ON SOME TRIPLE SERIES EQUATIONS

by K. N. Srivastava, *M. A. College of Technology, Bhopal, Madhya Pradesh*

(Communicated by R. S. Varma, F.N.I.)

(*Received October 25, 1966*)

Formal solutions of certain triple series equations, which can be regarded as extensions of dual series equations involving Laguerre polynomials and generalized Bateman K-functions, have been obtained. The main results of the paper are that the solution of any one set of triple series equations can be reduced to that of solving Fredholm integral equation of second kind.

1. Introduction

Certain mixed boundary value problems of mathematical physics lead to triple series equations. A method of solving such problems has recently been developed by the author (1967). A slight modification of the method renders even other similar problems amenable to this type of treatment. In this paper we shall discuss the solutions of the triple series equations

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+\alpha+1)\}L_n^{\alpha}(x) = 0, \qquad (0 \leqslant x < a,\ b < x \leqslant \infty), \quad .. \quad (1.1)$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+\alpha+\tfrac{1}{2})\}L_n^{\alpha}(x) = g(x), \quad (a < x < b),\ \alpha > -\tfrac{1}{2}, \quad .. \quad (1.2)$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+\tfrac{1}{2})\}K_{2n}^{2l}(x) = 0, \qquad (0 \leqslant x < a,\ b < x < \infty), \quad .. \quad (1.3)$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+1)\}K_{2n}^{2l}(x) = g(x), \quad (a < x < b),\ l > -\tfrac{1}{2}, \quad .. \quad (1.4)$$

defined as the series equations of the first kind and

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+\alpha+\tfrac{1}{2})\}L_n^{\alpha}(x) = \begin{cases} f(x), & (0 \leqslant x < a) \\ h(x), & (b < x \leqslant \infty), \end{cases} \quad .. \quad .. \quad (1.5)$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+\alpha+1)\}L_n^{\alpha}(n) = 0, \qquad (a < x < b),\ \alpha < -\tfrac{1}{2}, \quad .. \quad (1.6)$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+1)\}K_{2n}^{2l}(x) = \begin{cases} f(x), & (0 \leqslant x < a) \\ h(x), & (b < x \leqslant \infty) \end{cases} \quad .. \quad .. \quad (1.7)$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+\tfrac{1}{2})\}K_{2n}^{2l}(x) = 0, \qquad (a < x < b),\ l > -\tfrac{1}{2}, \quad .. \quad (1.8)$$

defined as the series equations of the second kind. Here $L_n^{\alpha}(x)$ is a Laguerre polynomial and $K_{2n}^{2l}(x)$ is a generalized Bateman K-function (Srivastava 1966*a*). In the above equations the function $f(x)$, $g(x)$ and $h(x)$ are prescribed and they are to be solved for the unknown coefficients A_n. These triple series equations are extensions of the dual series equations considered by the author in previous papers (Srivastava 1966*a*, *b*).

The main results of this paper are that the solution of each set of triple series equations can be reduced to that of solving a set of Fredholm integral equations. The analysis is purely formal and no attempt is made to justify various limiting processes.

Sections 5 and 6 are devoted to the problem of determining the coefficients A_n for triple series equations involving series of Laguerre polynomials. The last section deals with the problem for generalized Bateman K-functions where only the main results are given. These results can be derived by the method given in §§ 5 and 6.

2. Some Results and Notations

Here we list some results for ready reference. These results can be easily derived from the more general results known (Erdelyi 1954; Srivastava 1966*a*). For $\alpha > -1$, $l > -1$, we have

$$\int_0^\infty x^{\alpha} e^{-x} L_n^{\alpha}(x) L_m^{\alpha}(x)\, dx = (n+1)_{\alpha} \delta_{mn}, \quad \ldots \quad (2.1)$$

$$\int_0^\infty x^{-2l-1} K_{2m}^{2l}(x) K_{2n}^{2l}(x)\, dx = \frac{2^{2l}\Gamma(n-l)}{\Gamma(n+l+1)} \delta_{mn}, \quad \ldots \quad (2.2)$$

where δ_{mn} is the Kronecker delta, the orthogonal property for Laguerre polynomials and generalized Bateman K-functions. We also have

