GREEN'S FUNCTION OF AN ELASTIC RING BOUNDED BY SIMILAR PASCAL'S LIMAÇONS AND CLAMPED ALONG EACH BOUNDARY

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In this paper an expression is found for the transverse displacement due to an isolated load in a thin elastic ring bounded by similar Pascal's limaçons and clamped along each boundary. The displacement is found as an infinite convergent series. The method for determining the coefficients occurring in the series is not direct since the number of unknown coefficients is greater than the number of equations. The coefficients can be determined in terms of one of the unknown coefficients and the value of this unknown coefficient can be determined by applying a condition of convergence (Singh 1960). The numerical evaluation has been done for a few simple cases. Expressions for the reactions on the boundaries have been obtained.

1. Introduction

The plate (Fig. 1) is bounded by the limaçons whose parametric equations are

$$\left.\begin{aligned} x = A(\rho \cos \phi + m\rho^2 \cos 2\phi), \quad y = A(\rho \sin \phi + m\rho^2 \sin 2\phi); \\ \rho = \rho_1, \rho_2. \end{aligned}\right\}, \quad \ldots \quad (1)$$

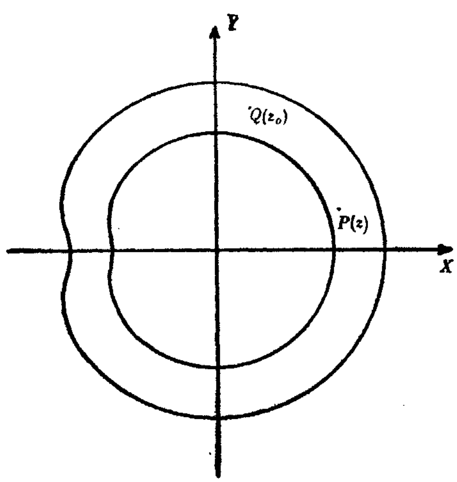

Fig. 1

The given region can be mapped conformally on the one lying between the two concentric circles

$$\rho = b, \ \ \rho = a, \ \ (1 \geqslant b \geqslant a)$$

in the ζ-plane (Fig. 2) by the transformation (Mushkhelishvili 1953)

$$z = A(\zeta + m\zeta^2), \ \zeta = \rho e^{i\phi}, \qquad \qquad (2)$$

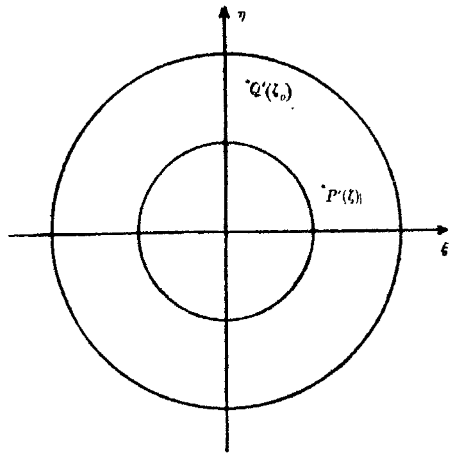

FIG. 2

where A is real and positive and $0 < m < \frac{1}{2}$, giving

$$\left.\begin{aligned} \frac{x}{a} &= \rho \cos \phi + m\rho^2 \cos 2\phi, \\ \frac{y}{A} &= \rho \sin \phi + m\rho^2 \sin 2\phi, \\ \frac{r^2}{A^2} &= \rho^2 + m^2\rho^4 + 2m\rho^3 \cos \phi. \end{aligned}\right\} \qquad (3)$$

A transverse force F is supposed applied at the point $Q(z_0)$ and w denotes the small transverse displacement of the plate at the point $P(z)$. Then w is a biharmonic function of the variables x, y and is (apart from a constant factor) the sum of $|z-z_0|^2 \log |z-z_0|$ and a function with no singularity at Q. The boundary conditions require that the displacement and its normal derivative must vanish on the clamped boundaries.

2. DISPLACEMENT

The problem of finding the displacement w has been broken up into two parts:

(i) to determine the displacement w_1 caused by two forces, each equal to F at the points $(x_0, \pm y_0)$;

(ii) to determine the displacement w_2 caused by a force F at (x_0, y_0) and an opposite force $(-F)$ at the point $(x_0, -y_0)$.

Then Green's function G and the actual displacement w at $P(x, y)$ for a transverse force F applied at $Q(x_0, y_0)$ are given by

$$\left.\begin{aligned} w_r &= \frac{F}{8\pi D} G_r, \\ w &= \tfrac{1}{2}(w_1+w_2),\ G = \tfrac{1}{2}(G_1+G_2). \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

3. Determination of G_1

To determine G_1 we start with

$$G_1' = |z-z_0|^2 \log |z-z_0| + |z-\bar{z}_0|^2 \log |z-\bar{z}_0|. \quad \ldots \quad \ldots \quad (5)$$

If $\zeta_0 = ce^{i\lambda}$ is the image point corresponding to $z_0 = x_0+iy_0$ and $1 \geqslant c \geqslant \rho$, we have

$$\left.\begin{aligned} |z-z_0|^2 \log |z-z_0| &= \sum_{n=0}^{\infty} P_n \cos n\phi + \sum_{n=1}^{\infty} Q_n \sin n\phi, \\ |z-\bar{z}_0|^2 \log |z-\bar{z}_0| &= \sum_{n=0}^{\infty} P_n \cos n\phi - \sum_{n=1}^{\infty} Q_n \sin n\phi \end{aligned}\right\} \quad \ldots \quad (6)$$

