ON THE TEMPERATURE DISTRIBUTION OF A VISCOUS LIQUID UNDER EXPONENTIAL RATE OF HEAT ADDITION SUPERPOSED ON THE STEADY TEMPERATURE OF INCOMPRESSIBLE FLUID BETWEEN TWO CONFOCAL ELLIPTIC CYLINDERS

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In the present paper the temperature distribution in a channel bounded by two confocal elliptic cylinders is obtained when viscous incompressible fluid is flowing through it and the exponential rate of heat addition is superposed on the steady temperature. Solutions are obtained for the two cases: (i) When the rate of heat addition is exponentially increasing; (ii) when the rate of heat addition is exponentially decreasing. The solutions are obtained in terms of Mathieu functions and from them results for the two extreme cases of very small and very large frequencies are deduced.

INTRODUCTION

Solutions for the temperature distribution in a circular pipe have been given by many authors, namely Graetz, Nusselt, Goldstein; all these have been cited in Goldstein's book (1938, §266). Lal (1964) has considered the temperature distribution in a channel bounded by two co-axial circular cylinders when viscous incompressible fluid is flowing through it and the rate of heat addition is an exponential function of time. In the present paper the expression for the temperature distribution in a channel bounded by two confocal elliptic cylinders is discussed when viscous incompressible fluid is flowing through it, the dissipation due to friction is neglected and the exponential rate of heat addition is superposed on the steady temperature. Solutions are obtained in two cases:

(i) When the rate of heat addition is exponentially increasing.
(ii) When the rate of heat addition is exponentially decreasing.

The solutions are obtained in terms of Mathieu functions and from them the results in the two extreme cases of very small and very large frequencies are also deduced.

1. Equation of Energy and its Solution

The equation of energy (Pai 1956) in the present case reduces to

$$\frac{\partial T}{\partial t} = \frac{1}{\rho C_v}\frac{\partial Q}{\partial t} + k'\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \qquad (1.1)$$

where $k' = \frac{k}{\rho C_v}$ is a constant and the dissipation due to friction is neglected.

2. Flow under Exponentially Increasing Rate of Heat Addition

Let us assume

$$\frac{1}{\rho C_v}\frac{\partial Q}{\partial t} = \sum_{n=1}^{\infty} a_n e^{nt}, \qquad (2.1)$$

and

$$T = T_0 + \sum_{n=1}^{\infty} T_n e^{nt}, \qquad (2.2)$$

where a_n, T_0 and T_n are real, and T_0 and T_n are functions of x and y.

Substituting (2.1), (2.2) in (1.1) and comparing the terms of the same family, we get the differential equations for the coefficients as

$$\frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_0}{\partial y^2} = 0, \qquad (2.3)$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{n}{k'} v = 0, \qquad (2.4)$$

where

$$v = \frac{n}{k'} T_n - \frac{a_n}{k'}.$$

If the boundary of the tube be given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we introduce elliptic coordinates ξ, η defined by

$$x + iy = c\cosh(\xi + i\eta), \quad c = \sqrt{a^2 - b^2}.$$

In these coordinates, (2.3) and (2.4) transform into

$$\frac{\partial^2 T_0}{\partial \xi^2} + \frac{\partial^2 T_0}{\partial \eta^2} = 0, \qquad (2.5)$$

and

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - 2p^2(\cosh 2\xi - \cos 2\eta)v = 0, \qquad (2.6)$$

where

$$2p = c\sqrt{\frac{n}{k'}}.$$

Integrating (2.5) we have

$$T_0 = \cos 2\eta[Ae^{2\xi}+Be^{-2\xi}]. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.7)$$

Before superposing the exponential rate of heat addition we must have the fully developed steady temperature. With this condition and with the following boundary conditions

$$\xi = \xi_1,\ T_0 = T_1,$$

and

$$\xi = \xi_2,\ T_0 = T_2,$$

the unknown constants A and B in (2.7) are determined. Hence

$$T_0 = \frac{T_1 \sinh 2(\xi_2-\xi)+T_2 \sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}.$$

For the integration of (2.6), let us put

$$v = \phi(\xi)\,.\,\psi(\eta)$$

as a solution of (2.6); we see that ϕ and ψ will satisfy the differential equations

$$\frac{d^2\phi}{d\xi^2}-(a+2q\cosh 2\xi)\phi = 0, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.8)$$

and

$$\frac{d^2\psi}{d\eta^2}+(a+2q\cos 2\eta)\psi = 0, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.9)$$

where a is a constant and $q = p^2$.

