1(s) \equiv [s]^2$, $\psi_2 \equiv (\psi_1(t)-\psi_1(s))/(t-s) \equiv ([t]+[s])[ts]$,

$\psi_3 \equiv (\psi_2(u,t)-\psi_2(t,s))/(u-s) \equiv ([u]+[t])[uts]+[us][ts]$,

$\psi_4 \equiv (\psi_3(v,u,t)-\psi_3(u,t,s))/(v-s) \equiv ([u]+[t])[vuts]+[vt][vus]+[us][vts]$,

$\psi_5 \equiv (\psi_4(w,v,u,t)-\psi_4(v,u,t,s))/(w-s) \equiv ([v]+[u])[wvuts]+[vt][wuts]+[wu][wvts]$
$\quad +[vts][wus]$,

$\alpha_4 \equiv ([vut][us]-[uts][uv])/(v-s) \equiv [vu][vuts]-[vut][vus]$,

$\alpha_5 \equiv ([vts][wvut]-[wvt][vuts])/(w-s) \equiv [vts][wvuts]-[wvts][vuts]$,

$\beta_4 \equiv ([vut][us][ts]-[uts][vt][vu])/(v-s) \equiv [us][ts][vuts]-[uts]([vus][vt]+[vts][su])$,

$\beta_5 \equiv ([wvut]([ts][vus]+[vu][uts])-[vuts]([wt][wvu]-[vu][wut]))/(w-s)$

$\quad \equiv [wvuts]([ts][vus]+[vu][uts])-[vuts]([wuts][vu]+[wvus][ts]+[wvu][wts])$.

§ 7. *Condition I′.*—It is well known that A satisfies Condition I′ at p if and only if $\lim\limits_{t,s\to 0}[ts]$ exists (allowing $\pm\infty$) (Lane and Singh 1965, 3.1). We may assume, without loss of generality, that if Condition I′ holds, then

$$(\mathrm{I}') \qquad \lim_{t,s\to 0}[ts] = 0.$$

§ 8. *Condition II′.*—Let A satisfy Condition I′ at p; thus we may assume (I′). If p, P , P are not collinear and P_1, $P_2 \notin \mathfrak{T}$, then the equation of the conic through P_1, P_2 and three mutually distinct points of A near p can be put in the form (cf. 6)

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ s^2 & \phi_1 & \psi_1 & s & [s] & 1 \\ t+s & \phi_2 & \psi_2 & 1 & [ts] & 0 \\ 1 & \phi_3 & \psi_3 & 0 & [uts] & 0 \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad (9)$$

The arc A will satisfy Condition II′ at p if and only if (9) converges as u, t, s tend to 0. It follows readily that A satisfies Condition II′ (in particular, Condition II) at p with p of Type 1 [2, 3] if and only if (9) converges to (2) with $k \neq 0$ [to (3) to (2) with $k = 0$]. Finally, assuming (I′) and $f(t) > 0$ for $t > 0$ when p is of Type 1 or 2, one can prove

$$\left.\begin{matrix}(\mathrm{II}_1') \\ (\mathrm{II}_2') \\ (\mathrm{II}_3')\end{matrix}\right\} \textit{Condition II}' \textit{ with } p \textit{ of Type} \begin{cases} 1 \\ 2 \\ 3 \end{cases} \longleftrightarrow \lim_{u,\, t,\, s \to 0} [uts] = \begin{cases} k > 0 \\ +\infty \\ 0. \end{cases}$$

§ 9. *Condition III′*.—Let A satisfy Condition II′ at p and let $P_1 \notin \mathfrak{T}$. From now on we may assume (I′), $f(t) > 0$ if p is of Type 1 or 2; thus $k > 0$ if p is of Type 1.

The conic through P_1 and four mutually distinct points of A near p has the equation

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ s^2 & \phi_1 & \psi_1 & s & [s] & 1 \\ t+s & \phi_2 & \psi_2 & 1 & [ts] & 0 \\ 1 & \phi_3 & \psi_3 & 0 & [uts] & 0 \\ 0 & \phi_4 & \psi_4 & 0 & [vuts] & 0 \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Let A satisfy Condition III′ (in particular, Condition III) at p. Then (10) must converge as v, u, t, s tend to 0.

If p is of Type 1a, then (10) will have to converge to (5). This implies

$$(\mathrm{III}_a') \qquad \lim\, [vuts] = h.$$

Conversely, if (I′), (II_1') and (III_a') hold, then $\lim\, \phi_4 = k$, $\lim\, \psi_4 = 0$, and (10) will converge to (5).

