STEADY MHD COUETTE FLOW OF AN ELECTRICALLY CONDUCTING, VISCOUS, INCOMPRESSIBLE RAREFIED GAS UNDER TRANSVERSE MAGNETIC FIELD

by V. M. Soundalgekar, *Department of Mathematics, Indian Institute of Technology, Bombay 76*

(Communicated by F. C. Auluck, F.N.I.)

(*Received August 9, 1966*)

An investigation of the combined influence of constant external magnetic field and electric field on the laminar steady Couette flow of an electrically conducting, incompressible, viscous rarefied gas is carried out, in the slip-flow régime. It is found that velocity profiles, velocity gradient at the upper wall, current distribution, slip-velocity and skin-friction are affected by magnetic field, electric field and the rarefaction of the gaseous medium.

1. Introduction

Couette flow with or without magnetic field has been studied by a number of researchers (Illingworth 1950; Lehnert 1952; Schlichting 1955; Liepmann and Bleviss 1956; Bleviss 1958). In all these studies, a continuum flow of compressible or incompressible fluid was assumed with no-slip boundary conditions. In some of them the effects of only magnetic field on the flow were studied. It is well known that at high temperatures gases become ionized. The ionized gases are also electrically conducting. But the density of such ionized gases is slightly reduced and there is accordingly some departure from continuum gasdynamics, hence the interest in the flow of low-density or rarefied gases in channels. The first effect of the gas rarefaction is displayed as a slip of the gas over the bounding walls. So no-slip boundary conditions fail to describe such a flow, and slip-flow boundary conditions are suggested, the details about which are summarized by Schaaf and Chambré (1961).

Recently such an attempt was made by Inmann (1965). He showed that, in the case of an electrically conducting, viscous, incompressible rarefied gas flowing between two stationary non-conducting walls, the velocity profiles, skin-friction and the rate of mass flow are affected by the rarefaction of the gas. But the current distribution and, therefore, also the magnetic field are not affected by the rarefaction of the gas. It was also observed by Inmann that the velocity gradient at the upper wall was not affected by the magnetic field and the rarefaction of the gaseous medium. This study was extended by Soundalgekar (*unpublished*) to the case of the flow in a channel with conducting walls. It was observed by the author that, in the case of a channel with conducting walls, the velocity profiles, the induced magnetic field distribution, the current distribution and the rate of mass flow are affected by the magnetic field, the conductivity of walls and the rarefaction of the gaseous medium. Velocity gradient at the upper wall and skin-friction coefficient were not affected by the rarefaction of the gas. However, they were affected by both wall-conductivity and the applied magnetic field.

Now the Couette flow of a rarefied gas without magnetic field was also studied by Inmann (1962) in the case of porous walls. The slip-flow boundary conditions of Inmann (1962) were later corrected by Dix (1963). The problem of a steady MHD Couette flow under the combined action of externally applied electric and magnetic fields, with no-slip boundary conditions, has been presented in a recently published book by Sutton and Sherman (1965).

The object of this paper is to extend the problem of Couette flow, considered by Sutton and Sherman (1965), to the case of a steady, laminar flow of a rarefied gas between two non-conducting parallel walls under the slip-flow boundary conditions of Dix (1963). In the case of the subsonic flows of a relatively hot gas, the assumptions of incompressibility and of uniform electrical conductivity are physically realizable. The gas is assumed to be flowing parallel to the channel walls under the action of a transversely applied uniform magnetic field. The method of analysis for slip flows utilized here is that of the continuum equations of motion which have been used throughout the gas together with the first-order slip-velocity boundary conditions at the walls.

In § 2, the problem is posed mathematically under suitable assumptions and solutions are derived for velocity field, current distribution, slip-velocity, velocity gradient and skin-friction. Velocity profiles and current distribution under different conditions are shown graphically and numerical values for velocity gradient, skin-friction, slip-velocity are entered in the tables. In § 3, conclusions are derived.

The problem of MHD Couette flow of rarefied gas between conducting walls will soon be published elsewhere.