Hence ψ is a periodic Mathieu function and ϕ is the modified Mathieu function.

The complete solution of (2.6) is (McLachlan 1947, p. 160)

$$v = \sum_{m=0}^{\infty} D'_{2m} Ce_{2m}(\xi,-q)ce_{2m}(\eta,-q)+\sum_{m=0}^{\infty} E'_{2m} Fey_{2m}(\xi,-q)ce_{2m}(\eta,-q),$$

where (McLachlan 1947, pp. 21, 27, 165)

$$ce_{2m}(\eta,-q) = (-1)^m \sum_{r=0}^{\infty} (-1)^r A_{2r}^{2m} \cos 2r\eta,$$

$$Ce_{2m}(\xi,-q) = (-1)^m \sum_{r=0}^{\infty} (-1)^r A_{2r}^{2m} \cosh 2r\xi,$$

and

$$Fey_{2m}(\xi,-q) = (-1)^m \frac{ce_{2m}(o,\,q)}{A_0^{2m}} \sum_{r=0}^{\infty} A_{2r}^{2m} Y_{2r}(2ip\cosh\xi),$$

where Y_{2r} is a Bessel function and the coefficients A_{2r}^{2m} are functions of q and are real. Thus (2.2) gives

$$T = \frac{T_1 \sinh 2(\xi_2-\xi)+T_2 \sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)} + \sum_{n=1}^{\infty}\left[\frac{a_n}{n}+\sum_{m=0}^{\infty} D_{2m}Ce_{2m}(\xi,-q)ce_{2m}(\eta,-q)+\sum_{m=0}^{\infty} E_{2m}Fey_{2m}(\xi,-q)ce_{2m}(\eta,-q)\right]e^{nt}. \quad (2.10)$$

The boundary conditions are

$$\left.\begin{aligned} \xi = \xi_1,\ \ T &= T_1e^{nt}+T_1, \\ \text{and} \quad \xi = \xi_2,\ \ T &= T_2e^{nt}+T_2. \end{aligned}\right\} \quad (2.11)$$

By applying the first of these boundary conditions we have

$$T_1-\frac{a_n}{n} = \sum_{m=0}^{\infty} D_{2m}Ce_{2m}(\xi_1,-q)ce_{2m}(\eta,-q)+\sum_{m=0}^{\infty} E_{2m}Fey_{2m}(\xi_1,-q)ce_{2m}(\eta,-q).$$

Now multiplying both the sides by $ce_{2m}(\eta,-q)$ and integrating with respect to η from 0 to 2π and using the orthogonality relations and normalization (McLachlan 1947, pp. 23-24), we have

$$\left(T_1-\frac{a_n}{n}\right)L_{2m} = D_{2m}Ce_{2m}(\xi_1,-q)+E_{2m}Fey_{2m}(\xi_1,-q), \quad (2.12)$$

where

$$L_{2m} = 2(-1)^m A_0^{2m}.$$

Now by applying the second condition of (2.11) and proceeding exactly in the same manner as before, we have

$$\left(T_2-\frac{a_n}{n}\right)L_{2m} = D_{2m}Ce_{2m}(\xi_2,-q)+E_{2m}Fey_{2m}(\xi_2,-q). \quad (2.13)$$

From (2.12) and (2.13), we have

$$D_{2m} = \frac{a_nL_{2m}}{n}\,\frac{Fey_{2m}(\xi_1,-q)-Fey_{2m}(\xi_2,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)} + L_{2m}\,\frac{T_1Fey_{2m}(\xi_2,-q)-T_2Fey_{2m}(\xi_1,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)},$$

and

$$E_{2m} = \frac{a_nL_{2m}}{n}\,\frac{Ce_{2m}(\xi_2,-q)-Ce_{2m}(\xi_1,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)} + L_{2m}\,\frac{T_2Ce_{2m}(\xi_1,-q)-T_1Ce_{2m}(\xi_2,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)}.$$