$$\int_0^x (x-y)^{-\frac{1}{2}} y^{\alpha-\frac{1}{2}} L_n^{\alpha-\frac{1}{2}}(y)\, dy = \frac{\Gamma(\frac{1}{2})\Gamma(n+\alpha+\frac{1}{2})}{\Gamma(n+\alpha+1)} x^{\alpha} L_n^{\alpha}(x), \quad \ldots \quad (2.3)$$

$$\int_x^\infty (y-x)^{-\frac{1}{2}} e^{-y} L_n^{\alpha}(y)\, dy = \Gamma(\tfrac{1}{2}) e^{-x} L_n^{\alpha-\frac{1}{2}}(x), \quad \ldots \quad (2.4)$$

$$\int_0^x (x-y)^{-\frac{1}{2}} e^{y} K_{2n}^{2l}(y)\, dy = \{\Gamma(\tfrac{1}{2})/2^{\frac{1}{2}}\} e^{x} K_{2n+\frac{1}{2}}^{2l+\frac{1}{2}}(x), \quad \ldots \quad (2.5)$$

$$\int_x^\infty (y-x)^{-\frac{1}{2}} e^{-y} y^{-l-1} K_{2n}^{2l}(y)\, dy = \frac{\Gamma(\frac{1}{2})\Gamma(n+l+\frac{1}{2})}{2^{\frac{1}{2}}\Gamma(n+l+1)} x^{-l-\frac{1}{2}} e^{-x} K_{2n-\frac{1}{2}}^{2l-\frac{1}{2}}(x). \quad (2.6)$$

We shall use the notations

$$E(t) = e^{t} t^{\alpha-\frac{1}{2}}, \; E_1(t) = e^{2t} t^{l+\frac{1}{2}}, \; K_a(u, x) = \int_0^a \frac{E(t)\, dt}{\{(u-t)(x-t)\}^{\frac{1}{2}}}.$$

3. Variants of Abel's Integral Equations

We shall use the following two forms. They are: If $f'(x)$ and $f(x)$ are continuous functions in $a \leqslant x \leqslant b$, then the integral equations

$$f(x) = \int_a^x (x-t)^{-\frac{1}{2}} g(t)\, dt, \qquad (3.1)$$

$$f(x) = \int_x^b (t-x)^{-\frac{1}{2}} g(t)\, dt \qquad (3.2)$$

have the solution

$$g(t) = \frac{1}{\pi} \frac{d}{dt} \int_a^t (t-x)^{-\frac{1}{2}} f(x)\, dx, \qquad (3.3)$$

$$g(t) = -\frac{1}{\pi} \frac{d}{dt} \int_t^b (x-t)^{-\frac{1}{2}} f(x)\, dx, \qquad (3.4)$$

respectively.

4. Two Summation Results

As a preparation for the study of triple series equations involving series of Laguerre polynomials or series of generalized Bateman K-functions, we establish two summation formulas. The results in question are

$$K(u, x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+1)}{\{\Gamma(n+\alpha+1)\}^2} u^{\alpha} L_n^{\alpha}(u) L_n^{\alpha}(x) = \frac{x^{-\alpha}}{\pi} \int_0^{\min(u,x)} \frac{E(t)\, dt}{\{(u-t)(x-t)\}^{\frac{1}{2}}} \qquad (4.1)$$

$$K_1(u, x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+l+\frac{1}{2})}{2^{2l}\Gamma(n-l)} e^{u} K_{2n}^{2l}(u) K_{2n}^{2l}(x) = \frac{2^{\frac{1}{2}} e^{-x} x^{l}}{\pi} \int_0^{\min(u,x)} \frac{E_1(t)\, dt}{\{(u-t)(x-t)\}^{\frac{1}{2}}}. \qquad (4.2)$$

These results are easily proved. We shall obtain (4.1) while (4.2) can be proved in the same way. Substituting the value of $L_n^{\alpha}(u)$ from (2.3) in (4.1) we have

$$K(u, x) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^u (u-y)^{-\frac{1}{2}} y^{\alpha-\frac{1}{2}} L(x, y)\, dy, \qquad (4.3)$$

where

$$L(x, y) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x) L_n^{\alpha-\frac{1}{2}}(y).$$