2. Mathematical Analysis

The steady flow is described by the following equations:
The momentum equation is

$$\rho(\overline{V} \cdot \nabla)\overline{V} + \nabla \cdot p = \mu \nabla^2 \overline{V} + \overline{J} \times \overline{B}; \qquad (1)$$

the continuity equation is

$$\nabla \cdot \overline{V} = 0; \qquad (2)$$

Maxwell's equations of electromagnetism are

$$\bar{\mathbf{J}} = \left(\frac{1}{\mu_c}\right) \nabla \times \bar{\mathbf{B}}, \qquad (3)$$

$$\nabla \cdot \vec{\mathbf{B}} = 0, \qquad (4)$$

$$\nabla \times \vec{\mathbf{E}} = 0, \qquad (5)$$

$$\nabla \cdot \vec{\mathbf{E}} = \rho_e/K_0; \qquad (6)$$

and Ohm's law is

$$\vec{\mathbf{J}} = \sigma\left(\vec{\mathbf{E}} + \vec{\mathbf{V}} \times \vec{\mathbf{B}}\right). \qquad (7)$$

Here the symbols $\bar{\mathbf{V}}$, p, $\bar{\mathbf{J}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{E}}$, σ, ρ_e, K_0 are as described in the book of Sutton and Sherman (1965, p. xv) and μ and μ_c are respectively the coefficient of viscosity and magnetic permeability. Also the rationalized MKS system of units is used throughout.

We make the following assumptions: (i) The bounding walls are infinite in extent in both x- and y-directions. Hence all the physical variables are functions of z only except p, the pressure, which in plane Couette flow, $\partial p/\partial x = 0$. (ii) The excess charge density is neglected.

Under these assumptions, the unknown variables have the form

$$\left.\begin{aligned} &\bar{\mathbf{V}} \equiv [u, 0, 0], \ \bar{J} \equiv [0, j_y, 0] \\ &\bar{\mathbf{B}} = B_0 + b \equiv [b_x, 0, B_0], \ \bar{\mathbf{E}} \equiv [0, E_y, 0] \end{aligned}\right\}, \qquad (8)$$

where E_y and B_0 are constants.

The boundary conditions are

$$\left.\begin{aligned} u(0) &\equiv u_{s,(0)} = \xi_u\left(\frac{du}{dz}\right)_{z=0} \\ u(h) &= \bar{\mathbf{U}} + u_{s,(h)} = \bar{\mathbf{U}} - \xi_u\left(\frac{du}{dz}\right)_{z=h} \end{aligned}\right\}, \qquad (9)$$

where $\bar{\mathbf{U}}$ is the uniform velocity of the upper wall, and the slip coefficient ξ_u is given by the expression (Schaaf and Chambré 1961),

$$\xi_u = \left[\frac{2-\beta}{\beta}\right] \bar{\mathbf{l}},$$

where $\bar{\mathbf{l}}$ is the mean free path (Schaaf and Chambré 1961) given by

$$\bar{\mathbf{l}} = (\sqrt{\pi/8}/0{\cdot}499)\mu(\sqrt{RT}/p),$$

and β is termed Maxwell's reflection coefficient, R is the gas constant.

In virtue of relations (8), we have from eqn. (1)

$$\mu \frac{d^2u}{dz^2} = -j_y B_0, \qquad (10)$$

$$\frac{dp}{dz} = -j_y b_x, \qquad (11)$$

and from eqn. (7), for Ohm's law,

$$j_y = \sigma(E_y - uB_0). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

In a steady state problem, it can be seen that E_y can be taken as constant, since $\nabla \times \bar{\mathbf{E}} = 0$. This leads to the most important fact that the fluid flow and the induced magnetic field are uncoupled and, hence, can be determined separately. In this analysis, we are mainly concerned with the effects of electric field and magnetic field on the flow of electrically conducting, viscous, incompressible rarefied gas with the distribution of current between the two walls. In the book of Sutton and Sherman (1965) no reference has been made to the effects of these fields on the skin-friction at the stationary wall. These are discussed here as a special case. Once the current distribution is known, one can determine the induced magnetic field under different boundary conditions from the relation obtained from Maxwell's eqn. (3), viz.

$$j_y = \frac{1}{\mu_c}\frac{db_x}{d_y}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Then eliminating j_y between (10) and (12), we get

$$\frac{d^2u}{dz^2} = -\frac{\sigma}{\mu}(E_y - uB_0)B_0. \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

Introducing the following non-dimensional quantities:

$$U = u/\bar{\mathrm{U}},\ Z = z/h,\ K = E_y/\bar{\mathrm{U}}B_0$$

in eqn. (14), we get, on rearrangement,

$$\frac{d^2U}{dZ^2} - M^2U = -M^2K, \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

where

$$M = B_0h\left(\frac{\sigma}{\mu}\right)^{\frac{1}{2}}$$

and is called the Hartmann number.