Hence (2.10) becomes

$$T = \frac{T_1 \sinh 2(\xi_2-\xi)+T_2 \sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}$$

$$+\sum_{n=1}^{\infty}\frac{a_n}{n}\Bigg[1-\sum_{m=0}^{\infty} L_{2m}\frac{\{Fey_{2m}(\xi_2,-q)-Fey_{2m}(\xi_1,-q)\}Ce_{2m}(\xi,-q)ce_{2m}(\eta,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)}$$

$$-\sum_{m=0}^{\infty} L_{2m}\frac{\{Ce_{2m}(\xi_1,-q)-Ce_{2m}(\xi_2,-q)\}Fey_{2m}(\xi,-q)ce_{2m}(\eta,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)}\Bigg]e^{nt}$$

$$+\sum_{n=1}^{\infty}\Bigg[\sum_{m=0}^{\infty}\frac{\{T_1Fey_{2m}(\xi_2,-q)-T_2Fey_{2m}(\xi_1,-q)\}Ce_{2m}(\xi,-q)ce_{2m}(\eta,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)}$$

$$+\sum_{m=0}^{\infty} L_{2m}\frac{\{T_2Ce_{2m}(\xi_1,-q)-T_1Ce_{2m}(\xi_2,-q)\}Fey_{2m}(\xi,-q)ce_{2m}(\eta,-q)}{Ce_{2m}(\xi_1,-q)Fey_{2m}(\xi_2,-q)-Ce_{2m}(\xi_2,-q)Fey_{2m}(\xi_1,-q)}\Bigg]e^{nt} \quad \text{.. (2.14)}$$

When q is small, we have (McLachlan 1947, pp. 15, 382) as $q \to 0$

$$ce_0(\eta,-q) \simeq (1+\tfrac{1}{2}q \cos 2\eta)A_0^0,$$

$$ce_2(\eta,-q) \simeq \cos 2\eta+q(\tfrac{1}{12} \cos 4\eta-\tfrac{1}{4}),$$

$$Ce_0(\xi,-q) \simeq (1+\tfrac{1}{2}q \cosh 2\xi)A_0^0,$$

$$Ce_2(\xi,-q) \simeq \cosh 2\xi+q(\tfrac{1}{12} \cosh 4\xi-\tfrac{1}{4}),$$

$$Fey_0(\xi,-q) \simeq \frac{\sqrt{2}}{\pi}(\tfrac{1}{2}\pi i+\xi+\tfrac{1}{2}\log_e q),$$

and

$$Fey_2(\xi,-q) \simeq \frac{16e^{-2\xi}}{\pi q^2}.$$

Also we have (McLachlan 1947, pp. 25, 46)

$$A_0^0 = \frac{1}{\sqrt{2}}, \quad A_0^2 = \tfrac{1}{4}q+O(q^3), \quad A_0^{2m} = O(q^m),$$

$$L_0 = \sqrt{2} \text{ and } L_2 = -\tfrac{1}{2}q.$$

Substituting these values in (2.14), and putting $T_1 = T_2$, we obtain

$$T = T_1\frac{\sinh 2(\xi_2-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}$$

$$+\frac{c^2}{8k'}\sum_{n=1}^{\infty} a_n\Bigg[\frac{\cos 2\eta\{\sinh 2(\xi_2-\xi)+\sinh 2(\xi-\xi_1)\}}{\sinh 2(\xi_2-\xi_1)}-(\cosh 2\xi+\cos 2\eta)$$

$$+\xi\frac{\cosh 2\xi_1}{\xi_1}\Bigg]e^{nt}$$

$$+T_1\sum_{n=1}^{\infty}\Bigg[1+\frac{q}{2}\Bigg\{(\cosh 2\xi+\cos 2\eta)-\xi\frac{\cosh 2\xi_1}{\xi_1}$$