With the help of (2.1) and (2.4) it can be shown that

$$L(x, y) = \{\Gamma(\tfrac{1}{2})\}^{-1} (x-y)^{-\frac{1}{2}} x^{-\alpha} e^{y} H(x-y), \qquad (4.4)$$

where $H(t)$ is Heaviside's unit function. The relation (4.4) is easily proved. Let

$$L(x, y) = \sum_{n=0}^{\infty} a_n L_n^{\alpha}(x),$$

where the coefficients a_n are given by

$$a_n = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\int_0^\infty L(x, y)x^\alpha e^{-x}L_n^\alpha(x)\,dx = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\frac{e^y}{\Gamma(\frac{1}{2})}\int_y^\infty (x-y)^{-\frac{1}{2}}e^{-x}L_n^\alpha(x)\,dx$$
$$= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}L_n^{\alpha-\frac{1}{2}}(y).$$

Substituting the value of $L(x, y)$ from (4.4) in the equation (4.3) we obtain the result.

5. The Solution of the Triple Series Equations involving Series of Laguerre Polynomials

The series equations of the first kind. Let us suppose that for $a < x < b$

$$\sum_{n=0}^\infty \{A_n/\Gamma(n+\alpha+\tfrac{1}{2})\}L_n^\alpha(x) = e^x n(x). \qquad \text{(5.1)}$$

Using the orthogonal relation (2.1), the above equation along with (1.1), we have

$$A_n = \frac{\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\alpha+1)}\int_a^b n(u)u^\alpha L_n^\alpha(u)\,du. \qquad \text{(5.2)}$$

If in eqn. (1.2) we substitute for the coefficients A_n, we get, on interchanging the order of summation and integration, the equation

$$g(x) = \int_a^b n(u)K(u, x)\,du, \quad a < x < b. \qquad \text{(5.3)}$$

From (4.1) we get

$$\pi x^\alpha g(x) = \int_a^b n(u)\,du\int_0^{\min(u,x)} \frac{E(y)\,dy}{\{(u-y)(x-y)\}^{\frac{1}{2}}} \quad a < x < b. \qquad \text{(5.4)}$$

We note that the integral in the above equation is

$$\int_a^b du\int_0^{\min(u,x)} dy = \int_a^x du\int_0^u dy + \int_x^b du\int_0^x dy$$

and inverting the order of integration we may write this as

$$\int_a^x dy\int_{aj}^b du + \int_0^a dy\int_a^b du,$$

the integrands understood. Hence we have

$$\int_a^x \frac{E(y)\,dy}{(x-y)^{\frac{1}{2}}}\int_y^b \frac{n(u)\,du}{(u-y)^{\frac{1}{2}}} = \pi x^\alpha g(x) - \int_0^a \frac{E(y)\,dy}{(x-y)^{\frac{1}{2}}}\int_a^b \frac{n(u)\,du}{(u-y)^{\frac{1}{2}}}, \quad a < x < b. \qquad \text{(5.5)}$$

Let

$$N(y) = \int_y^b \frac{n(u)\,du}{(u-y)^{\frac{1}{2}}}.$$

From (3.4) we have

$$n(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^b \frac{N(y)}{(y-u)^{\frac{1}{2}}}\,dy. \qquad \ldots \quad \ldots \quad \ldots \quad (5.6)$$

Equation (5.6) is of the form

$$\int_a^x \frac{E(y)N(y)}{(y-x)^{\frac{1}{2}}}\,dy = M(x).$$

From (3.3) we have

$$E(y)N(y) = \frac{d}{du}\int_a^y \frac{x^\alpha g(x)}{(y-x)^{\frac{1}{2}}}\,dx - \frac{1}{\pi}I, \quad a < y < b, \qquad \ldots \quad \ldots \quad (5.7)$$

where

$$I = \frac{d}{dy}\int_a^y \frac{dx}{(y-x)^{\frac{1}{2}}}\int_0^a \frac{E(t)\,dt}{(x-t)^{\frac{1}{2}}}\int_a^b \frac{n(u)}{(u-t)^{\frac{1}{2}}}\,du.$$

It is easy to show that

$$\frac{d}{dy}\int_a^y \frac{dx}{\{(y-x)(x-t)\}^{\frac{1}{2}}} = \frac{(a-t)^{\frac{1}{2}}}{(y-a)^{\frac{1}{2}}(y-t)}.$$

Hence

$$I = \int_0^a \frac{E(t)\,dt}{(y-a)^{\frac{1}{2}}(y-t)}\int_a^b \frac{n(u)}{(u-t)^{\frac{1}{2}}}\,du. \qquad \ldots \quad \ldots \quad \ldots \quad (5.8)$$