where it can be shown that

$$\begin{aligned} P_0 = A^2[&\{(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4) \log \overline{Ac(1+m\rho)}+\rho c U_1+m^2\rho^2c^2U_2\} \\ &+\{2mc^3 \log \overline{Ac(1+m\rho)}+(m\rho c^2-m\rho^3)U_1+m\rho^2cU_2\} \cos \lambda], \\ P_1 = A^2[&\{mc(\rho^2-c^2)U_1+m\rho c^2U_2+2m\rho^3 \log \overline{Ac(1+m\rho)}\} \\ &+\{\rho cU_2-(\rho^2+c^2+m^2\rho^2c^2+m^2\rho^4+m^2c^4)U_1+m^2\rho^2c^2U_3 \\ &\qquad -2\rho c \log \overline{Ac(1+m\rho)}\} \cos \lambda \\ &+\{m\rho^2cU_3-m\rho^3U_2-mc^3U_1-2m\rho c^2 \log \overline{Ac(1+m\rho)}\} \cos 2\lambda], \\ P_2 = A^2[&\{m\rho c^2U_3-mc^3U_2-m\rho^3U_1-2m\rho^2c \log \overline{Ac(1+m\rho)}\} \cos \lambda \\ &+\{m^2\rho^2c^2U_4+\rho cU_3-(\rho^2+c^2+m^2\rho^4+m^2c^4+2m^2\rho^2c^2)U_2+\rho cU_1 \\ &\qquad -2m^2\rho^2c^2 \log \overline{Ac(1+m\rho)}\} \cos 2\lambda \\ &+\{m\rho^2cU_4-m\rho^3U_3-mc^3U_2+m\rho c^2U_1\} \cos 3\lambda], \\ P_n = A^2[&\{m\rho c^2U_{n+1}-mc^3U_n-m\rho^3U_{n-1}+m\rho^2cU_{n-2}\} \cos (n-1)\lambda \\ &+\{m^2\rho^2c^2U_{n+2}+\rho cU_{n+1}-(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4)U_n \\ &\qquad +\rho cU_{n-1}+m^2\rho^2c^2U_{n-2}\} \cos n\lambda \\ &+\{m\rho^2cU_{n+2}-mc^3U_n+m\rho c^2U_{n-1}-m\rho^3U_{n+1}\} \cos (n+1)\lambda] \end{aligned}$$

$$\text{for } n > 3 \quad \ldots \quad (7)$$

and

$$Q_1 = A^2[\{m^2\rho^2c^2U_3+\rho cU_2-(\rho^2+c^2+m^2\rho^2c^2+m^2\rho^4+m^2c^4)U_1-2\rho c \log \overline{Ac(1+m\rho)}\} \sin \lambda$$
$$+\{m\rho^2cU_3-m\rho^3U_2-mc^3U_1-2m\rho c^2 \log \overline{Ac(1+m\rho)}\} \sin 2\lambda],$$
$$Q_2 = A^2[\{m\rho c^2U_3-mc^3U_2-m\rho^3U_1-2m\rho^2c \log \overline{Ac(1+m\rho)}\} \sin \lambda$$
$$+\{m^2\rho^2c^2U_4+\rho cU_3-(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4)U_2+\rho cU_1$$
$$-2m^2\rho^2c^2 \log \overline{Ac(1+m\rho)}\} \sin 2\lambda$$
$$+\{m\rho^2cU_4-m\rho^3U_3+m\rho c^2U_1-mc^3U_2\} \sin 3\lambda],$$
$$Q_n = A^2[\{m\rho^2cU_{n+1}-mc^3U_n-m\rho^3U_{n-1}+m\rho^2cU_{n-2}\} \sin (n-1)\lambda$$
$$+\{m^2\rho^2c^2U_{n+2}+\rho cU_{n+1}-(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4)U_n+\rho cU_{n-1}$$
$$+m^2\rho^2c^2U_{n-2}\} \sin n\lambda$$
$$+\{m\rho^2cU_{n+2}-m\rho^3U_{n+1}-mc^3U_n+m\rho c^2U_{n-1}\} \sin (n+1)\lambda]$$
$$\text{for } n \geqslant 3, \quad .. \quad (8)$$

where

$$U_n = \frac{1}{n}\left\{\left(\frac{\rho}{c}\right)^n+(-1)^n\left(\frac{mc}{1+m\rho}\right)^n\right\}$$

If $1 \geqslant \rho > c$, we have

$$\left.\begin{aligned} |z-z_0|^2 \log |z-z_0| &= \sum_{n=0}^{\infty} R_n \cos n\phi + \sum_{n=1}^{\infty} S_n \sin n\phi, \\ |z-\bar{z}_0|^2 \log |z-\bar{z}_0| &= \sum_{n=0}^{\infty} R_n \cos n\phi - \sum_{n=1}^{\infty} S_n \sin n\phi \end{aligned}\right\} \quad .. \quad (9)$$

where

$$R_0 = A^2[\{m^2\rho^2c^2V_2+\rho cV_1+(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4) \log \overline{A\rho(1+m\rho)}\}$$
$$+\{m\rho^2cV_2+m\rho(c^2-\rho^2)V_1+2mc^3 \log \overline{A\rho(1+m\rho)}\}\cos \lambda],$$
$$R_1 = A^2[\{m\rho c^2V_2+mc(\rho^2-c^2)V_1+2m\rho^3 \log \overline{A\rho(1+m\rho)}\}$$
$$+\{m^2\rho^2c^2V_3+\rho cV_2-(\rho^2+c^2+m^2\rho^2c^2+m^2\rho^4+m^2c^4)V_1-2\rho c \log \overline{A\rho(1+m\rho)}\} \cos \lambda$$
$$+\{m\rho^2cV_3-m\rho^3V_2-mc^3V_1-2\rho c^2m \log \overline{A\rho(1+m\rho)}\} \cos 2\lambda],$$
$$R_2 = A^2[\{m\rho c^2V_3-mc^3V_2-m\rho^3V_1-2m\rho^2c \log A\rho(1+m\rho)\} \cos \lambda$$
$$+\{m^2\rho^2c^2V_4+\rho cV_3-(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4)V_2+\rho cV_1$$
$$-2m^2\rho^2c^2 \log \overline{A\rho(1+m\rho)}\} \cos 2\lambda$$
$$+\{m\rho^2cV_4-m\rho^3V_3-mc^3V_2+m\rho c^2V_1\} \cos 3\lambda],$$
$$R_n = A^2[\{m\rho c^2V_{n+1}-mc^3V_n-m\rho^3V_{n-1}+m\rho^2cV_{n-2}\} \cos (n-1)\lambda$$
$$+\{m^2\rho^2c^2V_{n+2}+\rho cV_{n+1}-(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4)V_n+\rho cV_{n-1}$$
$$+m^2\rho^2c^2V_{n-2}\} \cos n\lambda$$
$$+\{m\rho^2cV_{n+2}-m\rho^3V_{n+1}-mc^3V_n+m\rho c^3V_{n-1}\} \cos (n+1)\lambda]$$
$$\text{for } n \geqslant 3, \quad .. \quad (10)$$