The boundary conditions in non-dimensional form are

$$\left.\begin{aligned} U(0) &= \frac{u_{s,(0)}}{\bar{\mathrm{U}}} = \bar{\mathrm{U}}_{s,(0)} = \lambda\left(\frac{dU}{dZ}\right)_{Z=0} \\ U(1) &= 1 - \frac{u_{s,(h)}}{\bar{\mathrm{U}}} = 1 - \lambda\left(\frac{dU}{dZ}\right)_{Z=1} \end{aligned}\right\}, \qquad \ldots \quad \ldots \quad (16)$$

where $\lambda = \xi_u/h$ is called the slip parameter.

The solution of eqn. (15), subject to the boundary conditions (16), is

$$U = K + \frac{[(1-K)\sinh MZ - K\sinh M(1-Z) - \lambda MK\cosh M(1-Z) + \lambda M(1-K)\cosh MZ]}{\sinh M(1+\lambda^2M^2) + 2\lambda M\cosh M}. \qquad \ldots \quad (17)$$

The velocity profiles for different values of λ, M, K are shown in Figs. 1, 2 and 3.

H ·

In the no-slip case, $\lambda = 0$ and (17) reduces to the equation given in Sutton and Sherman's book (1965). Also, for $M = 0$ and $\lambda = 0$ we get the classical solution, viz. $U = Z$.

The velocity-gradient at the moving wall is obtained from (17) as

$$\left.\frac{dU}{dZ}\right|_{Z=1} = \frac{(1-K)M \cosh M + KM + \lambda M^2(1-K) \sinh M}{\sinh M(1+\lambda^2M^2)+2\lambda M \cosh M}. \quad \quad .. \quad (18)$$

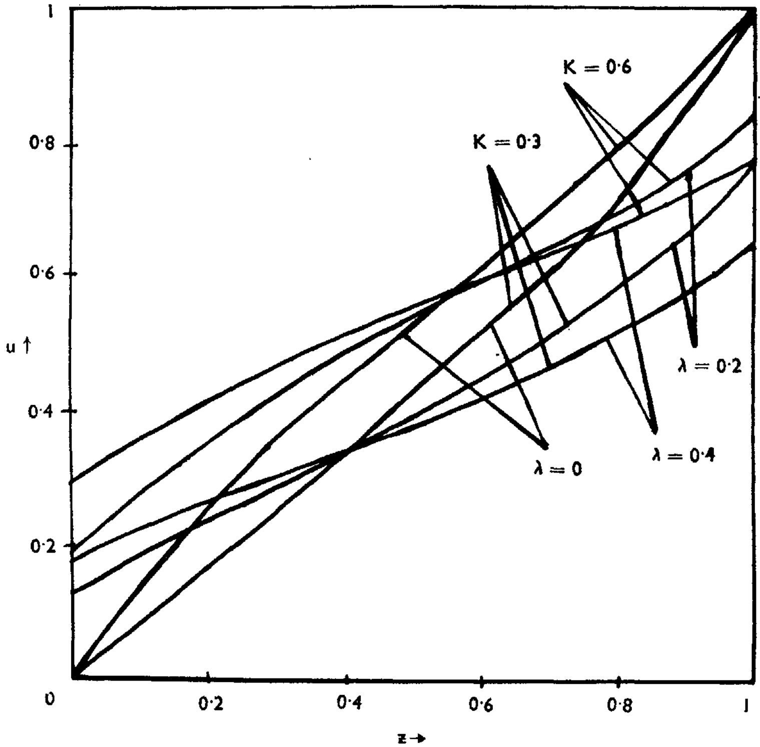

FIG. 1. Velocity profiles. $M = 2$, $\lambda = 0$, 0·2, 0·4, $K = 0{\cdot}3$, 0·6.

The numerical values for $\left.\frac{dU}{dZ}\right|_{Z=1}$ for different values of the parameters λ, M, K are entered in Table I.

For the current distribution, we have it in non-dimensional form

$$J_y = \frac{j_y}{\sigma B_0 \bar{u}} = \frac{\sigma}{\sigma B_0 \bar{u}}(E_y - uB_0) = K-U. \quad \quad .. \quad .. \quad (19)$$

Hence, from (17) and (19), we have

$$J_y = \frac{[K \sinh M(1-Z)-(1-K) \sinh MZ + \lambda MK \cosh M(1-Z) - \lambda M(1-K) \cosh MZ]}{\sinh M(1+\lambda^2M^2)+2\lambda M \cosh M}. \quad \quad .. \quad (20)$$

The variation of J_y with λ, M, K is shown in Figs. 4 and 5.