We write (5.6) in the form

$$n(u) = -\frac{1}{\pi}\frac{d\tau}{du},$$

where

$$\tau = \int_u^b \frac{N(s)}{(s-u)^{\frac{1}{2}}}\,ds,$$

and the last integral in (5.8) in the form

$$R = -\frac{1}{\pi}\int_a^b \frac{d\tau}{du}\frac{du}{(u-t)^{\frac{1}{2}}}.$$

On integrating by parts, we get

$$R = \frac{1}{\pi(a-t)^{\frac{1}{2}}}\int_a^b \frac{N(s)\,ds}{(s-a)^{\frac{1}{2}}} - \frac{1}{2\pi}\int_a^b N(s)\,ds\int_a^s \frac{du}{(u-t)^{\frac{3}{2}}(s-u)^{\frac{1}{2}}}$$

after substituting the value of J and inverting the order of integration in the last term. Hence

$$R = \frac{1}{\pi(a-t)^{\frac{1}{2}}}\int_a^b \frac{N(s)}{(s-a)^{\frac{1}{2}}}\left\{1 - \frac{s-a}{s-t}\right\}ds.$$

We have used the result

$$\int_a^s \frac{du}{(u-t)^{\frac{3}{2}}(s-u)^{\frac{1}{2}}} = \frac{2(s-a)^{\frac{1}{2}}}{(s-t)(a-t)^{\frac{1}{2}}}.$$

Hence

$$I = \frac{1}{\pi}\int_a^b N(s)R(s, y)\, ds \quad \ldots \quad (5.9)$$

where

$$R(s, y) = \int_0^a \frac{E(t)(a-t)\, dt}{(y-t)(s-t)\{(s-a)(y-a)\}^{\frac{1}{2}}}.$$

Hence

$$E(y)N(y) = \frac{d}{dy}\int_a^y \frac{x^\alpha g(x)}{(y-x)^{\frac{1}{2}}}\, dx - \frac{1}{\pi^2}\int_a^b N(s)R(s, y)\, ds, \quad a < y < b, \quad (5.10)$$

where $R(s, y)$ is a symmetric kernel. This is a Fredholm integral equation of the second kind which determines the function $N(y)$, $n(u)$ is found from eqn. (5.6) and the coefficients A_n are calculated from eqn. (5.2).

6. The Triple Series Equations of the Second Kind

We begin with the assumption

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+\alpha+\tfrac{1}{2})\}L_n^\alpha(x) = e^x l(x), \quad 0 \leqslant x < a, \quad \ldots \quad (6.1)$$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+\alpha+\tfrac{1}{2})\}L_n^\alpha(x) = e^x m(x), \quad b < x \leqslant \infty. \quad \ldots \quad (6.2)$$

These equations along with (1.6) and the orthogonality relation (2.1) give

$$A_n = \frac{\Gamma(n+\alpha+\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\alpha+1)}\left\{\int_0^a l(u) + \int_b^\infty m(u)\right\} u^\alpha L_n^\alpha(u)\, du. \quad \ldots \quad (6.3)$$

Substituting the value of the coefficients A_n from (6.3) in the eqns. (1.5), we find on interchanging the order of summation and integration that

$$f(x) = \left\{\int_0^a l(u) + \int_b^\infty m(u)\right\} K(u, x)\, du, \quad 0 \leqslant x < a, \quad \ldots \quad (6.4)$$

$$h(x) = \left\{\int_0^a l(u) + \int_b^\infty m(u)\right\} K(u, x)\, du, \quad b < x \leqslant \infty. \quad \ldots \quad (6.5)$$

The relation (6.4) can be written as

$$\pi x^\alpha f(x) = \int_0^a l(u)\, du \int_0^{\min(u,\, x)} \frac{E(y)\, dy}{\{(u-y)(x-y)\}^{\frac{1}{2}}} + \int_b^\infty m(u)K_x(u, x)\, du = I_1 + I_2.$$

We deal with I_1, $(0 \leqslant x < a)$ in the same way as in § 5 except that the limits now are from 0 to a instead of from a to b. We find that

$$\int_0^x \frac{E(y)\, dy}{(x-y)^{\frac{1}{2}}}\int_y^a \frac{l(u)}{(u-y)^{\frac{1}{2}}}\, du = \pi x^\alpha f(x) - I_2.$$