and

$$S_1 = A^2[\{m^2\rho^2c^2V_3+\rho cV_2-(\rho^2+c^2+m^2\rho^2c^2+m^2\rho^4+m^2c^4)V_1- \\ -2\rho c\log\overline{A\rho(1+m\rho)}\}\sin\lambda \\ +\{m\rho^2cV_3-m\rho^3V_2-2m\rho c^2\log\overline{A\rho(1+m\rho)}-mc^3V_1\}\sin 2\lambda],$$

$$S_2 = A^2[\{m\rho c^2V_3-mc^3V_2-m\rho^3V_1-2m\rho^3c\log\overline{A\rho(1+m\rho)}\}\sin\lambda \\ +\{m^2\rho^2c^2V_4+\rho cV_3-(\rho^2+c^2+2m^2\rho^2c^2+m^2\rho^4+m^2c^4)V_2+\rho cV_1 \\ -2m^2\rho^2c^2\log\overline{A\rho(1+m\rho)}\}\sin 2\lambda \\ +\{m\rho^2cV_4-m\rho^3V_3-mc^3V_2+m\rho c^2V_1\}\sin 3\lambda],$$

$$S_n = A^2[\{m\rho c^2V_{n+1}-mc^3V_n-m\rho^3V_{n-1}+m\rho^2cV_{n-2}\}\sin(n-1)\lambda \\ +\{\rho cV_{n+1}+m^2\rho^2c^2V_{n+2}-(\rho^2+c^2+m^2\rho^4+2m^2\rho^2c^2+m^2c^4)V_n+\rho cV_{n-1} \\ +m^2\rho^2c^2V_{n-2}\}\sin n\lambda \\ +\{m\rho^2cV_{n+2}-m\rho^3V_{n+1}-mc^3V_n+m\rho c^2V_n\}\sin(n+1)\lambda \\ \text{for } n \geqslant 3, \quad \ldots \quad (11)$$

where

$$V_n = \frac{1}{n}\left\{\left(\frac{c}{\rho}\right)^n+(-1)^n\left(\frac{mc}{1+m\rho}\right)^n\right\}.$$

It follows that

$$\left.\begin{aligned} G_1' &= \sum_{n=0}^{\infty} A_n\cos n\phi, \quad \frac{\partial G_1'}{\partial\rho} = \sum_{n=0}^{\infty} A_n'\cos n\phi, \text{ on } \rho = a, \\ G_1' &= \sum_{n=0}^{\infty} B_n\cos n\phi, \quad \frac{\partial G_1'}{\partial\rho} = \sum_{n=0}^{\infty} B_n'\cos n\phi, \text{ on } \rho = b, \end{aligned}\right\} \quad \ldots \quad (12)$$

where

$$\left.\begin{aligned} A_n &= 2(P_n)_{\rho=a}, \quad A_n' = 2\left(\frac{\partial P_n}{\partial\rho}\right)_{\rho=a} \\ B_n &= 2(R_n)_{\rho=b}, \quad B_n' = 2\left(\frac{\partial R_n}{\partial\rho}\right)_{\rho=b} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (13)$$

The function G_1 can be expressed as

$$G_1 = G_1'-G_1'', \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where G_1'' is a biharmonic function with no singularity in the region covered by the plate and which satisfies

$$\left.\begin{aligned} G_1'' &= \sum_{n=0}^{\infty} A_n\cos n\phi, \quad \frac{\partial G_1''}{\partial\rho} = \sum_{n=0}^{\infty} A_n'\cos n\phi \text{ when } \rho = a, \\ G_1'' &= \sum_{n=0}^{\infty} B_n\cos n\phi, \quad \frac{\partial G''}{\partial\rho} = \sum_{n=0}^{\infty} B_n'\cos n\phi \text{ when } \rho = b. \end{aligned}\right\} \quad \ldots \quad (15)$$

It is clear that G_1'' is an even function of x and y. We can therefore assume

$$G_1'' = \frac{r^2}{A^2}\left\{(J_0'+K_0' \log \rho)+\sum_{n=1}^{\infty}(J_n'\rho^n+K_n'\rho^{-n}) \cos n\phi\right\}+\frac{x}{A}K \log \rho$$
$$+\left\{(L_0+M_0 \log \rho)+\sum_{n=1}^{\infty}(L_n\rho^n+M_n\rho^{-n}) \cos n\phi\right\}. \quad .. \quad .. \quad (16)$$