$$-\cos 2\eta\,\frac{\sinh 2(\xi_2-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}\Bigg\}\Bigg]e^{nt},$$

when both the walls of the channel are at the same temperature, and

$$\frac{\cosh 2\xi_1}{\xi_1} = \frac{\cosh 2\xi_2}{\xi_2}.$$

The importance of $\dfrac{\cosh 2\xi_1}{\xi_1} = \dfrac{\cosh 2\xi_2}{\xi_2}$ has been given by Verma (1960). For large q, we have the asymptotic formula (McLachlan 1947, pp. 230, 385)

$$Ce_0(\xi,-q) \simeq \tfrac{1}{2}C_0\left[\frac{\exp(2p\cosh\xi)}{\cosh\frac{1}{2}\xi} - \frac{i\exp(-2p\cosh\xi)}{\sinh\frac{1}{2}\xi}\right],$$

$$Fey_0(\xi,-q) \simeq \tfrac{1}{2}C_0\left[\frac{i\exp(2p\cosh\xi)}{\cosh\frac{1}{2}\xi} - \frac{\exp(-2p\cosh\xi)}{\sinh\frac{1}{2}\xi}\right],$$

$$ce_0(\eta,-q) \simeq \frac{C_0}{\sin\eta}[\exp(2p\cos\eta)\sin\eta + \exp(-2p\cos\eta)],$$

and

$$C_0 = \frac{ce(0)ce\left(\frac{\pi}{2}\right)}{A_0^0(2p\pi)^{\frac{1}{2}}} \text{ (McLachlan 1947, p. 201)}$$

$$= \frac{1}{\sqrt{\pi p}},$$

where

$$2p = c\sqrt{\frac{n}{k'}}.$$

Substituting these in (2.14) and again putting $T_1 = T_2$, we have

$$T = T_1\frac{\sinh 2(\xi_2-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}$$

$$+\sum_{n=1}^{\infty}\frac{a_n e^{nt}}{n}\left[1-\left\{\frac{\cosh\frac{1}{2}\xi_2}{\cosh\frac{1}{2}\xi}\exp[-2p(\cosh\xi_2-\cosh\xi)]\right.\right.$$

$$\left.\left.+\frac{\sinh\frac{1}{2}\xi_1}{\sinh\frac{1}{2}\xi}\exp[-2p(\cosh\xi-\cosh\xi_1)]\right\}ce_0(\eta,-q)\right]$$

$$+T_1\sum_{n=1}^{\infty}e^{nt}\left[\left\{\frac{\cosh\frac{1}{2}\xi_2}{\cosh\frac{1}{2}\xi}\exp[-2p(\cosh\xi_2-\cosh\xi)]\right.\right.$$

$$\left.\left.+\frac{\sinh\frac{1}{2}\xi_1}{\sinh\frac{1}{2}\xi}\exp[-2p(\cosh\xi-\cosh\xi_1)]\right\}ce_0(\eta,-q)\right].$$

Therefore maximums of temperature distribution exist in the neighbourhood of the wall when q is large.

3. Flow under Exponentially Decreasing Rate of Heat Addition

Let us now assume

$$\frac{1}{\rho C_v}\frac{\partial Q}{\partial t} = \sum_{n=1}^{\infty} a_n e^{-nt}, \quad \ldots \quad (3.1)$$

and

$$T = T_0 + \sum_{n=1}^{\infty} T_n e^{-nt}, \quad \ldots \quad (3.2)$$

where a_n, T_0 and T_n are real and T_0 and T_n are functions of x and y.