If p is of Type 1b, 2 or 3, then (10) will have to converge to (6). In particular, if p is of Type 1b, this implies $\lim|[vuts]| = \infty$, and by the continuity of $[vuts]$

$$(\mathrm{III}_b') \qquad \lim\,[vuts] = \pm\infty.$$

If p is of Type 3, the convergence of (10) to (6) implies

$$(\mathrm{III}_3') \qquad \lim\,[uts]/[vuts] = 0.$$

Conversely, if either (I′), (II_1'), and (III_b'), or (II_3'), and (III_3') hold, then

$$\lim \phi_4/[vuts] = \lim \psi_4/[vuts] = 0,$$

and (10) will converge to (6).

If p is of Type 2, then $[uts]$ tends to $+\infty$ and it is convenient to divide the fifth row of (10) by $[uts]$ and put the last row in the alternative form $(1\ \alpha_4/[vuts]\ \beta_4/[vuts]\ 0\ 0\ 0)$, cf. 6. Since the resulting equation must still converge to (6) as s, t, u, v tend to 0, we obtain

$$(\mathrm{III}_2') \qquad \lim\,[vut][vus]/[vuts] = 0.$$

Conversely, if (I′), (II_2') and (III_2') hold, the new form of (10) will tend to (6) as v, u, t, s tend to 0.

Altogether if (I′) holds, and we assume $f(t) > 0$, then

$$\left.\begin{matrix}(\mathrm{II}_1', \mathrm{III}_a')\\ (\mathrm{II}_1', \mathrm{III}_b')\\ (\mathrm{II}_2', \mathrm{III}_2')\\ (\mathrm{II}_3', \mathrm{III}_3')\end{matrix}\right\} \begin{matrix}\textit{Condition}\ \ \mathrm{III}'\\ \textit{with } p \textit{ of Type}\end{matrix} \left\{\begin{matrix}1a\\ 1b\\ 2\\ 3\end{matrix}\right. \longleftrightarrow \lim\,[uts] = \left\{\begin{matrix}k > 0,\ \lim\,[vuts] = \begin{cases}h\\ \pm\infty\end{cases}\\ +\infty,\ \lim\,[vut][vts]/[vuts] = 0\\ 0,\ \lim\,[vts]/[vuts] = 0.\end{matrix}\right.$$

§ 10. *Condition IV′*.—Assume Condition III′ (and (I′)). The equation of the conic through five points of A near p can be expressed in the form

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1\\ s^2 & \phi_1 & \psi_1 & s & [s] & 1\\ t+s & \phi_2 & \psi_2 & 1 & [ts] & 0\\ 1 & \phi_3 & \psi_3 & 0 & [uts] & 0\\ 0 & \phi_4 & \psi_4 & 0 & [vuts] & 0\\ 0 & \phi_5 & \psi_5 & 0 & [wvuts] & 0 \end{vmatrix} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Let A satisfy Condition IV′ (in particular, Condition IV) at p. Then (11) must converge as w, v, u, t, s tend to 0.

If p is of Type 1$a(i)$, then (II_1') and (III_a') will hold and the convergence of (11) to (8) implies

$$(\mathrm{IV}_{(i)}') \qquad \lim\,[wvuts] = n.$$

Conversely if (I′), (II_1'), (III_1') and $(\mathrm{IV}_{(i)}')$ hold, then (11) converges to (8) and p is of Type $1a(i)$.

If p is of Type $1a(ii)$ or $1a(iii)$, then (II_1') and (III_a') hold and the convergence of (11) to $y^2 = 0$ implies $\lim |[wvuts]| = \infty$.

Now, Condition IV implies $\lim [wooo] = \lim (f(w)-kw^2-hw^3)/w^4 = +\infty$ $[-\infty]$ if p is of Type $1a(ii)$ [Type $1a(iii)$]. Since $[wvuts]$ is continuous we obtain

$$\left.\begin{matrix}(\mathrm{IV}_{(ii)}')\\(\mathrm{IV}_{(iii)}')\end{matrix}\right\} \lim\,[wvuts] = \left\{\begin{matrix}+\infty\\-\infty\end{matrix}\right\} \textit{ if } p \textit{ is of } \left\{\begin{matrix}\textit{Type } 1\mathrm{a}\ (ii)\\\textit{Type } 1\mathrm{a}\ (iii).\end{matrix}\right.$$

If p is of Type $1b$ [Type 3], then (II_1') and (III_b') $[(\mathrm{II}_3')$ and $(\mathrm{III}_3')]$ hold and the convergence of (11) of $y^2 = 0$ implies

$$(\mathrm{IV}_b', \mathrm{IV}_3') \qquad \lim \beta_5/\alpha_5 = 0,$$

or, alternatively,

$$\lim \psi_5/\phi_5 = 0.$$

Presumably points of Type 2 do not satisfy Condition (IV′) (Lane and Singh 1965, 5, 4, Remark).

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