TABLE I

Values of $\left.\frac{dU}{dZ}\right|_{Z=1}$

K	M/λ	2	4
0	0·2	2·904	2·225
	0·4	1·114	1·008
0·3	0	1·620	2·845
	0·2	2·255	10·61
0·6	0	1·161	1·991
	0·2	0·6098	9·162

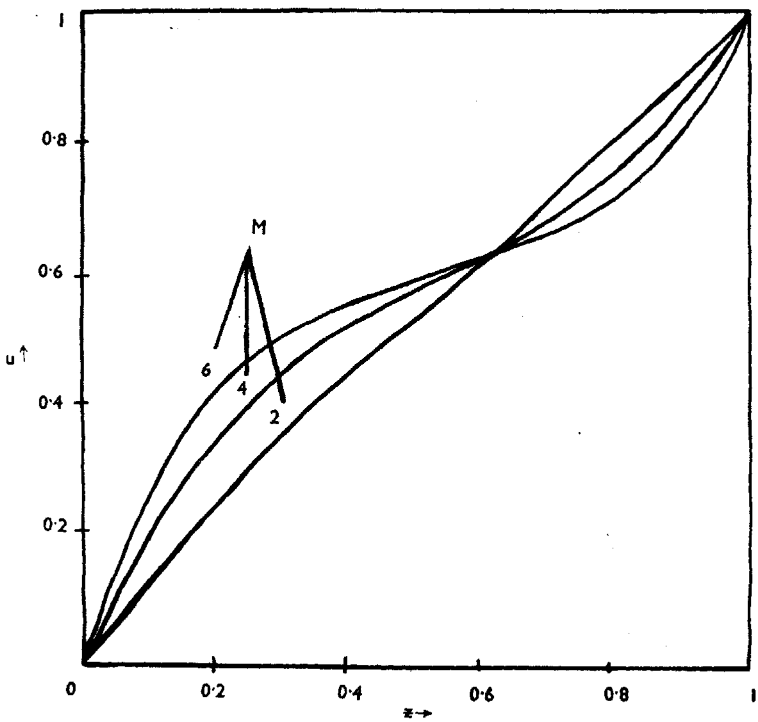

FIG. 2. Velocity profiles. $K = 0{\cdot}6$, $\lambda = 0$.

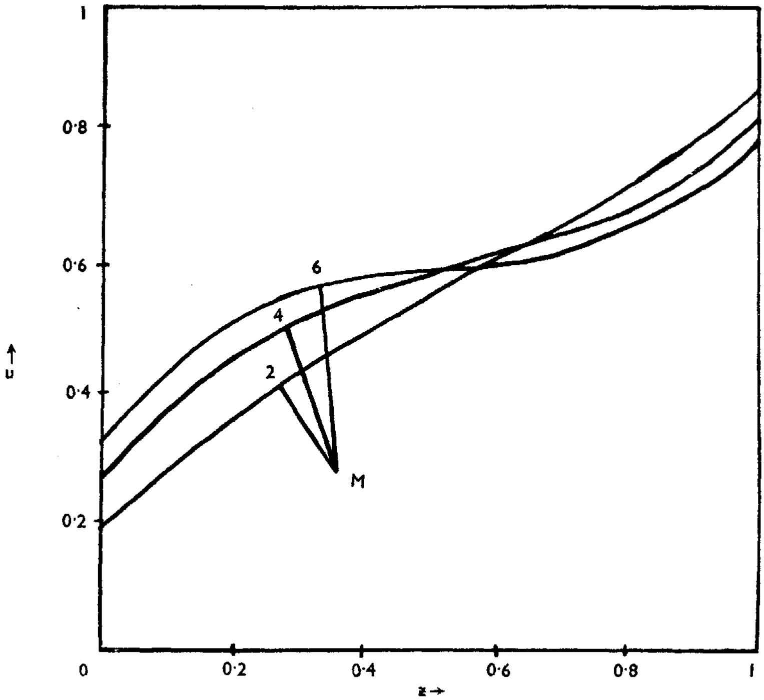

FIG. 3. Velocity profiles. $K = 0{\cdot}6$, $\lambda = 0{\cdot}2$.