Hence

$$E(y)\int_y^a \frac{l(u)}{(u-y)^{\frac{1}{2}}}\,du = -\frac{d}{dy}\int_0^y \frac{x^\alpha f(x)}{(y-x)^{\frac{1}{2}}}\,dx+\frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{I_2\,dx}{(y-x)^{\frac{1}{2}}}.$$

Consider the second integral

$$\frac{d}{dy}\int_0^y \frac{dx}{(y-x)^{\frac{1}{2}}}\int_b^\infty m(u)K_x(u,x)\,du = \int_b^\infty m(u)\,du\,\frac{d}{dy}\int_0^y \frac{E(t)\,dt}{(u-t)^{\frac{1}{2}}}\int_t^y \frac{dx}{\{(y-x)(x-t)\}^{\frac{1}{2}}}$$

$$= \int_b^\infty \frac{m(u)E(y)}{(u-y)^{\frac{1}{2}}}\pi\,du$$

since

$$\int_t^y \frac{dx}{\{(y-x)(x-t)\}^{\frac{1}{2}}} = \pi.$$

Hence

$$\int_y^a \frac{l(u)}{(u-y)^{\frac{1}{2}}}\,du = F(y)-\int_b^\infty \frac{m(u)}{(u-y)^{\frac{1}{2}}}\,du, \quad 0 \leqslant y < a, \qquad (6.6)$$

where

$$F(y) = -\frac{1}{E(y)}\frac{d}{dy}\int_0^y \frac{x^\alpha f(x)}{(y-x)^{\frac{1}{2}}}\,dx$$

is a known function since $f(x)$ is prescribed. Again using (3.4) we get

$$l(u) = F_1(u)-\frac{1}{\pi}\frac{d}{du}\int_b^\infty m(t)\,dt\int_u^a \frac{dy}{(y-u)^{\frac{1}{2}}(t-u)^{\frac{1}{2}}}$$

$$= F_1(u)-\frac{1}{\pi}\int_b^\infty \frac{m(t)(t-a)^{\frac{1}{2}}}{(a-u)^{\frac{1}{2}}(t-u)}\,dt, \quad 0 \leqslant u < a, \qquad (6.7)$$

where

$$F_1(u) = \frac{1}{\pi}\frac{d}{du}\int_u^a \frac{F(y)}{(y-u)^{\frac{1}{2}}}\,dy.$$

Here we have used the result

$$\frac{d}{dy}\int_u^a \frac{dy}{\{(y-u)(t-u)\}^{\frac{1}{2}}} = \frac{(t-a)^{\frac{1}{2}}}{(a-u)^{\frac{1}{2}}(t-u)}.$$

We next consider the eqn. (6.5) which can be written as

$$\pi x^\alpha h(x) = \int_0^a l(u)K_u(u,x)\,du+\int_b^\infty m(u)\,du\int_0^{\min(u,\,x)} \frac{E(y)\,dy}{\{(u-y)(x-y)\}^{\frac{1}{2}}}\,dy,$$

$$b < x \leqslant \infty.$$

We deal with the last integral in the same way as in § 5 except that the limits in the present case are from b to ∞ instead of from a to b. We find that

$$\int_b^x \frac{E(y)M(y)\,dy}{(x-y)^{\frac{1}{2}}} = \pi x^\alpha h(x)-\int_0^a l(u)K_u(u,x)\,du-\int_b^\infty m(u)K_b(u,x)\,du = z(x) \text{ (say)} \qquad (6.8)$$

where

$$M(y) = \int_y^\infty \frac{m(u)}{(u-y)^{\frac{1}{2}}}\,du.$$

From (3.4) we have

$$m(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^\infty \frac{M(y)}{(y-u)^{\frac{1}{2}}}\,dy. \qquad (6.9)$$

The eqn. (6.7) is an Abel integral equation and its solution is

$$\pi E(y)M(y) = \frac{d}{dy}\int_b^y \frac{z(x)}{(y-x)^{\frac{1}{2}}}\,dx = I_0 - I_1 - I_2, \qquad (6.10)$$

where

$$I_0 = \frac{d}{dy}\int_b^y \frac{\pi x^\alpha h(x)}{(y-x)^{\frac{1}{2}}}\,dx,$$

$$I_1 = \frac{d}{dy}\int_b^y \frac{dx}{(y-x)^{\frac{1}{2}}}\int_0^a l(u)K_u(u, x)\,du,$$

$$I_2 = \frac{d}{dy}\int_b^y \frac{dx}{(y-x)^{\frac{1}{2}}}\int_b^\infty m(u)K_b(u, x)\,du.$$