Substituting for $\frac{r^2}{A^2}$ and $\frac{x}{A}$ from (3), we can write

$$\begin{aligned} G_1'' = {} & \{m\rho^4 J_1+\rho^2 J_0+(\rho^2+m^2\rho^2)K_0 \log \rho+L_0+M_0 \log \rho\} \\ & +\{m\rho^5 J_2+\rho^3(mJ_0+J_1)+\rho K_1+(2m\rho^3 \log \rho)K_0+K\rho \log \rho+M_1\rho^{-1}\} \cos \phi \\ & +\{m\rho^6 J_3+\rho^4 J_2+K_3+\rho^2 K_2+Km\rho^2 \log \rho+M_2\rho^{-2}\} \cos 2\phi \\ & +\sum_{n\geqslant 3}^{\infty}\{m\rho^{n+4}J_{n+1}+\rho^{n+2}J_n+\rho^{-(n-2)}K_{n+1}+m\rho^{-(n-u)}K_n+L_n\rho^n+M_n\rho^{-n}\} \cos n\phi \end{aligned} \quad .. \quad (17)$$

and

$$\begin{aligned} \left(\rho\frac{\partial G_1''}{\partial \rho}\right) = {} & \{4m\rho^4 J_1+2\rho^2 J_0+((2\rho^2+4m^2\rho^4) \log \rho+(\rho^2+m^2\rho^4))K_0+M_0\} \\ & +\{5m\rho^5 J_2+3\rho^3(mJ_0+J_1)+\rho K_1+2m\rho^3(3 \log \rho+1)K_0+K\rho (\log \rho+1)-M_1\rho^{-1}\} \cos \phi \\ & +\{6m\rho^6 J_3+4\rho^4 J_2+2\rho^2 K_2+mK\rho^2 (2 \log \rho+1)-2M_2\rho^{-2}\} \cos 2\phi \\ & +\sum_{n\geqslant 3}^{\infty}\{m(n+4)\rho^{n+4}J_{n+1}+(n+2)\rho^{n+2}J_n-(n-2)\rho^{-(n-2)}K_{n+1}-(n-4)m\rho^{-(n-4)}K_n \\ & \qquad +nL_n\rho^n-nM_n\rho^{-n}\} \cos n\phi, \end{aligned} \quad .. \quad (18)$$

where

$$J_0'+mK_1' = J_0, \quad mJ_n'+J_{n+1}' = J_{n+1} \quad (n = 0, 1, 2, \ldots),$$
$$K_0' = K_0, \; mK_2'+L_1 = K_1, \quad m(K_1'+mK_2')+L_2 = K_2,$$
$$K_n'+mK_{n+1}' = K_{n+1} \quad (n = 2, 3, \ldots).$$

The equations to determine the unknown coefficients J_n, K_n, L_n, M_n are obtained by substituting (17), (18) in (15) and equating the coefficients of $\cos n\phi$ ($n = 0, 1, 2 \ldots$). We have

$$\left.\begin{aligned} ma^4 J_1+a^2 J_0+K_0(a^2+m^2a^4) \log a+L_0+M_0 \log a &= A_0, \\ 4ma^4 J_1+2a^2 J_0+K_0\{(2a^2+4m^2a^4) \log a+(a^2+m^2a^4)\}+M_0 &= aA_0', \\ mb^4 J_1+b^2 J_0+K_0(b^2+m^2b^4) \log b+L_0+M_0 \log b &= B_0, \\ 4mb^4 J_1+2b^2 J_0+K_0\{(2b^2+4m^2b^4) \log b+(b^2+m^2b^4)\}+M_0 &= bB_0', \end{aligned}\right\} \quad .. \quad (19)$$

$$\left.\begin{aligned} ma^5J_2+a^3(mJ_0+J_1)+K_0(2ma^3\log a)+aK_1+Ka\log a+M_1a^{-1} &= A_1,\\ 5ma^5J_2+3a^3(mJ_0+J_1)+2ma^3K_0(3\log a+1)+aK_1+aK(\log a+1)-M_1a^{-1} &= aA_1',\\ mb^5J_2+b^3(mJ_0+J_1)+K_0(2mb^3\log b)+bK_1+Kb\log b+M_1b^{-1} &= B_1,\\ 5mb^5J_2+3b^3(mJ_0+J_1)+2mb^3K_0(3\log b+1)+bK_1+bK(\log b+1)-M_1b^{-1} &= bB_1'; \end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned} ma^6J_3+a^4J_2+K_3+a^2K_2+(ma^2\log a)K+M_2a^{-2} &= A_2,\\ 6ma^6J_3+4a^4J_2+2a^2K_2+Kma^2(2\log a+1)-2M_2a^{-2} &= aA_2',\\ mb^6J_3+b^4J_2+K_3+b^2K_2+(mb^2\log b)K+M_2b^{-2} &= B_2,\\ 6mb^6J_3+4b^4J_2+2b^2K_2+Kmb^2(2\log b+1)-2M_2b^{-2} &= bB_2'; \end{aligned}\right\} \quad \ldots \quad (21)$$

and for $n \geqslant 3$

$$\left.\begin{aligned} &ma^{n+4}J_{n+1}+a^{n+2}J_n+a^{-(n-2)}K_{n+1}+ma^{-(n-4)}K_n+L_na^n+M_na^{-n} = A_n,\\ &m(n+4)a^{n+4}J_{n+1}+(n+2)a^{n+2}J_n-(n-2)a^{-(n-2)}K_{n+1}\\ &\qquad -m(n-4)a^{-(n-4)}K_n+nL_na^n-nM_na^{-n} = aA_n',\\ &mb^{n+4}J_{n+1}+b^{n+2}J_n+b^{-(n-2)}K_{n+1}+mb^{-(n-4)}K_n+L_nb^n+M_nb^{-n} = B_n,\\ &m(n+4)b^{n+4}J_{n+1}+(n+2)b^{n+2}J_n-(n-2)b^{-(n-2)}K_{n+1}\\ &\qquad -m(n-4)b^{-(n-4)}K_n+nL_nb^n-nM_nb^{-n} = bB_n'. \end{aligned}\right\} \quad (22)$$

The solution of these equations is not direct, for there are five unknown constants J_0, J_1, K_0, L_0, M_0 in the set of four equations (19). The solution of the actual set of equations at first gives the constants L_0, M_0, K_0, J_1 in terms of the given constants A_0, A_0' together with J_0. The correct value of J_0 has then to be determined from a condition of convergence. It is clear that if J_0 is known, all the constants are uniquely determined by the sets of eqns. 19 to 22.