Substituting (3.1) and (3.2) in (1.1) and comparing the terms of the same family, we get the differential equations for the coefficients as

$$\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2} = 0, \quad \ldots \quad (3.3)$$

and

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{n}{k'}v = 0, \quad \ldots \quad (3.4)$$

where

$$v = \frac{n}{k'}T_n+\frac{a_n}{k'}.$$

If the boundary of the tube be given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,$$

we introduce elliptic coordinates ξ, η defined by

$$x+iy = c\cosh(\xi+i\eta),\ c = \sqrt{a^2-b^2}.$$

In these coordinates (3.3) and (3.4) transform into

$$\frac{\partial^2 T_0}{\partial \xi^2}+\frac{\partial^2 T_0}{\partial \eta^2} = 0, \quad \ldots \quad (3.5)$$

and

$$\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}+2p^2(\cosh 2\xi-\cos 2\eta)v = 0, \quad \ldots \quad (3.6)$$

where

$$2p = c\sqrt{\frac{n}{k'}}.$$

Integrating (3.5) and applying the boundary conditions in a way similar to that in the previous section, we have

$$T_0 = \frac{T_1\sinh 2(\xi_2-\xi)+T_2\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}.$$

For the integration of (3.6), let us put

$$v = \phi(\xi) \,.\, \psi(\eta)$$

as a solution of (3.6); we see that ϕ and ψ will satisfy the differential equations

$$\frac{d^2\phi}{d\xi^2} - (a - 2q \cosh 2\xi)\phi = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

and

$$\frac{d^2\psi}{d\eta^2} + (a - 2q \cos 2\eta)\psi = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

where a is a constant and $q = p^2$.

Hence ψ is a periodic Mathieu function and ϕ is the modified Mathieu function. The complete solution of (3.6) is (McLachlan 1947, p. 160)

$$v = \sum_{m=0}^{\infty} D'_{2m} Ce_{2m}(\xi, q) ce_{2m}(\eta, q) + \sum_{m=0}^{\infty} E'_{2m} Fey_{2m}(\xi, q) ce_{2m}(\eta, q),$$

where (McLachlan 1947, pp. 21, 26, 159)

$$ce_{2m}(\eta, q) = \sum_{r=0}^{\infty} A_{2r}^{2m} \cos 2r\eta,$$

$$Ce_{2m}(\xi, q) = \sum_{r=0}^{\infty} A_{2r}^{2m} \cosh 2r\xi,$$

and

$$Fey_{2m}(\xi, q) = \frac{ce_{2m}(0, q)}{A_0^{2m}} \sum_{r=0}^{\infty} A_{2r}^{2m}\, Y_{2r}(2p \sinh \xi), \; (|\sinh \xi| > 1),$$

where Y_{2r} is a Bessel function and the coefficients A_{2r}^{2m} are functions of q and are real.

Then (3.2) gives

$$T = \frac{T_1 \sinh 2(\xi_2 - \xi) + T_2 \sinh 2(\xi - \xi_1)}{\sinh 2(\xi_2 - \xi_1)}$$
$$+ \sum_{n=1}^{\infty} \left[-\frac{a_n}{n} + \sum_{m=0}^{\infty} D_{2m} Ce_{2m}(\xi, q) ce_{2m}(\eta, q) + \sum_{m=0}^{\infty} E_{2m} Fey_{2m}(\xi, q) ce_{2m}(\eta, q) \right] e^{-nt} \qquad \ldots \quad \ldots \quad (3.9)$$

The boundary conditions are

$$\xi = \xi_1, \quad T = T_1 e^{-nt} + T_1,$$

$$\xi = \xi_2, \quad T = T_2 e^{-nt} + T_2.$$

Proceeding in exactly the same way as in the previous section we can easily obtain D_{2m} and E_{2m}.

Thus

$$T = \frac{T_1 \sinh 2(\xi_2-\xi)+T_2 \sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}$$

$$-\sum_{n=1}^{\infty} \frac{a_n}{n}\left[1-\sum_{m=0}^{\infty} L_{2m} \frac{\{Fey_{2m}(\xi_2, q)-Fey_{2m}(\xi_1, q)\}Ce_{2m}(\xi, q)ce_{2m}(\eta, q)}{Ce_{2m}(\xi_1, q)Fey_{2m}(\xi_2, q)-Ce_{2m}(\xi_2, q)Fey_{2m}(\xi_1, q)}\right.$$