First consider

$$I_1 = \int_0^a l(u)\,du \int_0^u \frac{E(t)\,dt}{(u-t)^{\frac{1}{2}}}\frac{d}{dy}\int_b^y \frac{dx}{\{(y-x)(x-t)\}^{\frac{1}{2}}}$$

$$= \frac{1}{(y-b)^{\frac{1}{2}}}\int_0^a \frac{E(t)(b-t)^{\frac{1}{2}}}{y-t}\,dt\int_t^a \frac{l(u)}{(u-t)^{\frac{1}{2}}}\,du.$$

Introducing the value of the last integral in the above equation from (6.6) we get

$$(y-b)^{\frac{1}{2}}I_1 = F_2(y) - \int_b^\infty m(u)\,du\int_0^a \frac{E(t)(b-t)^{\frac{1}{2}}}{(y-t)(u-t)^{\frac{1}{2}}}\,dt,$$

where

$$F_2(y) = \int_0^a \frac{E(y)F(t)(b-t)^{\frac{1}{2}}}{(y-t)}\,dt.$$

Next consider

$$I_2 = \int_b^\infty m(u)\,du\int_0^b \frac{E(t)\,dt}{(u-t)^{\frac{1}{2}}}\frac{d}{dy}\int_b^y \frac{dx}{\{(y-x)(x-t)\}^{\frac{1}{2}}}$$

$$= \int_b^\infty m(u)\,du\int_0^b \frac{E(t)(b-t)^{\frac{1}{2}}}{\{(y-b)(u-t)\}^{\frac{1}{2}}(y-t)}\,dt.$$

Hence we have

$$(I_1+I_2)(y-b)^{\frac{1}{2}} = F_2(y) + \int_b^\infty m(u)\,du\int_a^b \frac{E(t)(b-t)^{\frac{1}{2}}}{(u-t)^{\frac{1}{2}}(y-t)}\,dt$$

$$= F_2(y) + \int_a^b \frac{E(t)(b-t)^{\frac{1}{2}}}{(y-t)}R\,dt,$$

where

$$R = \int_b^\infty \frac{m(u)}{(u-t)^{\frac{1}{2}}}\,du.$$

From eqn. (6.9) we have

$$\pi R = -\int_b^\infty \frac{dJ}{du}\,\frac{du}{(u-t)^{\frac{1}{2}}},$$

where

$$J = \int_u^\infty \frac{M(s)}{(s-u)^{\frac{1}{2}}}\,ds.$$

We deal with the above integral as in § 5. This yields

$$\pi R = \frac{1}{(b-t)^{\frac{1}{2}}}\int_b^\infty \frac{M(s)(b-t)}{(s-b)^{\frac{1}{2}}(s-t)}\,ds.$$

The eqn. (6.10) can be written as

$$E(y)M(y) = \frac{d}{dy}\int_b^y \frac{x^{\alpha}h(x)}{(y-x)^{\frac{1}{2}}}\,dx - \frac{1}{\pi}\,\frac{F_2(y)}{(y-b)^{\frac{1}{2}}} - \frac{1}{\pi^2}\int_b^\infty M(s)T(s,\,y)\,ds, \quad b < y \leqslant \infty, \qquad (6.11)$$

where

$$T(s,\,y) = \int_b^\infty \frac{E(t)(b-t)}{(s-t)(y-t)\{(y-b)(s-b)\}^{\frac{1}{2}}}\,dt$$

is a symmetric kernel. In the above equation, the first two expressions on the right-hand side are known functions. This is a Fredholm integral equation which determines $M(y)$; and $m(u)$ is obtained from eqn. (6.9). Then eqn. (6.7) determines $l(u)$ and the coefficients A_n are calculated from (6.3).