Equation (19) gives

$$m\left\{(b^4-a^4)-4a^4\log\frac{b}{a}\right\}J_1+\left\{[(b^2-a^2)+m^2(b^4-a^4)]\right.$$
$$\left.-[(2a^2+4m^2a^4)\log a+(a^2+m^2a^4)]\log\frac{b}{a}\right\} = X_0,$$

$$4mJ_1(b^4-a^4)+\{(2b^2+4m^2b^4)\log b-(2a^2+4m^2a^4)\log a+(b^2+m^2b^4)$$
$$-(a^2+m^2a^4)\} = Y_0,$$

where

$$X_0 = -J_0\left\{(b^2-a^2)-2a^2\log\frac{b}{a}\right\}+B_0-A_0-aA_0'\log\frac{b}{a},$$

$$Y_0 = -2J_0(b^2-a^2)+bB_0'-aA_0'.$$

Equations (20) and (21) give

$$6mJ_2(b^4-a^4)+2K\log\frac{b}{a} = X_1,$$

$$4mJ_2\left(\frac{b^5}{a}-\frac{a^5}{b}\right)+K\left(\frac{b}{a}-\frac{a}{b}\right) = Y_1,$$

where

$$X_1 = -4(b^2-a^2)(mJ_0+J_1)-2mK_0\{b^2(4\log b+1)-a^2(4\log a+1)\}$$

$$+\frac{1}{b}(bB_1'+B_1)-\frac{1}{a}(aA_1'+A_1),$$

$$Y_1 = -2(mJ_0+J_1)\left(\frac{b^3}{a}-\frac{a^3}{b}\right)-2mK_0\left\{2\left(\frac{b^3}{a}\log b-\frac{a^3}{b}\log a\right)+\left(\frac{b^3}{a}-\frac{a^3}{b}\right)\right\}$$

$$+\frac{1}{a}(bB_1'-B_1)-\frac{1}{b}(aA_1'-A_1)$$

and

$$8mJ_3a^2b^2(b^4-a^4)-2K_3(b^2-a^2) = X_2,$$

$$4mJ_3\left(\frac{b^6}{a^2}-\frac{a^6}{b^2}\right)+2K_3\left(\frac{1}{b^2}-\frac{1}{a^2}\right) = Y_2,$$

where

$$X_2 = -6a^2b^2(b^2-a^2)J_2-mK\{b(4\log b+1)-a(4\log a+1)\}a^2b^2$$

$$+a^2(bB_2'+2B_2)-b^2(aA_2'+2A_2),$$

$$Y_2 = -2J_2\left(\frac{b^4}{a^2}-\frac{a^4}{b^2}\right)-mK\left(\frac{b^2}{a^2}-\frac{a^2}{b^2}\right)+\frac{1}{a^2}(bB_2'-2B_2)-\frac{1}{b^2}(aA_2'-2A_2).$$

Eliminating L_n and M_n from (22), we have

$$\left.\begin{aligned} 2m(n+2)J_{n+1}a^nb^n(b^4-a^4)+2K_{n+1}\left(\frac{a^n}{b^{n-2}}-\frac{b^n}{a^{n-2}}\right) &= X_n,\\ 4mJ_{n+1}\left(\frac{b^{n+4}}{a^n}-\frac{a^{n+4}}{b^n}\right)-2(n-1)\frac{1}{a^nb^n}K_{n+1}(b^2-a^2) &= Y_n \end{aligned}\right\} \quad \ldots \quad (23)$$

where

$$X_n = -2(n+1)J_na^nb^n(b^2-a^2)-4mK_n\left(\frac{a^n}{b^{n-4}}-\frac{b^n}{a^{n-4}}\right)+a^n(bB_n'+nB_n)$$

$$-b^n(aA_n'+nA_n),$$

$$Y_n = 2(n-2)mK_n\left(\frac{a^n}{b^{n-4}}-\frac{b^n}{a^{n-4}}\right)-2J_n\left(\frac{b^{n+2}}{a^n}-\frac{a^{n+2}}{b^n}\right)+\frac{1}{a^n}(bB_n'-nB_n)$$

$$-\frac{1}{b^n}(aA_n'-nA_n).$$

Solving eqns. (23) and substituting for X_n, Y_n, we have

$$\left.\begin{aligned} J_{n+1} &= [J_{n+1}]/m\Delta_{n+1},\\ K_{n+1} &= [K_{n+1}]/\Delta_{n+1}, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where

$$[J_{n+1}] = K_n\left\{2(n-2)m\left(\frac{a^n}{b^{n-4}}-\frac{b^n}{a^{n-4}}\right)\left(\frac{a^n}{b^{n-2}}-\frac{b^n}{a^{n-2}}\right)-4m(n-1)(b^2-a^2)\left(\frac{1}{b^{2n-4}}-\frac{1}{a^{2n-4}}\right)\right.$$

$$-J_n\left\{2\left(\frac{b^{n+2}}{a^n}-\frac{a^{n+2}}{b^n}\right)\left(\frac{a^n}{b^{n-2}}-\frac{b^n}{a^{n-2}}\right)+2(n^2-1)(b^2-a^2)^2\right\}$$

$$+\left(\frac{a^n}{b^{n-2}}-\frac{b^n}{a^{n-2}}\right)\left\{\frac{1}{a^n}(bB_n'-nB_n)-\frac{1}{b^n}(aA_n'-nA_n)\right\}$$

$$+(n-1)(b^2-a^2)\left\{\frac{1}{b^n}(bB_n'+nB_n)-\frac{1}{a^n}(aA_n'+nA_n)\right\}, \qquad \ldots \quad \ldots \quad (24A)$$