$$\left.-\sum_{m=0}^{\infty} L_{2m} \frac{\{Ce_{2m}(\xi_1, q)-Ce_{2m}(\xi_2, q)\}Fey_{2m}(\xi, q)ce_{2m}(\eta, q)}{Ce_{2m}(\xi_1, q)Fey_{2m}(\xi_2, q)-Ce_{2m}(\xi_2, q)Fey_{2m}(\xi_1, q)}\right]e^{-nt}$$

$$+\sum_{n=1}^{\infty}\left[\sum_{m=0}^{\infty} L_{2m} \frac{\{T_1 Fey_{2m}(\xi_2, q)-T_2 Fey_{2m}(\xi_1, q)\}Ce_{2m}(\xi, q)ce_{2m}(\eta, q)}{Ce_{2m}(\xi_1, q)Fey_{2m}(\xi_2, q)-Ce_{2m}(\xi_2, q)Fey_{2m}(\xi_1, q)}\right.$$

$$\left.+\sum_{m=0}^{\infty} L_{2m} \frac{\{T_2 Ce_{2m}(\xi_1, q)-T_1 Ce_{2m}(\xi_2, q)\}Fey_{2m}(\xi, q)ce_{2m}(\eta, q)}{Ce_{2m}(\xi_1, q)Fey_{2m}(\xi_2, q)-Ce_{2m}(\xi_2, q)Fey_{2m}(\xi_1, q)}\right]e^{-nt}. \quad (3.10)$$

When q is small, we have (McLachlan 1947, pp. 15, 382)

$$ce_0(\eta, q) \simeq A_0^0(1-\tfrac{1}{2}q \cos 2\eta),$$
$$ce_2(\eta, q) \simeq \cos 2\eta - q(\tfrac{1}{12} \cos 4\eta - \tfrac{1}{4}),$$
$$Ce_0(\xi, q) \simeq A_0^0(1-\tfrac{1}{2}q \cosh 2\xi),$$
$$Ce_2(\xi, q) \simeq \cosh 2\xi - q(\tfrac{1}{12} \cosh 4\xi - \tfrac{1}{4}),$$
$$Fey_0(\xi, q) \simeq \frac{\sqrt{2}}{\pi}(\xi + \tfrac{1}{2} \log_e q),$$

and

$$Fey_2(\xi, q) \simeq -\frac{16e^{-2\xi}}{\pi q^2}.$$

Also we have (McLachlan 1947, pp. 25, 46)

$$A_0^0 = \frac{1}{\sqrt{2}},\ A_0^2 = \tfrac{1}{4}q + O(q^3),\ A_0^{2m} = O(q^m),$$

and

$$L_0 = \sqrt{2} \text{ and } L_2 = \tfrac{1}{2}q.$$

Substituting these in (3.10) and following the same procedure as that of the last section, we have

$$T = T_1 \frac{\sinh 2(\xi_2-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)}$$

$$+\frac{c^2}{8k'}\sum_{n=1}^{\infty} a_n\left[\frac{\cos 2\eta\{\sinh 2(\xi_2-\xi)+\sinh 2(\xi-\xi_1)\}}{\sinh 2(\xi_2-\xi_1)} - (\cosh 2\xi + \cos 2\xi) + \xi \frac{\cosh 2\xi_1}{\xi_1}\right]e^{-nt}$$

$$+T_1 \sum_{n=1}^{\infty}\left[1+\frac{q}{2}\left\{\cos 2\eta \frac{\sinh 2(\xi_2-\xi)+\sinh 2(\xi-\xi_1)}{\sinh 2(\xi_2-\xi_1)} - (\cosh 2\xi + \cos 2\eta) + \xi \frac{\cosh 2\xi_1}{\xi_1}\right\}\right]e^{-nt},$$

when

$$T_1 = T_2, \text{ and } \frac{\cosh 2\xi_1}{\xi_1} = \frac{\cosh 2\xi_2}{\xi_2}.$$