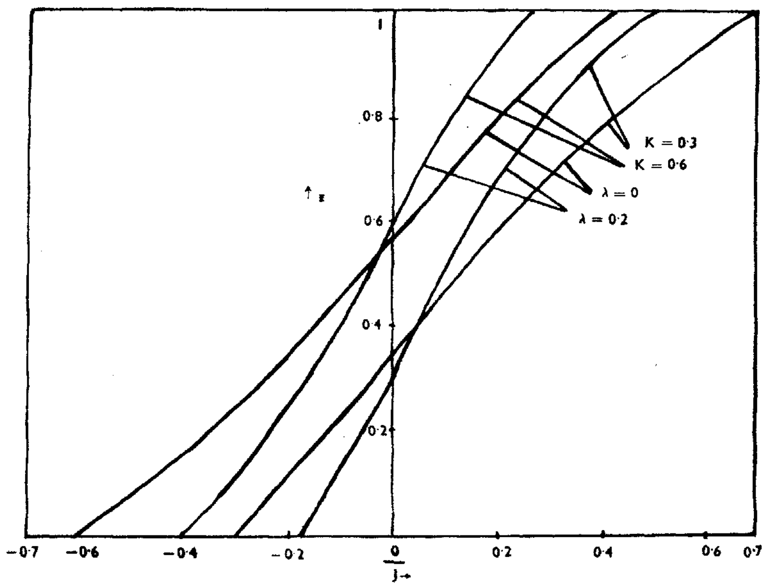

FIG. 4. Variation of induced current with λ and K. $M = 2$.

From (13) and (20), the induced magnetic field can be determined under different conditions.

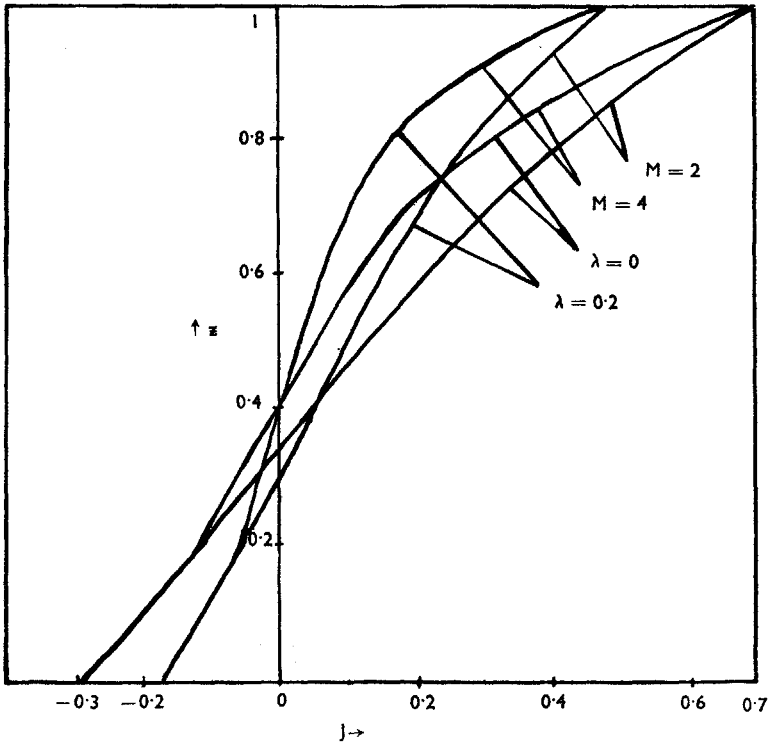

FIG. 5. **Variation of induced current with M. $K = 0{\cdot}3$.**

The slip velocity at the stationary wall is given by

$$\bar{\mathbf{U}}_{s,(0)} = \lambda\left(\frac{dU}{dZ}\right)_{Z=0}$$

$$= \lambda\left[\frac{M(1-K)+MK\cosh M+\lambda M^2K\sinh M}{\sinh M(1+\lambda^2M^2)+2\lambda M\cosh M}\right]. \quad \ldots \quad \ldots \quad (21)$$

The skin-friction at the stationary wall is defined as

$$\tau^* = \mu\left(\frac{du}{dz}\right)_{z=0},$$

which in non-dimensional form is given by

$$R_T = \left(\frac{dU}{dZ}\right)_{Z=0}$$

$$= \frac{\bar{\mathbf{U}}_{s,(0)}}{\lambda}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Hence, from (21) and (22), we get

$$R_T = \frac{M(1-K)+MK\cosh M+\lambda M^2K\sinh M}{\sinh M(1+\lambda^2M^2)+2\lambda M\cosh M}. \quad \cdots \quad \cdots \quad (23)$$

The numerical values for R_T and $\overline{\mathrm{U}}_{s,(0)}$ are entered in Table II.