$$[K_{n+1}] = 2a^nb^n(b^2-a^2)J_n\left\{(n+2)\left(\frac{b^{n+2}}{a^n}-\frac{a^{n+2}}{b^n}\right)-2(n+1)\left(\frac{b^{n+4}}{a^n}-\frac{a^{n+4}}{b^n}\right)\right\}$$

$$-2mK_n\left\{4\left(\frac{a^n}{b^{n-4}}-\frac{b^n}{a^{n-4}}\right)\left(\frac{b^{n+4}}{a^n}-\frac{a^{n+4}}{b^n}\right)+(n^2-4)\left(\frac{a^n}{b^{n-4}}-\frac{b^n}{a^{n-4}}\right)a^nb^n(b^4-a^4)\right\}$$

$$+2\left(\frac{b^{n+4}}{a^n}-\frac{a^{n+4}}{b^n}\right)\left\{a^n(bB_n'+nB_n)-b^n(aA_n'+nA_n)\right\}$$

$$-(n+2)a^nb^n(b^4-a^4)\left\{\frac{1}{a^n}(bB_n'-nB_n)-\frac{1}{b^n}(aA_n'-nA_n)\right\}, \qquad \ldots \quad (24B)$$

$$\Delta_{n+1} = \left\{4\left(\frac{a^n}{b^{n-2}}-\frac{b^n}{a^{n-2}}\right)\left(\frac{b^{n+4}}{a^n}-\frac{a^{n+4}}{b^n}\right)+2(n+2)(n-1)(b^4-a^4)(b^2-a^2)\right\}. \qquad (24C)$$

It follows from eqns. 24A to 24C that for large values of n

$$-4mJ_{n+1} \doteq \frac{2a^2}{b^4}m\left\{(n-2)a^2+\frac{2}{b^{2n}}(n-1)(b^2-a^2)\right\}K_n-\frac{2}{b^2}J_n-\frac{1}{b^{n+4}}$$

$$\times(bB_n'-nB_n)-(n-1)(b^2-a^2)\frac{a^{n-2}}{b^{2n+4}}(aA_n'+nA_n),$$

$$-4K_{n+1} \doteq 2a^2\left\{4m+(n^2-4)(b^4-a^4)\frac{a^{2n}}{b^4}\right\}K_n$$

$$-2(b^2-a^2)a^{2n-2}\left\{2(n+1)-\frac{(n+2)}{b^2}(b^2+a^2)\right\}J_n$$

$$-a^{n-2}(aA_n'+nA_n)-(n-2)(b^4-a^4)\frac{a^{2n-2}}{b^{n+4}}(bB_n'-nB_n).$$

Now putting

$$J_n^* = b^nJ_n,\ K_n^* = \frac{K_n}{a^n},$$

we have

$$-4mJ^*_{n+1} \doteq 2mb^{n-2}a^{n-2}\left\{(n-2)a^8+\frac{2}{b^{2n}}(n-1)(b^2-a^2)\right\}K^*_n-\frac{2}{b}J^*_n$$

$$-\frac{1}{b^3}(bB'_n-nB_n)-(n-1)(b^2-a^2)\frac{a^{n-2}}{b^{n+3}}(aA'_n+nA_n), \quad \ldots \quad (25A)$$

$$-4K^*_{n+1} \doteq 2a\left\{4m+(n^2-4)(b^4-a^4)\frac{a^{2n}}{b^4}\right\}K^*_n$$

$$-2(b^2-a^2)b^na^{n-3}\{2(n+1)-(n+2)(b^2+a^2)\}J^*_n$$

$$-\frac{1}{a^3}(aA'_n+nA_n)-(n+2)(b^4-a^4)\frac{a^{n-3}}{b^{n+4}}(bB'_n-nB_n). \quad \ldots \quad (25B)$$

For the convergence of the series (17), it is necessary that $J^*_{n+1} \to 0$ and $K^*_{n+1} \to 0$ as $n \to \infty$. The terms in (25*A*) and (25*B*) which depend on A_n, A'_n, B_n, B'_n tend to zero because of the convergence of the original series (12). The coefficient of K^*_n in eqn. 25*A* is of order $n(ab)^n$ and tends to zero, as does the coefficient of J^*_n in eqn. 25*B*. If these terms are left out

$$J^*_{n+1} \doteq \frac{1}{2mb}J^*_n, \; K^*_{n+1} \doteq -2amK^*_n. \quad \ldots \quad \ldots \quad (26)$$

Since $\frac{1}{b} \geqslant 1$ and $a < 1$, it is the coefficient J^*_{n+1} that may not tend to zero unless J_0 has the correct value. This value of J_0 is determined approximately by taking $J^*_n = 0$ for successively larger and larger. Calculating numerically for $a = \frac{1}{4}$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $m = \frac{1}{5}$, $\lambda = \frac{\pi}{2}$ and $A = 1$, it is seen that

$$J_1 = 0 \text{ gives } J_0 = 2\ (0{\cdot}7989),$$

$$J_2 = 0 \text{ gives } J_0 = 2\ (0{\cdot}7864),$$

$$J_3 = 0 \text{ gives } J_0 = 2\ (0{\cdot}7873),$$

$$J_4 = 0 \text{ gives } J_0 = 2\ (0{\cdot}7873),$$

$$J_5 = 0 \text{ gives } J_0 = 2\ (0{\cdot}7872).$$

These values give the successive approximations to the correct value of J_0.