For large q we have the asymptotic formula (McLachlan 1947, pp. 229, 385)

$$Ce_0(\xi, q) \simeq \left(\frac{2}{\cosh \xi}\right)^{\frac{1}{2}} C_0 \cos [2p \sinh \xi - \tan^{-1} (\tanh \tfrac{1}{2}\xi)],$$

$$Fey_0(\xi, q) \simeq \left(\frac{2}{\cosh \xi}\right)^{\frac{1}{2}} C_0 \sin [2p \sinh \xi - \tan^{-1} (\tanh \tfrac{1}{2}\xi)],$$

$$ce_0(\eta, q) \simeq \frac{C_0}{\cos \eta}\left[\exp (2p \sin \eta) \cos \left(\tfrac{1}{2}\eta + \frac{\pi}{4}\right) + \exp (-2p \sin \eta) \sin \left(\tfrac{1}{2}\eta + \frac{\pi}{4}\right)\right],$$

where

$$C_0 = \frac{ce(0)ce\left(\frac{\pi}{2}\right)}{A_0^0 (2\pi p)^{\frac{1}{2}}}.$$

Substituting these in (3.10) and again putting $T_1 = T_2$, we get

$$= T_1 \frac{\sinh 2(\xi_2 - \xi) + \sinh 2(\xi - \xi_1)}{\sinh 2(\xi_2 - \xi_1)}$$

$$- \sum_{n=1}^{\infty} \frac{a_n}{n}\left[1 - \left\{\frac{\cosh \xi_2}{\cosh \xi}\right.\right.$$

$$\times \frac{\sin 2p(\sinh \xi - \sinh \xi_1) \cosh \tfrac{1}{2}(\xi + \xi_1) - \cos 2p(\sinh \xi - \sinh \xi_1) \sinh \tfrac{1}{2}(\xi - \xi_1)}{\sin 2p(\sinh \xi_2 - \sinh \xi_1) \cosh \tfrac{1}{2}(\xi_1 + \xi_2) - \cos 2p(\sinh \xi_2 - \sinh \xi_1) \sinh \tfrac{1}{2}(\xi_2 - \xi_1)}$$

$$\left.\left. + \frac{\cosh \xi_1}{\cosh \xi} \frac{\sin 2p(\sinh \xi_2 - \sinh \xi) \cosh \tfrac{1}{2}(\xi + \xi_2) - \cos 2p(\sinh \xi_2 - \sinh \xi) \sinh \tfrac{1}{2}(\xi_2 - \xi)}{\sin 2p(\sinh \xi_2 - \sinh \xi_1) \cosh \tfrac{1}{2}(\xi_1 + \xi_2) - \cos 2p(\sinh \xi_2 - \sinh \xi_1) \sinh \tfrac{1}{2}(\xi_2 - \xi_1)}\right\} ce_0(\eta, q)\right] e^{-nt}$$

$$+ T_1 \sum_{n=1}^{\infty} \left[\left\{\frac{\cosh \xi_2}{\cosh \xi} \frac{\sin 2p(\sinh \xi - \sinh \xi_1) \cosh \tfrac{1}{2}(\xi + \xi_1) - \cos 2p(\sinh \xi - \sinh \xi_1) \sinh \tfrac{1}{2}(\xi - \xi_1)}{\sin 2p(\sinh \xi_2 - \sinh \xi_1) \cosh \tfrac{1}{2}(\xi_1 + \xi_2) - \cos 2p(\sinh \xi_2 - \sinh \xi_1) \sinh \tfrac{1}{2}(\xi_2 - \xi_1)}\right.\right.$$

$$\left.\left. + \frac{\cosh \xi_1}{\cosh \xi} \cdot \frac{\sin 2p(\sinh \xi_2 - \sinh \xi) \cosh \tfrac{1}{2}(\xi + \xi_2) - \cos 2p(\sinh \xi_2 - \sinh \xi) \sinh \tfrac{1}{2}(\xi_2 - \xi)}{\sin 2p(\sinh \xi_2 - \sinh \xi_1) \cosh \tfrac{1}{2}(\xi_1 + \xi_2) - \cos 2p(\sinh \xi_2 - \sinh \xi_1) \sinh \tfrac{1}{2}(\xi_2 - \xi_1)}\right\} ce_0(\eta, q)\right] e^{-nt}.$$

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