7. Triple Series Equations involving Series of Generalized Bateman K-functions

For triple series equations of the first kind we begin with the assumption that for $a < x < b$

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+\tfrac{1}{2})\}K_{2n}^{2l}(x) = e^x x^{2l+1} z(x). \qquad (7.1)$$

The coefficients A_n are given by

$$A_n = \frac{\Gamma(n+l+\frac{1}{2})\Gamma(n+l+1)}{2^{2l}\Gamma(u-l)}\int_a^b e^u z(u)K_{2n}^{2l}(u)\,du, \qquad (7.2)$$

where the function $z(u)$ is determined from the integral equations

$$E_1(y)Z(y) = \frac{1}{2^{\frac{1}{2}}}\,\frac{d}{dy}\int_a^y \frac{x^{-l}e^x g(x)}{(y-x)^{\frac{1}{2}}}\,dx - \frac{1}{\pi^2}\int_a^b Z(s)R_1(s,\,y)\,ds, \quad a < y < b, \qquad (7.3)$$

$$z(u) = -\frac{1}{\pi}\,\frac{d}{du}\int_u^b \frac{Z(u)\,du}{(y-u)^{\frac{1}{2}}}. \qquad (7.4)$$

Here

$$R_1(s, y) = \int_0^a \frac{E_1(t)(a-t)}{(y-t)(s-t)\{(y-a)(s-a)\}^{\frac{1}{2}}} dt$$

is a symmetric kernel.

In like manner, for triple series equations of the second kind we start with the assumption that

$$\sum_{n=0}^{\infty} \{A_n/\Gamma(n+l+\tfrac{1}{2})\}K_{2n}^{2l}(x) = e^x x^{2l+1} \begin{cases} p(x), & 0 \leqslant x < a, \\ q(x), & b < x \leqslant \infty. \end{cases} \quad \text{.. (7.5)}$$

The coefficients A_n are determined from the equation

$$A_n = \frac{\Gamma(n+l+\frac{1}{2})\Gamma(n+l+1)}{2^{2l}\Gamma(n-l)} \left\{ \int_0^a p(u) + \int_b^{\infty} q(u) \right\} e^u K_{2n}^{2l}(u)\, du. \quad \text{(7.6)}$$

The functions $p(u)$ and $q(u)$ are determined from the integral equations

$$E_1(y)Q(y) = \phi_1(y) - \frac{1}{\pi^2}\int_b^{\infty} Q(s)T_1(s, y)\, ds, \quad b < y \leqslant \infty \quad \text{.. (7.7)}$$

$$q(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^{\infty} \frac{Q(y)}{(y-u)^{\frac{1}{2}}} dy \quad \text{.. .. .. .. (7.8)}$$

$$p(u) = \psi_1(u) - \frac{1}{\pi}\int_b^{\infty} \frac{q(t)(t-a)^{\frac{1}{2}}}{(a-u)^{\frac{1}{2}}(t-u)} dt, \quad 0 \leqslant u < a, \quad \text{.. (7.9)}$$

where

$$\phi_1(y) = \frac{1}{2^{\frac{1}{2}}}\frac{d}{dy}\int_b^y \frac{x^{-l}e^x h(x)}{(y-x)^{\frac{1}{2}}} dx + \frac{1}{\pi 2^{\frac{1}{2}}(y-b)^{\frac{1}{2}}}\int_0^a \frac{(b-t)^{\frac{1}{2}}}{y-t}\left\{\frac{d}{dt}\int_0^t \frac{x^{-l}e^x f(x)\, dx}{(t-x)^{\frac{1}{2}}}\right\} dt$$

$$\psi_1(u) = -\frac{1}{2^{\frac{1}{2}}\pi}\frac{d}{du}\int_u^a \frac{dy}{E_1(y)(y-u)^{\frac{1}{2}}}\left\{\frac{d}{dy}\int_0^y \frac{e^x x^{-l} f(x)}{(y-x)^{\frac{1}{2}}} dx\right\}$$

are known functions since $f(x)$ and $h(x)$ are prescribed and

$$T_1(s, y) = \int_a^b \frac{E_1(t)(b-t)}{(s-t)(y-t)\{(s-b)(y-b)\}^{\frac{1}{2}}} dt$$

is a symmetric kernel.

References

Erdelyi, A. (1954). Tables of Integral Transforms. Vol. 2. McGraw-Hill Book Co. Inc., New York.

Srivastava, K. N. (1966*a*). On dual series relations involving series of generalized Bateman *K*-functions. *Proc. Am. math. Soc.*, **17**, 796–802.

——— (1966*b*). On dual series relations involving series of Laguerre polynomials. *Pacif. J. Math.*, **19**, 529–533.

——— (1967). On triple series equations involving series of Jacobi polynomials. *Proc. Edinb. math. Soc.*, **15**.