4. Determination of G_2

To determine G_2, we start with

$$G'_2 = |z-z_0|^2 \log|z-z_0| - |z-\bar{z}_0|^2 \log|z-\bar{z}_0|. \quad \ldots \quad \ldots \quad (27)$$

Then

$$G_2' = \sum_{n=1}^{\infty} C_n \sin n\phi,\ \frac{\partial G_2'}{\partial \rho} = \sum_{n=1}^{\infty} C_a' \sin n\phi, \text{ on } \rho = a,$$

and

$$\left.\begin{aligned} G_2' = \sum_{n=1}^{\infty} D_n \sin n\phi,\ \frac{\partial G_2'}{\partial \rho} = \sum_{n=1}^{\infty} D_n' \sin n\phi, \text{ on } \rho = b, \end{aligned}\right\} \quad .. \quad (28)$$

where

$$\left.\begin{aligned} C_n &= 2(Q_n)_{\rho=a}, \quad C_2' = 2\left(\frac{\partial Q_n}{\partial \rho}\right)_{\rho=a}, \\ D_n &= 2(S_n)_{\rho=b}, \quad D_n' = 2\left(\frac{\partial S_n}{\partial \rho}\right)_{\rho=b}. \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (29)$$

The function G_2 can be written as

$$G_2 = G_2' - G_2'', \qquad .. \quad .. \quad .. \quad .. \quad (30)$$

where G_2'' is a biharmonic function with no singularity in the plate and which satisfies (28) when G_2' is replaced by G_2''. We can assume

$$G_2'' = \frac{r^2}{A^2} \sum_{n=1}^{\infty} (j_n' \rho^n + k_n' \rho^{-n}) \sin n\phi + \frac{y}{A} k \log \rho + \sum_{n=1}^{\infty} (l_n \rho^n + m_n \rho^{-n}) \sin n\phi \qquad (31)$$

giving

$$\begin{aligned} G_2'' = &\{m\rho^5 j_2 + \rho^3 j_1 + \rho k_1 + k\rho \log \rho + m_1 \rho^{-1}\} \sin \phi \\ &+ \{m\rho^6 j_3 + \rho^4 j_2 + k_3 + \rho^2 k_2 + mk\rho^2 \log \rho + m_2 \rho^{-2}\} \sin 2\phi \\ &+ \sum_{n=3}^{\infty} \{m\rho^{n+4} j_{n+1} + \rho^{n+2} j_n + \rho^{-(n-2)} k_{n+1} + m\rho^{-(n-4)} k_n + l_n \rho^n + m_n \rho^{-n}\} \sin n\phi, \end{aligned} \quad .. \quad (32)$$

and

$$\begin{aligned} \left(\rho \frac{\partial G_2''}{\partial \rho}\right) = &\{5m\rho^5 j_2 + 3\rho^3 j_1 + k_1 \rho + k\rho(\log \rho + 1) - m_1 \rho^{-1}\} \sin \phi \\ &+ \{6m\rho^6 j_3 + 4\rho^4 j_2 + 2\rho^2 k_2 + m\rho^2 (2 \log \rho + 1)k - 2\rho^{-2} m_2\} \sin 2\phi \\ &+ \sum_{n=3}^{\infty} \{m(n+4)\rho^{n+4} j_{n+1} + (n+2)\rho^{n+2} j_n - (n-2)\rho^{-(n-2)} k_{n+1} \\ &- m(n-4)\rho^{-(n-4)} k_n + n\rho^n l_n - n\rho^{-n} m_n\} \sin n\phi, \end{aligned} \quad .. \quad .. \quad .. \quad (33)$$

where

$$\begin{aligned} j_1' + m^2 k_1' &= j_1, \\ mj_n' + j_{n+1}' &= j_{n+1} \qquad (n = 1, 2, 3 \ldots); \\ l_1 + k_1' + mk_2' &= k_1, \quad mk_1' + m^2 k_2' + l_2 = k_2, \\ k_n' + mk_{n+1}' &= k_{n+1} \quad (n = 3, 4, 5 \ldots). \end{aligned}$$

The equations to determine the unknown coefficients are

$$\left.\begin{aligned} ma^5 j_2+a^3 j_1+ka\log a+ak_1+m_1a^{-1} &= C_1,\\ 5ma^5 j_2+3a^3 j_1+ka(\log a+1)+ak_1-m_1a^{-1} &= aC_1',\\ mb^5 j_2+b^3 j_1+kb\log b+bk_1+m_1b^{-1} &= D_1,\\ 5mb^5 j_2+3b^3 j_1+kb(\log b+1)+bk-m_1b^{-1} &= bD_1'; \end{aligned}\right\} \quad \ldots \quad (34)$$

$$\left.\begin{aligned} ma^6 j_3+a^4 j_2+k_3+a^2k_2+ma^2k\log a+m_2a^{-2} &= C_2,\\ 6ma^6 j_3+4a^4 j_2+2a^2k_2+ma^2k(2\log a+1)-2m_2a^{-2} &= aC_2',\\ mb^6 j_3+b^4 j_2+k_3+b^2k_2+mb^2k\log b+m_2b^{-2} &= D_2,\\ 6mb^6 j_3+4b^4 j_2+2b^2k_2+mb^2k(2\log b+1)-2m_2b^{-2} &= bD_2'; \end{aligned}\right\} \quad \ldots \quad (35)$$

and for $n \geqslant 3$

$$\left.\begin{aligned} ma^{n+4} j_{n+1}+a^{n+2} j_n+a^{-(n-2)}k_{n+1}+ma^{-(n-4)}k_n+l_na^n+m_na^{-n} &= C_n,\\ m(n+4)a^{n+4} j_{n+1}+(n+2)a^{n+2} j_n-(n-2)a^{-(n-2)}k_{n+1}-m(n-4)a^{-(n-4)}k_n &\\ +nl_na^n-nm_na^{-n} &= aC_n',\\ mb^{n+4} j_{n+1}+b^{n+2} j_n+b^{-(n-2)}k_{n+1}+mb^{-(n-4)}k_n+l_nb^n+m_nb^{-n} &= D_n,\\ m(n+4)b^{n+4} j_{n+1}+(n+2)b^{n+2} j_n-(n-2)b^{-(n-2)}k_{n+1}-m(n-4)b^{-(n-4)}k_n &\\ +nl_nb^n-nm_nb^{-n} &= bD_n' \end{aligned}\right\} \quad (36)$$