TABLE II

		R_T			$\overline{\mathrm{U}}_{s,(0)}$	
M	K/λ	0	0·2	0·4	0·2	0·4
2	0·3	1·0084	0·6273	0·4510	0·1254	0·1804
	0·6	1·4653	0·9776	0·7350	0·1955	0·2940
4	0·3	1·3034	0·6983	0·4766	0·1396	0·1906
	0·6	2·4602	1·3514	0·9316	0·2702	0·3726
6	0·3	1·8208	0·8224	0·5312	0·1644	0·2124
	0·6	3·6119	1·6388	1·0598	0·3277	0·4239

CONCLUSIONS

From TABLE I, we have the following conclusions:

Case I: $K = 0$ (short-circuited). (1) It has been observed in the work of Sutton and Sherman (1965) that, in the absence of rarefaction of the gaseous medium, the velocity gradient at the moving wall increases with increase in the value of Hartmann number M. But when the gaseous medium is rarefied, say for $\lambda = 0{\cdot}2$, the velocity gradient at the moving wall decreases with increase in the value of the Hartmann number M. In other words, in the absence of rarefaction, the larger the M, the greater the force necessary to move this wall; whereas in the case of a rarefied gas, the larger the M, the smaller is the force.

(2) Also, for $M = 2$, the velocity gradient at the moving wall decreases with increase in the value of slip-parameter λ. Hence, the larger the λ, the smaller the force necessary to move the wall.

Case II: $k \neq 0$ (open-circuited). (1) One can conclude from the study of Table I that in general an increase in the value of the Hartmann number leads to an increase in the value of the velocity gradient. But the rate of increase is more rapid in the case of rarefied gas than that in the case of an ordinary one. In other words, the force necessary to move the upper wall is more when rarefaction is present than when it is not present. Moreover, in the case of rarefied gas, the force to move the wall has to be increased rapidly with the increase in the value of the Hartmann number.

(2) In the presence or absence of rarefaction of the gas, for the same value of M, the velocity gradient decreases with the increase in the value of K or, in other words, the force necessary to move the upper wall decreases with the increase in the value of K, both in the case of rarefied or non-rarefied gas.

Conclusions

From Table II, we have the following conclusions:

(1) For all λ, the skin-friction decreases with increase in the value of K or M.

(2) For the same value of M and K, the skin-friction decreases with increase in the value of the slip-parameter λ.

(3) The slip velocity also decreases with increase in the value of λ, K or M.

References

Bleviss, Z. O. (1958). Magnetogasdynamics of hypersonic Couette flow. *J. Aero/Space Sci.*, **25**, 601–615.

Dix, D. M. (1963). Comment on porous wall Couette flow under slip-flow conditions. *Am. Inst. Aeronaut. Astronaut.*, **1**, 1233.

Illingworth, C. R. (1950). Some solutions of the equations of a viscous compressible fluid. *Proc. Camb. phil. Soc. math. phys.*, **46**, 469–478.

Inmann, R. M. (1962). On porous wall Couette flow under slip-flow conditions. *J. Aero/Space Sci.*, **29**, 1123.

——— (1965). Laminar slip flow of an electrically conducting incompressible rarefied gas in a channel with a transverse magnetic field. *Appl. scient. Res.*, **11 B**, 391–400.

Lehnert, B. O. (1952). On the behavior of an electrically conductive liquid in a magnetic field. *Ark. Fys.*, **5**, 69–90.

Liepmann, H. W., and Bleviss, Z. O. (1956). The Effects of Dissociation and Ionization on Compressible Couette Flow. Douglas Aircraft Co. Inc., Report No. SM–19831.

Schaaf, S. A., and Chambré, P. L. (1961). Flow of Rarefied Gases. Princeton University Press.

Schlichting, H. (1955). Boundary Layer Theory. McGraw-Hill Book Co. Inc., New York.

Soundalgekar, V. M. (*in press*). MHD channel flow of an electrically conducting, incompressible, viscous, rarefied gas as affected by wall electrical conductances (to be published in *Proc. natn. Inst. Sci. India*).

Sutton, G. W., and Sherman, A. (1965). Engineering Magnetohydrodynamics. McGraw-Hill Book Co. Inc., New York, pp. 352–356.