From equations (34) and (35) we get

$$\left.\begin{aligned} 6mab(b^4-a^4) j_2+2kab\log\frac{b}{a} &= X_1',\\ 4m\left(\frac{b^5}{a}-\frac{a^5}{b}\right) j_2+k\left(\frac{b}{a}-\frac{a}{b}\right) &= Y_1', \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

where

$$X_1' = -4ab(b^2-a^2) j_1+a(bD_1'+D_1)-b(aC_1'+C_1),$$

$$Y_1' = -2\left(\frac{b^3}{a}-\frac{a^3}{b}\right) j_1+\frac{1}{a}(bD_1'-D_1)-\frac{1}{b}(aC_1'-C_1),$$

and

$$\left.\begin{aligned} 8ma^3b^3(b^3-a^3) j_3-2k_3(b^3-a^3) &= X_2',\\ 4m\left(\frac{b^6}{a^2}-\frac{a^6}{b^2}\right) j_3+2k_3\left(\frac{1}{b^2}-\frac{1}{a^2}\right) &= Y_2', \end{aligned}\right\} \quad \ldots \quad \ldots \quad (38)$$

where

$$X_2' = -6a^3b^3(b-a) j_2-m\{4a^2b^2(a\log b-b\log a)+a^2b^2(a-b)\}k$$
$$+a^3(bD_2'+2D_2)-b^3(aC_2'+2C_2),$$

$$Y_2' = -2\left(\frac{b^4}{a^2}-\frac{a^4}{b^2}\right) j_2-m\left(\frac{b^2}{a^2}-\frac{a^2}{b^2}\right)k+\frac{1}{a^2}(bD_2'-2D_2)$$
$$-\frac{1}{b}(aC_2'-2C_2).$$

Equations (36) are identical with (22) when the A's and B's of the latter are replaced by C's and D's. Thus j_1 can be evaluated in the same way as J_0 and j_{n+1}, k_{n+1} ($n \geqslant 3$) are given by equations of the form 24, 24A to 24C.

The value of j_1 calculated from $j^*_n = 0$ for $a = \frac{1}{3}$, $b = \frac{3}{4}$, $c = \frac{1}{2}$, $m = \frac{1}{8}$, $A = 1$, $\lambda = 0$ are given below:

$$j_2 = 0 \text{ gives } j_1 = 2\ (0{\cdot}02396),$$

$$j_3 = 0 \text{ gives } j_1 = 2\ (0{\cdot}02619),$$

$$j_4 = 0 \text{ gives } j_1 = 2\ (0{\cdot}02621),$$

$$j_5 = 0 \text{ gives } j_1 = 2\ (0{\cdot}02621).$$

The actual displacement w for a transverse force F acting at $Q(x_0, y_0)$ is given by (4) and may now be expressed as

$$w = \frac{F}{8\pi D}\{|z-z_0|^2 \log |z-z_0| - \tfrac{1}{2}(G_1''+G_2'')\}. \quad \ldots \quad \ldots \quad (39)$$

5. Reactions on the Boundaries

If C is any closed curve drawn in the interior of the plate and $V(x, y)$ denotes the function conjugate to the plane harmonic function $\nabla^2 w$, so that

$$\nabla^2 w + iV = f(z), \quad (z = x+iy)$$

then the resultant force N^*, perpendicular to the plate, on the closed curve C is given by

$$N^* = -D[V]_c = iD[f(z)]_c, \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

where $(U)_c$ denotes the change in the value of U by going once round C (Singh 1961). The resultant moments M_x and M_y of the forces acting on the curve C are given by

$$M_x + iM_y = D\left\{[zf(z)]_c - \int_c f(z)\,dz\right\}. \quad \ldots \quad \ldots \quad (41)$$

To find the reaction on the inner boundary, we know that only the terms containing $r^2 \log \rho$ will contribute to the resultant force and that only the terms containing $x \log \rho$, $y \log \rho$ will contribute to the moment (Singh 1961). Thus to find the resultant force N^*, we can assume

$$W = -\frac{F}{32\pi D}\frac{K_0}{A^2} r^2 \log \rho$$

and obtain

$$\nabla^2 w = -\frac{F}{8\pi D}\frac{K_0}{A^2}\left\{\log \rho + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\log \rho\right\}$$

$$= -\frac{F}{8\pi D}\frac{K_0}{A^2} R\left\{\frac{d}{dz}(z \log \zeta)\right\},$$

where $R(X)$ stands for the real part of X. We can therefore write

$$V = -\frac{F}{8\pi D}\frac{K_0}{A^2}\left\{\phi - x\frac{\partial}{\partial y}\log\rho + y\frac{\partial}{\partial x}\log\rho\right\}$$

and obtain

$$N^* = -D[V]_c = \frac{K_0 F}{4A^2}. \qquad \ldots \qquad (42)$$

To find the resultant moment, we can put

$$W = -\frac{F}{16\pi D}\left\{\frac{K}{A}x\log\rho + \frac{k}{A}y\log\rho\right\}$$

and therefore

$$\nabla^2 w = -\frac{F}{8\pi D}\left\{\frac{K}{A}\frac{\partial}{\partial x}\log\rho + \frac{k}{A}\frac{\partial}{\partial y}\log\rho\right\}$$

$$= -\frac{F}{8\pi D}R\left\{\frac{K}{A}\frac{d}{dz}\log\zeta + i\frac{k}{A}\frac{d}{dz}\log\zeta\right\}.$$

Hence we may take

$$f(z) = -\frac{F}{8\pi D}\frac{K+ik}{A}\frac{d}{dz}\log\zeta$$

and get from (41)

$$M_x + iM_y = -\frac{k+iK}{4A}F.$$

Therefore

$$M_x = -\frac{k}{4A}F,\ M_y = \frac{K}{4A}F. \qquad \ldots \qquad (43)$$

The reactions on the outer boundary can now be easily obtained from statical considerations.

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